18.781 Problem Set 4: Due Wednesday, March 15 (*Second edition*).

1. (a) Show that (as has been claimed in lecture) any rational number has exactly two expressions as a finite proper continued fraction.

(b) At the start of Davenport's chapter on continued fractions, he explains how the Euclidean algorithm leads to an expression of a rational number as a finite continued fraction. In **1(b)(iii)** of Problem Set 1 you worked out the Euclidean algorithm for the pair 1024 and 729. Use that work to express $\frac{1024}{729}$ as a finite continued fraction.

2. Find the continued fraction for $\sqrt{31}$.

3. Let $\alpha = \sqrt[3]{2}$.

(a) Show that

$$\frac{1}{a\alpha^2 + b\alpha + c} = \frac{(b^2 - ac)\alpha^2 + (2a^2 - bc)\alpha + (c^2 - 2ab)}{4a^3 + 2b^3 + c^3 - 6abc}.$$

(b) Find the first four terms of the continued fraction for α, and the first four convergents. Besides **(a)**, it may be useful to note that $\alpha < 2$ and that $\alpha^2 + \alpha + 1 < 4$.

4. Let γ be the "golden ratio" $\frac{1+\sqrt{5}}{2}$. We will see in this problem that this number is "very nearly rational." The following result was mentioned in class.

Theorem 1. For any irrational real α and any $n > 0$,

$$|\alpha - c_m| < \frac{1/\sqrt{5}}{(\operatorname{ht} c_m)^2}$$

for some m with $|n - m| \leq 1$.

We showed that this is false for *rational* α. The number γ shows that the numerator $1/\sqrt{5}$ cannot be improved for general irrational numbers α:

Theorem 2. For any $\delta < 1/\sqrt{5}$ there are only finitely many nonzero rational numbers c for which

$$|\gamma - c| < \frac{\delta}{(\operatorname{ht} c)^2}.$$

(a) What is the continued fraction of γ?

(b) Recall the Fibonacci sequence ϕ_n: $\phi_0 = \phi_1 = 1$; $\phi_{n+1} = \phi_n + \phi_{n-1}$. Express the numerators a_n and the denominators b_n of the convergents of the continued fraction for γ in terms of the Fibonacci sequence, and find the nth "α_n."

We saw in class that for any irrational real α,

$$|\alpha - c_n| = \frac{1}{b_n(b_n\alpha_n + b_{n-1})}.$$

(c) Let $\delta < 1/\sqrt{5}$ and show that there are only finitely many n for which

$$|\alpha - c_n| < \frac{\delta}{(\mathrm{ht}\, c_n)^2}.$$

(Hint: Compute $\lim_{n\to\infty} \frac{b_n^2}{b_n(b_n\alpha_n + b_{n-1})}$.)

(d) Now use the fact that any nonzero rational c such that

$$|\gamma - c| < \frac{1/2}{(\mathrm{ht}\, c)^2}$$

is one of the continued fraction convergents to complete a proof of Theorem 2.

It turns out that $1/\sqrt{5}$ *can* be improved if (and only if) the irrational real α is not equivalent (under the $\mathrm{GL}_2(\mathbb{Z})$-action) to γ. Then it may be replaced by $1/\sqrt{8}$. This is just the beginning; for more see J. W. S. Cassels, *An Introduction to Diophantine Approximation.*

Challenge. Prove Theorem 1 above.

5. (a) What number has continued fraction $\langle 1, \overline{2, 3}\rangle$?

(b) Let n be a positive integer. What number has continued fraction $\langle \overline{n} \rangle$?

6. (a) Prove that there are infinitely many primes congruent to $-1 \pmod 4$. *Hint*: Suppose $\{p_1, \ldots, p_n\}$ is a complete list of such primes, and consider $4p_1 \cdots p_n - 1$.

(b) Prove that there are infinitely many primes congruent to $1 \pmod 4$. *Hint*: Suppose $\{p_1, \ldots, p_n\}$ is a complete list of such primes, and consider $(p_1 \cdots p_n)^2 + 1$.

I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.—Isaac Newton