# CHAPTER 7. POISSON DISTRIBUTIONS.

We have seen above that values of binomial probabilities can often be found with good accuracy by taking areas under the standard normal curve. There is a second way of getting approximate values for binomial probabilities. It is called the Poisson approximation. It gives good accuracy, like the normal approximation, and can often be used in cases where the normal approximation does not apply.

The formula:

$$\frac{e^{-m}\, m^{x}}{x!}$$

is called the Poisson probability formula, and we abbreviate it as:

$$\underline{p}(x;m) = \frac{e^{-m}\, m^{x}}{x!}$$

Poisson's formula can be used to get values for the binomial formula $b(x;n,p)$ when $n$ is large and $p$ is small (we explain this further below). To use Poisson's formula, we simply take $m = np$ and then have:

$$\underline{p}(x;np) \approx b(x;n,p),$$

where, as before, "$\approx$" means "approximately equals."

Example. If the probability of a given phone call being wrongly dialled is 0.001, what is the probability that there will be exactly three wrongly dialled calls in the next 2000 calls observed? If we can assume that the outcome wrongly dialled is independent from one call to the next (as seems reasonable), then we have a binomial experiment. The exact probability must be:

$$b(3;2000,0.001) = \binom{2000}{3}(0.001)^3(0.999)^{1997} .$$

For the Poisson approximation, we have $np = 2000(0.001) = 2$, and we get:

$$\underline{p}(3;2) = \frac{e^{-2}\,2^3}{3!} = 0.180.$$

(Direct calculation of $b(3;2000,0.001)$ gives $0.181$ .)

Note that since $\frac{np}{q} \approx 2$ and so is not $\geq 9$, this example does not meet the condition in Chapter 6 for using the normal approximation.

Remark. When can the Poisson approximation be used? By a detailed study one can show that $\underline{p}(x;np)$ will give the binomial probability $b(x;n,p)$ to an accuracy of two decimal places if the following condition holds: $p \leq 0.1$ and $n \geq 500p$. For $n \geq 100$, it will always be the case, for any value of $p$, that at least one of the two approximations (normal or Poisson) can be used with two decimal place accuracy (if $p$ is close to $1$, we may have to interchange $p$ and $q$ in order to use the Poisson approximation). If $n < 100$, the conditions for normal and for Poisson approximation to give two-decimal-place accuracy may occasionally both fail. In such cases, we should calculate the binomial formula directly (or use tables for it) if we want accuracy to two decimal places. (In many cases not covered by its condition, however, the Poisson formula gives an accuracy that is nearly as good as two decimal places, as does normal approximation in many cases not covered by the condition that $\frac{np}{q}$ and $\frac{nq}{p}$ be $\geq 9$.)

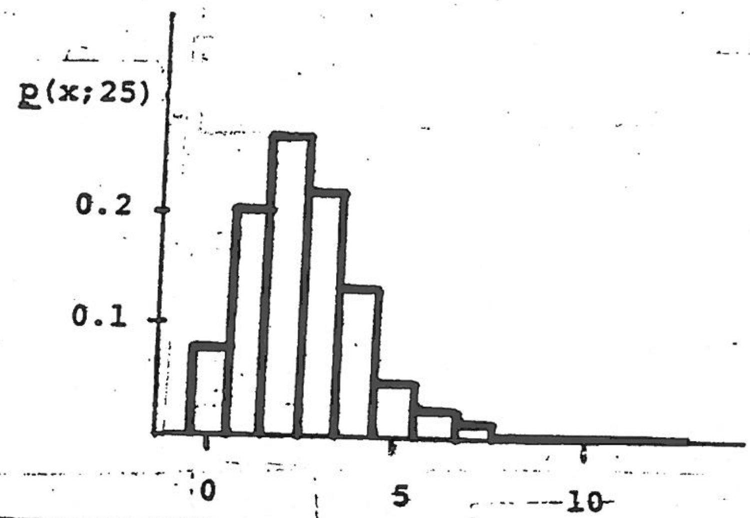
Remark. We can apply the Poisson approximation if we know the product $np$ and if we know that $n$ is large and $p$ is small. (It is enough, for example, to know that $n \geq 50$ and $p \leq 0.1$.) We do not need to know the exact values for $n$ and $p$ separately. Suppose for example, that for peak hours at a certain telephone exchange, the values of $n$ and $p$ are unknown, where $n$ is the number of calls in an hour and $p$ is the probability that a single call is wrongly dialed. Suppose also that we have observed, over a longer period, that the average number of wrongly dialed calls in a peak hour is 5. Then $\frac{5}{n}$ is the relative frequency of wrong calls and must be close to $p$ (by the stability of relative frequencies). Hence we can say that $np$ must be close to 5. We can therefore apply the Poisson approximation, taking $m = 5$, even though we do not know $n$ and $p$. We get:

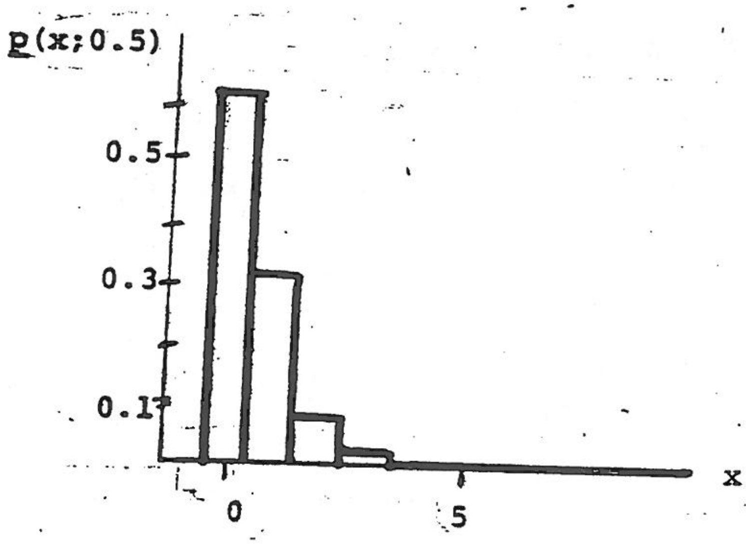
$$P(\text{exactly } x \text{ wrong calls in a peak hour}) \approx \frac{e^{-5}\,5^x}{x!} .$$

Hence we can conclude, for instance, that in any given peak hour, the probability of exactly two wrong calls $\approx \frac{e^{-5}\,5^2}{2!} = 0.084$, and the probability of no wrong calls $\approx \frac{e^{-5}\,5^0}{0!} = e^{-5} = 0.007$.

