18.781 Problem Set 7: Due Wednesday, April 19.

1. Think of a quadratic form as taking a column vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ as input. If $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, one defines a new quadratic form $f\gamma$ by

$$(f\gamma)(v) = (\det\gamma) f(\gamma v).$$

This defines a "right action" of $\mathrm{GL}_2(\mathbb{Z})$ on the set of quadratic forms of discriminant d: $(f\gamma)\delta = f(\gamma\delta)$ and $fI = f$ if I is the identity matrix. Call two quadratic forms f and g *sign-equivalent* if $g = f\gamma$ for some $\gamma \in \mathrm{GL}_2(\mathbb{Z})$.

(a) Check that if f is primitive, so is $f\gamma$.

Thus we have a right action of $\mathrm{GL}_2(\mathbb{Z})$ on $F(d)$. Define a *left* action by $\mathrm{GL}_2(\mathbb{Z})$ by the formula

$$\gamma f = f\gamma^{-1}.$$

Recall the definition of the left action by $\mathrm{GL}_2(\mathbb{Z})$ on the set of quadratic irrationals of discriminant d:

$$\gamma\alpha = \frac{p\alpha + q}{r\alpha + s}, \quad \alpha = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

(b) Show that the map θ is equivariant with respect to these actions of $\mathrm{GL}_2(\mathbb{Z})$. (You may use the fact that $\mathrm{GL}_2(\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

so it's enough to check that θ commutes with the action of these three matrices.)

It follows that θ establishes a bijection between the set of equivalence classes of quadratic irrationals of discriminant d and the set of sign-equivalence classes of quadratic forms of discriminant d.

We write $h(d)$ for the number of equivalence classes in $X(d)$ (or $F(d)$) under this action of $\mathrm{GL}_2(\mathbb{Z})$.

(c) Alternatively, we can alter the action of $\mathrm{GL}_2(\mathbb{Z})$ on $X(d)$, by declaring that γ acts by sending α to $\gamma\alpha$ as before if $\det\gamma = 1$, but to $\gamma\alpha'$ if $\det\gamma = -1$.

Check that this is an action, and if $\mathrm{GL}_2(\mathbb{Z})$ acts without signs on $F(d)$, then θ is equivariant with respect to this action.

(d) The equivalence relation generated by the action of $\mathrm{SL}_2(\mathbb{Z})$
by the action described in **(c)** is generated by the action of S, T, and U:

$$\alpha \equiv \frac{1}{\alpha} \equiv \alpha + 1 \equiv -\alpha'.$$

The first two give the "strict" equivalence relation, which we know is generated by the square of the continued fraction operator. In terms of the set $R(d)$ of reduced quadratic irrationals of discriminant d—that is, those with purely periodic continued fractions—the strict equivalence relation is generated by rotating the period by 2 steps. The final relation is

2. **(a)** Let $p \neq 2, 5$ be prime. Show that p is represented by a positive definite quadratic form of discriminant -20 if and only if $p \equiv 1, 3, 7$ or $9 \pmod{20}$.

(b) Let $p \neq 2, 5$ be prime. Prove that p is represented by $x^2 + 5y^2$ if and only if $p \equiv 1$ or $9 \pmod{20}$, and by $2x^2 - 2xy + 3y^2$ if and only if $p \equiv 3$ or $7 \pmod{20}$. (Hint: Reduce mod 20.) This method of distinguishing between representability by different forms of the same discriminant constitutes Gauss's theory of "genera" of forms. It is totally effective only for finitely many discriminants.

(c) Let $n = 5^b p_1^{c_1} \cdots p_r^{c_r} q_1^{d_1} \cdots q_s^{d_s} u_1^{e_1} \cdots u_t^{e_t}$, where the p's are primes congruent to 1 or 9 mod 20, the q's are primes congruent to 3 or 7 mod 20, and the u's are other primes. Show that if $d_1 + \cdots + d_s$ and each e_i are even, then n is represented by $x^2 + 5y^2$. The following identity will be useful:

$$(2x^2-2xy+3y^2)(2z^2-2zw+3w^2) = (xw+yz+2yw-2xz)^2+5(xw+yz-yw)^2.$$

3. Find one reduced representative of each sign-equivalence class of quadratic forms of the following discriminants:

(a) 328 **(b)** -84 **(c)** -163