18.781 Problem Set 9: Due Friday, May 12

1. Let I be an ideal in a commutative ring A. Prove:

(a) I is prime if and only if A/I is an integral domain.

(b) I is maximal if and only if A/I is a field.

(c) $\alpha \in A$ is prime (i.e., if $\alpha|\beta\gamma$ then either $\alpha|\beta$ or $\alpha|\gamma$) if and only if the principal ideal $A\alpha$ is prime.

2. As we observed in class, the maximal order $A(-23)$ fails to be a unique factorization domain, because for example

$$\frac{1+\sqrt{-23}}{2} \cdot \frac{1-\sqrt{-23}}{2} = 2 \cdot 3.$$

Each of these four numbers is irreducible but not prime in $A(-23)$.

(a) Compute the class number $h(-23)$.

(b) Factor the principal ideals of these four numbers as products of (non-principal) ideals, and show how the equation above is consistent with unique factorization of ideals.

3. Last week I asked you to determine the group structure of $Cl(-164)$, and you found it to be cyclic of order 8. I did not ask you to write down an isomorphism, then, but I do so now. Thus: you have determined $R(-164)$. Each element $\alpha \in R(-164)$ determines a fractional ideal $\langle 1, \alpha \rangle$. (If you prefer, you may multiply through by the "denominator" a to obtain a true ideal rather than a fractional one.) Select one which generates the class group, and find its sequence of powers. (Needless to say, I want you to describe these ideal classes by writing down the corresponding elements of $R(-164)$.)

4. Davenport lists the reduced quadratic forms of discriminant -15: they are $f(x, y) = x^2 + xy + 4y^2$ and $g(x, y) = 2x^2 + xy + 2y^2$. Thus every positive-definite form of discriminant 15 is strictly equivalent to one of these two, and so takes on the same set of values as one of these two. Make a table of the values assumed by these two forms, up to the value 50. Then make three lists, of the products members of these sets. Formulate and prove a conjecture, concerning products of values of forms of a general discriminant.