The Poisson distribution. So far, in this chapter, we have seen that the Poisson formula gives approximate values for binomial probabilities (under certain circumstances), and that in order to use it, we only need to know the value of the product $np$. We now turn to a second and entirely separate use for the Poisson formula. Here, we use the Poisson formula directly to define, for each value of $m$, a probability function on the sample space $S = \{0,1,2,3,\ldots\}$. These new probability spaces will exist as probability spaces in their own right, quite apart from any binomial distribution. They are called Poisson distributions. Let $m$ be a fixed real number $> 0$. Consider a sample space with the points $0,1,2,\ldots$ . Let the probability value for the point $x$ be given by the Poisson formula $p(x;m)$. To check that we have a probability function, we must show that the sum of the probability values over the whole space $= 1$. This means that we must find the sum (of the infinite series):

We also speak of such a table of values as a <u>Poisson distribution</u>. The two Poisson distributions above can be graphed as:




These graphs are typical. When $m < 1$, the Poisson distribution decreases from left to right. When $m > 1$, the distribution increases for $x$ values up to the neighborhood of $m$, and then decreases. When $m = 1$, the first two values are equal, while later values decrease.

$$\sum_{x=0}^{\infty} \underline{p}(x;m) =$$

$$\frac{e^{-m} m^0}{0!} + \frac{e^{-m} m^1}{1!} + \frac{e^{-m} m^2}{2!} + \ldots$$

But this =

$$e^{-m}(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots)$$

and the last series in parentheses is the power series for $e^m$. Hence our original series $= e^{-m}e^m = 1$, and we do have a probability function. (Our probability space is hence an infinite, discrete probability space (see Chapter 2.) If an event A in this sample space has infinitely many points in it, we find P(A) by evaluating the corresponding infinite series (to whatever accuracy we need).)

Probability values for such a space can be given in a table. For example, when $m = 2.5$ we get:

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{p}(x;2.5)$ | 0.082 | 0.205 | 0.257 | 0.214 | 0.134 | 0.067 | 0.028 | 0.010 | 0.003 | 0.001 | 0.000 | ... |

and when $m = 0.5$:

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $\underline{p}(x;0.5)$ | 0.607 | 0.303 | 0.076 | 0.013 | 0.002 | 0.000 | 0.000 | ... |

Poisson experiments. As in the case of binomial distributions, we use the phrase "Poisson distribution" to refer both to the probability function given by the table and to the probability space described by the table. Poisson distributions are common and useful in applications of probability theory. Even when there is no reason to think that an experiment is binomial, the experiment may follow a Poisson distribution. Usually such an experiment takes the form of counting occurrences of a kind that can be informally described as "independent" and "random" in space or time. Such experiments, for which we believe that a Poisson distribution is a good probability space, are sometimes called Poisson experiments. Here are some examples of Poisson experiments:

(i) Count the number of typographical errors in a column of text in a given newspaper.

(ii) Count the number of red cells in a single grid-square of a blood smear on a ruled microscope slide.

(iii) Count the number of traffic accidents reported in a certain city in a single week-end.

(iv) Count the number of almonds in one bar for a certain brand of chocolate almond bar.

(v) Count the number of checks cashed in a certain store in a fixed time interval at a given time of day.

(vi) Count the number of clicks in a fixed time interval on a Geiger counter that is monitoring a simple one-stage radioactive decay process in a macroscopic source.

We give further examples in the text that follows.

In some but not all of these cases, it is possible to think of the underlying process as binomial or approximately binomial. For example, in the case of the almond bars, we can imagine a single bar as poured from a large vat in which there is some large number of almonds $n$, each having a small probability $p$ of being selected for our particular bar. (Of course, this situation is only approximately binomial since the "trials" are not fully independent - if one almond is chosen, there is less room in the bar for the other almonds.)

A common feature of these examples, and of virtually all Poisson experiments, is that they count occurrences that can be informally described as "accidental" and "occasional." In older texts, the Poisson formula is sometimes called the law of rare events.

We thus see that a Poisson distribution can be used in two distinct ways: (1) it can be used (and thought of) as an approximation to a binomial distribution; or (2) it can be used (and thought of ) as a probability space in its own right, apart from any binomial distribution. Although we have first introduced Poisson distributions in connection with binomial approximation, uses of the second kind (Poisson experiments) are perhaps more important. Poisson experiments will occur often in later examples in this book.

Fitting a Poisson distribution. If we have a Poisson experiment (that is, an experiment for which a Poisson distribution is a good probability space), what value of m should we take to get a distribution that best fits the data observed from the experiment? We answer this question by noting that if observed relative frequencies in a given experiment followed a Poisson distribution exactly (for some fixed m), then the average value of x observed in the long run would have to be:

$$0 \cdot \underline{p}(0;m) + 1 \cdot \underline{p}(1;m) + 2 \cdot \underline{p}(2;m) + \ldots$$

$$= 0 + 1 \cdot \frac{e^{-m}\, m}{1} + 2 \cdot \frac{e^{-m}\, m^2}{2!} + 3 \cdot \frac{e^{-m}\, m^3}{3!} + \ldots$$

$$= me^{-m}\left(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\right)$$

$$= me^{-m} \cdot e^{m} = m\ .$$

This suggests that when we start with observed data, we should choose our Poisson distribution by setting m = the average observed value of x. This is what is usually done in order to get a useful probability space for a given Poisson experiment.

Thus, if we believe that the number of fire-alarms per week in a certain town follows a Poisson distribution, and if the average number recently observed is 3 per week, then we use the distribution given by $\underline{p}(x;3)$ as our model. We can then conclude, for example, that the probability of a given week having more than 5 fire alarms is:

$$\underline{p}(6;3) + \underline{p}(7;3) + \underline{p}(8;3) + \underline{p}(9;3) + \underline{p}(10;3) + \underline{p}(11;3) + \ldots$$

$$= 0.050 + 0.022 + 0.008 + 0.003 + 0.001 + 0.000 + \ldots$$

$$= 0.084.$$

Normal approximation to the Poisson distribution. From the conditions given above for the Poisson approximation and given in Chapter 6 for the normal approximation, we can see that for $p \leq 0.1$ and $\frac{np}{q} \geq 9$, both the normal and Poisson approximations to the binomial distribution give accuracy to two decimal places. This means that for $p \leq 0.1$ and $\frac{np}{q} \geq 9$, the values obtained for a Poisson formula and for a corresponding normal area are approximately equal to each other (with an accuracy close to two decimal places.) In particular, normal areas can be used to find probability values for a Poisson distribution when $m \geq 9$. (Think of a binomial experiment with $n$ large, $p$ small and $np \geq 9$. Then $q \approx 1$ and hence both $\frac{np}{q} \geq 9$ and $\frac{nq}{p} \geq 9$.)

For example, if the number of fire alarms per week in a town averages 10, and if we wish to find the probability that there are 15 or more alarms in a given week, we can do the following. We imagine a binomial experiment with n large, p small, and np = 10. Then, following Chapter 6, we have

$$P(15 \le x) = \underline{\text{Normal Area}_z^{\infty}} = \tfrac{1}{2} - A(z),$$

$$\text{where} \qquad z = \frac{15 - \frac{1}{2} - 10}{\sqrt{10}} = 1.42$$

(Here we use $\sqrt{np}$ in place of $\sqrt{npq}$ in the denominator since q = 1-p is close to 1.)

Thus we get $P(15 \le x) = \frac{1}{2} - A(1.42) = \frac{1}{2} - 0.422 = 0.078$

(Direct use of the Poisson distribution for this example gives $\underline{p}(15;10) + \underline{p}(16;10) + \underline{p}(17;10) + \ldots = 0.083$.)

In general, to use normal approximation for a Poisson distribution with $m \ge 9$, we let $z = \dfrac{x \pm 1/2 - m}{\sqrt{m}}$, where the choice of + 1/2 or - 1/2 depends upon how we wish to treat the bar at x (in a bar graph of the Poisson distribution).

__Tables__. Most texts on probability and statistics contain tables of the Poisson distribution. Such tables usually give Poisson distributions for selected values of m ranging from 0 to 10 or 15; for larger values of m, the normal approximation is used as indicated just above. Such tables can be helpful in working with Poisson distributions as models for experiments. Tables for $\underline{p}(x;m)$ for certain values of m up to m = 15 are given at the end of this chapter. (A second column in each table gives, for each $k$, the value of $P(x \geq k)$; that is to say, the sum of the infinite series $\underline{p}(k;m) + \underline{p}(k+1;m) + \underline{p}(k+2;m) + \ldots$.) For values of $m$ not included in the tables, but lying between values of $m$ included in the tables, values of $\underline{p}(x;m)$ may be approximated by linear interpolation. For very small values of $m$, we have $\underline{p}(0;m) \approx 1$ and $\underline{p}(x;m) \approx 0$ for $x > 0$. If an electronic calculator possessing the exponential function is available, values of $\underline{p}(x;m)$ are easy to calculate directly, and values of $P(x \geq k)$ can be obtained by nothing that $P(x \geq k) = 1 - P(x < k)$ and that $P(x < k)$ is a finite sum of $\underline{p}(x;m)$ values.

Note on Accuracy. If one seeks accuracy greater than two decimal places, appropriate conditions for the normal and the Poisson approximations are difficult to state simply. We do not give them here.

Note on Proof. A proof that Poisson's formula approximates the binomial formula when $n$ is large and $p$ is small can be given as follows. Consider $b(x;n,p)$ for fixed $x$. Let $n$ increase, and let $p$ decrease so that $np$ has a constant value. Call this constant value $m$. By elementary algebra,

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x} = \frac{1}{x!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{n-x+1}{n} \cdot n^x \cdot p^x (1-p)^n (1-p)^{-x}.$$

Observing that $(1-p)^n = \left((1 - \frac{m}{n})^{\frac{n}{m}}\right)^m$, that $(1 - \frac{m}{n})^{\frac{n}{m}} \to e^{-1}$ as $n \to \infty$, and that $(1-p)^{-x} = (1 - \frac{m}{n})^{-x} \to 1$ as $n \to \infty$, we obtain

$$b(x;n,p) \to \frac{1}{x!} m^x e^{-m} = \underline{p}(x;m) .$$

Three examples of Poisson experiments.

(1) Deaths from horse-kicks. An early example of observations recognized as fitting a Poisson distribution was given by data on the number of soldiers killed each year by horse-kicks in army corps of the Prussian army. Ten corps were observed for a twenty-year period (1875-1894) giving a total of 200 yearly observations. In 109 of these observations there were no deaths, in 65 there was one death, in 22 there were two deaths, in 3 there were three deaths, and in 1 there were four deaths. This gives a total of 122 deaths. To fit a Poisson distribution, we take $m = \frac{122}{200} = 0.61$. We get the following:

| Number of deaths $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 & over |
|---|---|---|---|---|---|---|---|
| Number of observations | 109 | 65 | 22 | 3 | 1 | 0 | 0 |
| $200\,p(x;0.61)$ (Expected number of observations for the fitted Poisson distribution) | 108.7 | 66.3 | 20.2 | 4.1 | 0.6 | 0.1 | ... |

Here and below, the expected number of observations for a given $x$ is the number which would make the observed relative frequency of $x$ exactly agree with the relative frequency for $x$ predicted from the chosen model (the fitted Poisson distribution). We allow non-integer values for this expected number, even though only integer values can actually occur in observations.

(2) Flying-bomb hits. During Worls War II, data on flying-bomb hits on London were kept by dividing a 12 km. by 12 km. region into 1/4 km. squares and recording the number of hits in each square. 535 hits occurred in the 576 squares observed. To

fit a Poisson distribution, we take $m = \frac{535}{576} = 0.93$. Observed results together with expected results given by the fitted Poisson distribution are as follows:

| Number of hits x | 0 | 1 | 2 | 3 | 4 | 5 | 6 & over |
|---|---|---|---|---|---|---|---|
| Number of squares | 229 | 211 | 93 | 35 | 7 | 1 | 0 |
| 576 $p(x;0.93)$ (Expected number of squares for the fitted Poisson distribution) | 227.3 | 211.4 | 98.3 | 30.5 | 7.1 | 1.3 | ... |

(3) Ties in hockey. 720 games were played in the regular 1977-78 season of the National Hockey League. A total of 4,747 goals were scored in these games. We assume, as a crude model, that the number of goals scored per game by each team is given by a Poisson distribution. We assume further that this number is independent of the number of goals scored by the opposing team, and that the same Poisson distribution holds for each team. We fit this Poisson distribution by taking $m = \frac{4,747}{2(720)} = 3.3$. What is the probability, according to this model, that a game ends in a tie? The Poisson distribution for goals per game for each team is given by the table

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $p(x;3.3)$ | 0.037 | 0.122 | 0.201 | 0.221 | 0.182 | 0.120 | 0.066 | ... |

Then we have:

$$\begin{aligned} P(\text{tie game}) &= P(\text{both score } 0) + P(\text{both score } 1) + P(\text{both score } 2) + \ldots \\ &= (0.037)^2 + (0.122)^2 + (0.201)^2 + \ldots \\ &= 0.158 \end{aligned}$$

In the 720 games played, 132 ties actually occurred, giving an observed relative frequency of $\frac{132}{720} = 0.183$. This observed result appears to agree with our crude model. (For this model, the weak stability law, with $n = 720$, tells us that the observed relative frequency should lie within $\frac{1}{\sqrt{720}} = 0.037$ of the correct probability.)

Comments. In the case of the flying-bomb hits, some squares were hit four or five times while many others escaped completely. These data were seen by some observers at the time as evidence of non-randomness and deliberate aiming. In fact, we know from our discussion of Poisson experiments that the observed data confirm randomness. A more evenly spaced spread of hits would be non-random and would indicate a greater measure of deliberate control on the part of the attacker.

Similarly, the agreement with observation of our crude probabilistic model for hockey suggests that the performance of a team may follow a more random pattern than members of the team may themselves believe. Indeed, the feeling of participants in a game that they are not playing well and that this is somehow causing poor results may be, at least in part, incorrect. A more correct view may be that the poor results, occurring in a random way, are on some occasions causing the feeling of not playing well.

Combining independent Poisson experiments. Town A has an average of $m_1$ fire alarms per week. Assume that observing the number $x_1$ of fire alarms in a given week in Town A is a Poisson experiment. $x_1$ then has the probability distribution $p(x_1;m_1)$. Similarly, assume that observing the number $x_2$ of fire alarms in a given week in Town B is a Poisson experiment with average value $m_2$. Thus $x_2$ has the probability distribution $\underline{p}(x_2;m_2)$. We now perform the two experiments together. That is to say, we observe the total number $y = x_1 + x_2$ of fire alarms occurring in a given week in both towns together. Furthermore, we assume that in this combined experiment, the two original Poisson experiments are <u>independent</u>. (Physically, this means that the outcome of one experiment has no influence on the outcome of the other. Mathematically, this means that for any $j$ and $k$, the events $\underline{x_1 = j}$ and $\underline{x_2 = k}$ are independent events in the probability space for the combined experiment.) What can we conclude about the probability space for the outcome $y$ of the combined experiment?

As the average observed in Town A is $m_1$, and the average observed in Town B is $m_2$, the average observed in the two towns together must be $m_1 + m_2$. (To see this, assume that the average is obtained from the same 100 week period for both towns. Let $N_1$ be the total observed in A over this period and let $N_2$ be the total observed in B. Then the total observed in the two towns together must be $N_1 + N_2$. Hence the observed average for the two towns must be $\frac{N_1+N_2}{100}$. But this equals $\frac{N_1}{100} + \frac{N_2}{100} = m_1 + m_2$.) Furthermore, our intuitive view of Poisson experiments as experiments involving what could be informally described as "independent" and "random" occurrences suggests that the distribution for $y$ should also be a Poisson distribution. Hence, we intuitively expect the combined experiment to follow the distribution $\underline{p}(y;m_1+m_2)$

It is easy to verify mathematically that this intuitive conclusion is correct. For any given value $\ell$, the outcome $y = \ell$ can occur if $x_1 = 0$ and $x_2 = \ell$, or if $x_1 = 1$ and $x_2 = \ell-1$, or if $x_1 = 2$ and $x_2 = \ell-2, \ldots,$ or if $x_1 = \ell$ and $x_2 = 0$. Hence we have

$$P(y = \ell) = P(x_1 = 0 \text{ and } x_2 = \ell) + P(x_1 = 1 \text{ and } x_2 = \ell-1)$$
$$+ \ldots + P(x_1 = \ell \text{ and } x_2 = 0).$$

Using independence, we have, by the multiplication law,

$$P(y = \ell) = P(x_1 = 0)P(x_2 = \ell) + P(x_1 = 1)P(x_2 = \ell-1) + \ldots .$$

Introducing the Poisson formula for each of the original experiments, we get

$$\begin{aligned} P(y = \ell) &= \underline{p}(0;m_1)\underline{p}(\ell;m_2) + \underline{p}(1;m_1)\underline{p}(\ell-1;m_2) + \ldots \\ &= \frac{e^{-m_1}m_1^0}{0!}\,\frac{e^{-m_2}m_2^\ell}{\ell!} + \frac{e^{-m_1}m_1^1}{1!}\,\frac{e^{-m_2}m_2^{\ell-1}}{(\ell-1)!} + \ldots \\ &= e^{-(m_1+m_2)}\left(\frac{1}{0!\,\ell!}\,m_1^0 m_2^\ell + \frac{1}{1!\,(\ell-1)!}\,m_1^1 m_2^{\ell-1} + \ldots\right) \\ &= \frac{e^{-(m_1+m_2)}}{\ell!}\left(\frac{\ell!}{0!\,\ell!}\,m_1^0 m_2^\ell + \frac{\ell!}{1!\,(\ell-1)!}\,m_1^1 m_2^{\ell-1} + \ldots\right) \\ &= \frac{e^{-(m_1+m_2)}}{\ell!}\left(\binom{\ell}{0} m_1^0 m_2^\ell + \binom{\ell}{1} m_1^1 m_2^{\ell-1} + \ldots\right). \end{aligned}$$

Using the binomial theorem, we have

$$P(y = \ell) = \frac{e^{-(m_1+m_2)}}{\ell!}\,(m_1+m_2)^\ell.$$

But this last is simply

$$\underline{p}(\ell;m_1+m_2),$$

as desired.

The general theory of combining independent experiments (not necessarily Poisson) in this way is fundamental in probability and statistics. We consider it in Chapter 16.

The weak stability law in physics. We now show, through an example, how the Poisson and binomial formulas lead to a version of the weak stability of relative frequencies that provides a connection in physics between random fluctuations at the molecular level and such relatively steady measurable quantities as density and pressure.

Let C be a given container of gas at standard pressure and temperature. The volume of C is one litre. Let the mass of the gas in C be d grams. Hence d is also the value of the average density (in grams per litre) of the gas in C. Let M be the number of molecules of gas in C. (From Avogadro's number and principle, $M \approx 2.7 \times 10^{22}$.) We now consider a particular region R within C of volume = 1 cubic millimeter. Let us ask what variations of average density we can expect to occur in R as time passes.

We take as our basic probability model the following: at any instant of time, a given molecule of gas in C has probability $p = 10^{-6}$ of being in R (since $\frac{\text{volume R}}{\text{volume C}} = 10^{-6}$), and, furthermore, the event being in R for one molecule is independent of the event being in R for another different molecule. Hence $x$ = the number of molecules in R at a given instant is, in our model, the outcome of a binomial experiment with $n = M$ and $p = 10^{-6}$. Since $n$ is large and $p$ is small, the binomial distribution $b(x;n,p)$ can be approximated by the Poisson distribution $p(x;N)$ where $N = np = M(10^{-6}) \approx 2.7 \times 10^{16}$ is the average number of molecules per cubic millimeter in C at any moment, as well as the average number of molecules in R as time passes. This Poisson distribution can, in turn, be approximated by the normal distribution as described above. The normal distribution thus tells us, for example, that the probability is 0.95 that $x$ falls in a range such that

$$\frac{x + \frac{1}{2} - N}{\sqrt{N}} \leq 1.96 \quad \text{and} \quad \frac{x - \frac{1}{2} - N}{\sqrt{N}} \geq -1.96.$$

As the term 1/2 is negligible compared to the other quantities appearing, these inequalities become

$$\frac{x - N}{\sqrt{N}} \leq 1.96 \quad \text{and} \quad \frac{x - N}{\sqrt{N}} \geq -1.96,$$

or

$$|x-N| \leq 1.96\sqrt{N}.$$

We can round this off to the approximate form

$$|x-N| \leq 2\sqrt{N}.$$

Thus with probability 0.95 we can expect that the total number of molecules in R will nearly always lie within $2\sqrt{N}$ of N (or, in our specific example, within $3.3 \times 10^8$ of $2.7 \times 10^{16}$).

Now the density in R (in grams per litre) will be $(\frac{xd}{M}) \cdot 10^6 = \frac{xd}{N}$. Hence with probability 0.95, the density in R must nearly always fall within $\frac{2\sqrt{N}d}{N} = \frac{2d}{\sqrt{N}}$ of d (where d is the overall average density of gas in C). Thus in our specific example, the density in R must nearly always fall within $\frac{2 \times 10^{-8}}{1.6} d = 1.2 \times 10^{-8} d$ of d. Since d is also the average value of the density in R as time passes, the probability is 0.95 that at a given moment we will find the density in R to be within 0.0000012 percent of its average value as time passes. Finally, we note that the expression for fluctuations of density in R only involves N, the average number of molecules in R, and hence is independent of the size of C. Empirically observed density fluctuations in very small regions of gases closely confirm calculations of the kind given above and are strong evidence in favor of probabilistic molecular models for gases.

The result in the example above, and empirical observations confirming it, give another version of the weak stability of relative frequencies. This version says that under certain conditions of randomness: if there are, on the average, N events occurring in a given single observation, then we will nearly always observe a variation of less than $2\sqrt{N}$ from this average. This form of the law is also a convenient and useful way to estimate fluctuations in time of other physical quantities such as pressure and temperature.

Note. In the above statement, we used "nearly always" to mean 95 percent of the time. If, for example, we wished to have it mean 99 percent of the time, theory tells us (using areas under the normal curve) that we would have variations of less than $2.6\sqrt{N}$. Variations of less than $3\sqrt{N}$ should occur 99.7 percent of the time. Variations of more than $4\sqrt{N}$ should occur in fewer than one out of every $10^4$ observations, and a variation of more than $10\sqrt{N}$ should occur in fewer than one out of every $10^{22}$ observations.

Predicting variation. The above formulation of weak stability for physical situations gives us a more realistic way to approach the toll-bridge examples of Chapter 1 (pages 14 and 15). In the case of the trucks, if the average number crossing the bridge per week-night is $N_1 = 100$, we can now expect a variation of no more than $2\sqrt{N_1} = 20$ on each side of this average value. Hence the total money received from $10 truck tolls should lie in an interval of length $400 around $1000. Similarly, if the average number of cars is $N_2 = 10,000$ per week-night, then we can expect a variation of $2\sqrt{N_2} = 200$ around this average value. Hence the total money received from 10¢ car tolls should lie in an interval of length $40 around $1000.

These intervals are somewhat longer than the intervals obtained in Chapter 1. This is because, in Chapter 1, we assumed a specific model of decision by coin toss, while here we need assume only that there is some underlying mechanism (which need not be binomial) which justifies our looking upon the experiment as a Poisson experiment. We have lost some accuracy in our estimate, but we have gained greatly in the range of models that we allow.

# POISSON TABLES

| x | m = 0.1 Ind. | m = 0.1 Cum. | m = 0.2 Ind. | m = 0.2 Cum. | m = 0.3 Ind. | m = 0.3 Cum. | m = 0.4 Ind. | m = 0.4 Cum. | m = 0.5 Ind. | m = 0.5 Cum. | m = 0.6 Ind. | m = 0.6 Cum. | m = 0.7 Ind. | m = 0.7 Cum. | x |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 905 | 1. | 819 | 1. | 741 | 1. | 670 | 1. | 607 | 1. | 549 | 1. | 497 | 1. | 0 |
| 1 | 090 | 095 | 164 | 181 | 222 | 259 | 268 | 330 | 303 | 393 | 329 | 451 | 348 | 503 | 1 |
| 2 | 005 | 005 | 016 | 018 | 033 | 037 | 054 | 062 | 076 | 090 | 099 | 122 | 122 | 156 | 2 |
| 3 | | | 001 | 001 | 003 | 004 | 007 | 008 | 013 | 014 | 020 | 023 | 028 | 034 | 3 |
| 4 | | | | | | | 001 | 001 | 002 | 002 | 003 | 003 | 005 | 006 | 4 |
| 5 | | | | | | | | | | | | | 001 | 001 | 5 |

| x | m = 0.8 Ind. | m = 0.8 Cum. | m = 0.9 Ind. | m = 0.9 Cum. | m = 1.0 Ind. | m = 1.0 Cum. | m = 1.1 Ind. | m = 1.1 Cum. | m = 1.2 Ind. | m = 1.2 Cum. | m = 1.3 Ind. | m = 1.3 Cum. | m = 1.4 Ind. | m = 1.4 Cum. | x |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 449 | 1. | 407 | 1. | 368 | 1. | 333 | 1. | 301 | 1. | 273 | 1. | 247 | 1. | 0 |
| 1 | 359 | 551 | 366 | 593 | 368 | 632 | 366 | 667 | 361 | 699 | 354 | 727 | 345 | 753 | 1 |
| 2 | 144 | 191 | 165 | 228 | 184 | 264 | 201 | 301 | 217 | 337 | 230 | 373 | 242 | 408 | 2 |
| 3 | 038 | 047 | 049 | 063 | 061 | 080 | 074 | 100 | 087 | 121 | 100 | 143 | 113 | 167 | 3 |
| 4 | 008 | 009 | 011 | 013 | 015 | 019 | 020 | 026 | 026 | 034 | 032 | 043 | 039 | 054 | 4 |
| 5 | 001 | 001 | 002 | 002 | 003 | 004 | 004 | 005 | 006 | 008 | 008 | 011 | 011 | 014 | 5 |
| 6 | | | | | 001 | 001 | 001 | 001 | 001 | 002 | 002 | 002 | 003 | 003 | 6 |
| 7 | | | | | | | | | | | | | 001 | 001 | 7 |

| x | m = 1.5 Ind. | m = 1.5 Cum. | m = 1.6 Ind. | m = 1.6 Cum. | m = 1.7 Ind. | m = 1.7 Cum. | m = 1.8 Ind. | m = 1.8 Cum. | m = 1.9 Ind. | m = 1.9 Cum. | m = 2.0 Ind. | m = 2.0 Cum. | m = 2.1 Ind. | m = 2.1 Cum. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 223 | 1. | 202 | 1. | 183 | 1. | 165 | 1. | 150 | 1. | 135 | 1. | 122 | 1. |
| 1 | 335 | 777 | 323 | 798 | 311 | 817 | 298 | 835 | 284 | 850 | 271 | 865 | 257 | 878 |
| 2 | 251 | 442 | 258 | 475 | 264 | 507 | 268 | 537 | 270 | 566 | 271 | 594 | 270 | 620 |
| 3 | 126 | 191 | 138 | 217 | 150 | 243 | 161 | 269 | 171 | 296 | 180 | 323 | 189 | 350 |
| 4 | 047 | 066 | 055 | 079 | 064 | 093 | 072 | 109 | 081 | 125 | 090 | 143 | 099 | 161 |
| 5 | 014 | 019 | 018 | 024 | 022 | 030 | 026 | 036 | 031 | 044 | 036 | 053 | 042 | 062 |
| 6 | 004 | 004 | 005 | 006 | 006 | 008 | 008 | 010 | 010 | 013 | 012 | 017 | 015 | 020 |
| 7 | 001 | 001 | 001 | 001 | 001 | 002 | 002 | 003 | 003 | 003 | 003 | 005 | 004 | 006 |
| 8 | | | | | | | | 001 | 001 | 001 | 001 | 001 | 001 | 001 |

| x | m = 2.2 Ind. | m = 2.2 Cum. | m = 2.3 Ind. | m = 2.3 Cum. | m = 2.4 Ind. | m = 2.4 Cum. | m = 2.5 Ind. | m = 2.5 Cum. | m = 2.6 Ind. | m = 2.6 Cum. | m = 2.7 Ind. | m = 2.7 Cum. | m = 2.8 Ind. | m = 2.8 Cum. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 111 | 1. | 100 | 1. | 091 | 1. | 082 | 1. | 074 | 1. | 067 | 1. | 061 | 1. |
| 1 | 244 | 889 | 231 | 900 | 218 | 909 | 205 | 918 | 193 | 926 | 181 | 933 | 170 | 939 |
| 2 | 268 | 645 | 265 | 669 | 261 | 692 | 257 | 713 | 251 | 733 | 245 | 751 | 238 | 769 |
| 3 | 197 | 377 | 203 | 404 | 209 | 430 | 214 | 456 | 218 | 482 | 220 | 506 | 222 | 531 |
| 4 | 108 | 181 | 117 | 201 | 125 | 221 | 134 | 242 | 141 | 264 | 149 | 286 | 156 | 308 |
| 5 | 048 | 072 | 054 | 084 | 060 | 096 | 067 | 109 | 074 | 123 | 080 | 137 | 087 | 152 |
| 6 | 017 | 025 | 021 | 030 | 024 | 036 | 028 | 042 | 032 | 049 | 036 | 057 | 041 | 065 |
| 7 | 005 | 007 | 007 | 009 | 008 | 012 | 010 | 014 | 012 | 017 | 014 | 021 | 016 | 024 |
| 8 | 002 | 002 | 002 | 003 | 002 | 003 | 003 | 004 | 004 | 005 | 005 | 007 | 006 | 008 |
| 9 | | | | 001 | 001 | 001 | 001 | 001 | 001 | 001 | 001 | 002 | 002 | 002 |
| 10 | | | | | | | | | | | | 001 | | 001 |

| x | m = 2.9 Ind. | m = 2.9 Cum. | m = 3.0 Ind. | m = 3.0 Cum. | m = 3.1 Ind. | m = 3.1 Cum. | m = 3.2 Ind. | m = 3.2 Cum. | m = 3.3 Ind. | m = 3.3 Cum. | m = 3.4 Ind. | m = 3.4 Cum. | m = 3.5 Ind. | m = 3.5 Cum. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 055 | 1. | 050 | 1. | 045 | 1. | 041 | 1. | 037 | 1. | 033 | 1. | 030 | 1. |
| 1 | 160 | 945 | 149 | 950 | 140 | 955 | 130 | 959 | 122 | 963 | 113 | 967 | 106 | 970 |
| 2 | 231 | 785 | 224 | 801 | 216 | 815 | 209 | 829 | 201 | 841 | 193 | 853 | 185 | 864 |
| 3 | 224 | 554 | 224 | 577 | 224 | 599 | 223 | 620 | 221 | 641 | 219 | 660 | 216 | 679 |
| 4 | 162 | 330 | 168 | 353 | 173 | 375 | 178 | 397 | 182 | 420 | 186 | 442 | 189 | 463 |
| 5 | 094 | 168 | 101 | 185 | 107 | 202 | 114 | 219 | 120 | 237 | 126 | 256 | 132 | 275 |
| 6 | 045 | 074 | 050 | 084 | 056 | 094 | 061 | 105 | 066 | 117 | 072 | 129 | 077 | 142 |
| 7 | 019 | 029 | 022 | 034 | 025 | 039 | 028 | 045 | 031 | 051 | 035 | 058 | 039 | 065 |
| 8 | 007 | 010 | 008 | 012 | 010 | 014 | 011 | 017 | 013 | 020 | 015 | 023 | 017 | 027 |
| 9 | 002 | 003 | 003 | 004 | 003 | 005 | 004 | 006 | 005 | 007 | 006 | 008 | 007 | 010 |
| 10 | 001 | 001 | 001 | 001 | 001 | 001 | 001 | 002 | 002 | 002 | 002 | 003 | 002 | 003 |
| 11 | | | | | | | | | | 001 | 001 | 001 | 001 | 001 |

| $x$ | $m = 4.0$ Ind. | Cum. | $m = 4.5$ Ind. | Cum. | $m = 5.0$ Ind. | Cum. | $m = 5.5$ Ind. | Cum. | $m = 6.0$ Ind. | Cum. | $m = 6.5$ Ind. | Cum. | $m = 7.0$ Ind. | Cum. | $x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 018 | 1. | 011 | 1. | 007 | 1. | 004 | 1. | 002 | 1. | 002 | 1. | 001 | 1. | 0 |
| 1 | 073 | 982 | 050 | 989 | 034 | 993 | 022 | 996 | 015 | 998 | 010 | 998 | 006 | 999 | 1 |
| 2 | 147 | 908 | 112 | 939 | 084 | 960 | 062 | 973 | 045 | 983 | 032 | 989 | 022 | 993 | 2 |
| 3 | 195 | 762 | 169 | 826 | 140 | 875 | 113 | 912 | 089 | 938 | 069 | 957 | 052 | 970 | 3 |
| 4 | 195 | 567 | 190 | 658 | 175 | 735 | 156 | 798 | 134 | 849 | 112 | 888 | 091 | 918 | 4 |
| 5 | 156 | 371 | 171 | 468 | 175 | 560 | 171 | 642 | 161 | 715 | 145 | 776 | 128 | 827 | 5 |
| 6 | 104 | 215 | 128 | 297 | 146 | 384 | 157 | 471 | 161 | 554 | 157 | 631 | 149 | 699 | 6 |
| 7 | 060 | 111 | 082 | 169 | 104 | 238 | 123 | 314 | 138 | 394 | 146 | 473 | 149 | 550 | 7 |
| 8 | 030 | 051 | 046 | 087 | 065 | 133 | 085 | 191 | 103 | 256 | 119 | 327 | 130 | 401 | 8 |
| 9 | 013 | 021 | 023 | 040 | 036 | 068 | 052 | 106 | 069 | 153 | 086 | 208 | 101 | 271 | 9 |
| 10 | 005 | 008 | 010 | 017 | 018 | 032 | 029 | 054 | 041 | 084 | 056 | 123 | 071 | 170 | 10 |
| 11 | 002 | 003 | 004 | 007 | 008 | 014 | 014 | 025 | 023 | 043 | 033 | 067 | 045 | 099 | 11 |
| 12 | 001 | 001 | 002 | 002 | 003 | 005 | 007 | 011 | 011 | 020 | 018 | 034 | 026 | 053 | 12 |
| 13 | | | 001 | 001 | 001 | 002 | 003 | 004 | 005 | 009 | 009 | 016 | 014 | 027 | 13 |
| 14 | | | | | | 001 | 001 | 002 | 002 | 004 | 004 | 007 | 007 | 013 | 14 |
| 15 | | | | | | | | 001 | 001 | 001 | 002 | 003 | 003 | 006 | 15 |
| 16 | | | | | | | | | | 001 | 001 | 001 | 001 | 002 | 16 |
| 17 | | | | | | | | | | | | | 001 | 001 | 17 |

| $x$ | $m = 8.0$ Ind. | Cum. | $m = 9.0$ Ind. | Cum. | $m = 10.0$ Ind. | Cum. | $m = 11.0$ Ind. | Cum. | $m = 12.0$ Ind. | Cum. | $m = 13.0$ Ind. | Cum. | $m = 14.0$ Ind. | Cum. | $x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | 1. | | 1. | | 1. | | 1. | | 1. | | 1. | | 1. | 0 |
| 1 | 003 | 1− | 001 | 1− | | 1− | | 1− | | 1− | | 1− | | 1− | 1 |
| 2 | 011 | 997 | 005 | 999 | 002 | 1− | 001 | 1− | | 1− | | 1− | | 1− | 2 |
| 3 | 029 | 986 | 015 | 994 | 008 | 997 | 004 | 999 | 002 | 999 | 001 | 1− | | 1− | 3 |
| 4 | 057 | 958 | 034 | 979 | 019 | 990 | 010 | 995 | 005 | 998 | 003 | 999 | 001 | 1− | 4 |
| 5 | 092 | 900 | 061 | 945 | 038 | 971 | 022 | 985 | 013 | 992 | 007 | 996 | 004 | 998 | 5 |
| 6 | 122 | 809 | 091 | 884 | 063 | 933 | 041 | 962 | 025 | 980 | 015 | 989 | 009 | 994 | 6 |
| 7 | 140 | 687 | 117 | 793 | 090 | 870 | 065 | 921 | 044 | 954 | 028 | 974 | 017 | 986 | 7 |
| 8 | 140 | 547 | 132 | 676 | 113 | 780 | 089 | 857 | 066 | 910 | 046 | 946 | 030 | 968 | 8 |
| 9 | 124 | 407 | 132 | 544 | 125 | 667 | 109 | 768 | 087 | 845 | 066 | 900 | 047 | 938 | 9 |
| 10 | 099 | 283 | 119 | 413 | 125 | 542 | 119 | 659 | 105 | 758 | 086 | 834 | 066 | 891 | 10 |
| 11 | 072 | 184 | 097 | 294 | 114 | 417 | 119 | 540 | 114 | 653 | 101 | 748 | 084 | 824 | 11 |
| 12 | 048 | 112 | 073 | 197 | 095 | 303 | 109 | 421 | 114 | 538 | 110 | 647 | 098 | 740 | 12 |
| 13 | 030 | 064 | 050 | 124 | 073 | 208 | 093 | 311 | 106 | 424 | 110 | 537 | 106 | 642 | 13 |
| 14 | 017 | 034 | 032 | 074 | 052 | 136 | 073 | 219 | 090 | 318 | 102 | 427 | 106 | 536 | 14 |
| 15 | 009 | 017 | 019 | 041 | 035 | 083 | 053 | 146 | 072 | 228 | 088 | 325 | 099 | 430 | 15 |
| 16 | 005 | 008 | 011 | 022 | 022 | 049 | 037 | 093 | 054 | 156 | 072 | 236 | 087 | 331 | 16 |
| 17 | 002 | 004 | 006 | 011 | 013 | 027 | 024 | 056 | 038 | 101 | 055 | 165 | 071 | 244 | 17 |
| 18 | 001 | 002 | 003 | 005 | 007 | 014 | 015 | 032 | 026 | 063 | 040 | 110 | 055 | 173 | 18 |
| 19 | | 001 | 001 | 002 | 004 | 007 | 008 | 018 | 016 | 037 | 027 | 070 | 041 | 117 | 19 |
| 20 | | | 001 | 001 | 002 | 003 | 005 | 009 | 010 | 021 | 018 | 043 | 029 | 077 | 20 |
| 21 | | | | | 001 | 002 | 002 | 005 | 006 | 012 | 011 | 025 | 019 | 048 | 21 |
| 22 | | | | | | 001 | 001 | 002 | 003 | 006 | 006 | 014 | 012 | 029 | 22 |
| 23 | | | | | | | 001 | 001 | 002 | 003 | 004 | 008 | 007 | 017 | 23 |
| 24 | | | | | | | | | 001 | 001 | 002 | 004 | 004 | 009 | 24 |
| 25 | | | | | | | | | | 001 | 001 | 002 | 002 | 005 | 25 |
| 26 | | | | | | | | | | | 001 | 001 | 001 | 003 | 26 |
| 27 | | | | | | | | | | | | | 001 | 001 | 27 |
| 28 | | | | | | | | | | | | | | 001 | 28 |

## EXERCISES FOR CHAPTER 7.

7-1. A single card is drawn from a shuffled bridge deck. This experiment is repeated (with replacement) 13 times. Let $x$ = the number of times that the ace of spades appears. The probability distribution for $x$ is given by $b(x;13,\frac{1}{52})$.

(a) Show that the condition for two-decimal accuracy for the Poisson approximation holds.

(b) Show that the condition for two-decimal accuracy for the normal approximation does not hold.

(c) Use the Poisson approximation to estimate $P(x = 0)$, $P(x = 1)$, and $P(x \geq 1)$.

7-2. (a) The probability that a certain thumbtack land on its side when tossed is 0.6. The tack is tossed 15 times. Let $x$ = the number of times it lands on its side. Then the probability distribution for $x$ is given by $b(x;15,0.6)$. Does the condition for two-decimal accuracy for the normal approximation hold? Does the condition for two-decimal accuracy for the Poisson approximation hold?

(b) Answer the same two questions for the case where the same tack is tossed 10 times.

(c) Can Poisson approximation be used to estimate $b(x;30,0.95)$ to two-decimal accuracy? If so, explain how, and get an approximate value for $P(x = 28)$.

7-3. In a certain factory, the probability of one or more accidents occurring on any given day is 0.002. Use the Poisson approximation to estimate the following probabilities:

(a) P(one or more accidents occur on exactly one of the next 1000 days);

(b) P(1000 days go by without an accident);

(c) P(one or more accidents occur on 5 or more of the next 1000 days).

7-4. A certain issue of postage stamps is printed in sheets of 60 stamps. A collector is interested in acquiring corner stamps of this issue. (The corner stamps, occurring at the four corners of each sheet, are perforated on only two sides.) The collector buys a bundle of 100 individual used stamps of the issue. Use the Poisson approximation to find the following probabilities:

(a) P(no corner stamps);

(b) P(three or more corner stamps).

7-5. The number of accidents at a certain intersection averages 7 per week. Assume fairly constant traffic conditions from day to day.

(a) What is the probability that two or more accidents will occur on any given day?

(b) What is the probability that exactly one accident occur?

(c) What is the probability that no accident occur?

7-6. At a given intersection, an average of four cars pass by during each daylight hour.

(a) Find the probability that during a given hour exactly two cars pass the intersection.

(b) Find the probability that during a given four-hour period no cars pass.

(c) Estimate the probability that if a certain ten-hour daylight period is divided into 10 one-hour periods, then the number of these one-hour periods in which no cars pass is at most two.

7-7. During the busiest hours at a telephone exchange, the probability that there are no wrong numbers dialed in a one-minute period is 0.135. Find the probability that there will be exactly one wrong number dialed during a given two-and-a-half-minute period.

7-8. In 1800 innings of baseball, the Red Sox obtain 180 home runs. Assuming that occurrence of home runs follows a Poisson distribution (at least for periods of length nine innings) find the probability that 4 or more Red Sox home runs occur in a given nine-inning game.

7-9. A grocery store sells an average of one jar of caviar per week. Assuming that all purchases are of a single jar and occur independently, what is the minimum number of jars that the store should have on hand at the beginning of a given week in order that the probability be less than 0.05 that more than that number of jars will be required in the given week?

7-10. A baker plans to make 1000 chocolate-chip cookies. For any given cookie, he would like to have P(no chips) be less than 0.01, but he would like to use as few chocolate chips as possible. How many chocolate chips should he put in the cookie mix?

7-11. If newly manufactured automobiles of a certain make have, on the average, 100 defects, estimate the percentage of the autos that have fewer than 80 defects. What percentage have fewer than 60 defects?

7-12. An office switch-board receives an average of 25 calls per minute. It cannot satisfactorily handle more than 35 per minute. What is the probability that it cannot satisfactorily handle the calls received during a given minute?

7-13. A druggist estimates that the average demand for tubes of a particular type of toothpaste is 150 tubes per week. Assume that the daily demand has a Poisson distribution.

(a) Find the smallest number of tubes that the druggist should have in stock at the beginning of the week if the probability of running out before the end of the week is to be less than 0.1.

(b) If the druggist has 150 tubes in stock at the beginning of a week, what is the probability that he will run out by the end of the fifth day?

7-14. Let $\phi(z)$ be the function for the standard normal curve. Show that for $m \geq 9$, individual Poisson probabilities can be approximated by

$$\underline{p}(x;m) \approx \frac{1}{\sqrt{m}}\phi\left(\frac{x-m}{\sqrt{m}}\right).$$

7-15. Assume that when the Bruins hockey team plays the Kings hockey team, the number of goals the Bruins score can be viewed as the outcome of a Poisson experiment. Assume the same for the Kings against the Bruins. Assume further that the two experiments (for a single game) are independent. The Bruins average 3 goals per game against the Kings, and the Kings average 1 goal per game against the Bruins. Find the following probabilities for a game between the two teams:

(a) P(Bruins win);

(b) P(Kings win);

(c) P(tie occurs).

7-16. Make the same general assumptions as in Exercise 7-15 about the Montreal and Vancouver hockey teams. Assume that Montreal averages 6 goals per game against Vancouver and that Vancouver averages 3 goals per game against Montreal. Find the following probabilities for a game between the two teams:

(a) P(Montreal wins by a score of 4 to 2);

(b) P(Vancouver wins and Montreal scores no goals);

(c) P(Montreal leads at end of the second period | Vancouver is ahead 2 to 0 at the end of the first period) (a period is one-third of a game);

(d) P(both teams together score 12 or more goals).

7-17. It is known that 3/4 of the major political editorials in a given newspaper are written by the editor-in-chief and 1/4 by the city editor. The editor-in-chief uses the word "also" about 0.3 times in every 1000 words, and the city editor uses the word "also" about 0.65 times in every 1000 words. An editorial is randomly selected and proves to be 1800 words in length. It does not contain the word "also". What is the probability that the editorial was written by the editor-in-chief?

7-18. (a) In the gas container C described in the text, take a region $R^1$ of volume = one cubic micron (= $10^{-9}$ cubic millimeters). What percentage variation of density in $R^1$ would you expect to observe as time passes?

(b) What percentage variation would you expect to observe in a region of volume = $10^{-12}$ cubic millimeters?

(c) What percentage in a region of volume = $10^{-15}$ cubic millimeters?

7-19. A partial vacuum has been created in a closed container. Molecules of the remaining gas strike a small area A of the inside wall of the container at an average rate of $10^8$ per second. The average force F on A in any given time interval may be taken to be proportional to the number of molecules striking A during that interval divided by the length of the interval. The average inside pressure at A in that time interval is defined to be F/(Area of A). You obtain a pressure gauge which measures the pressure at A by counting the number of molecules that strike A in a microsecond ($10^{-6}$ second) interval. How much percentage variation do you expect to observe in the values that you obtain as you make successive measurements?

<u>Special Problem:</u> Julius Caesar died from stab wounds in 44 BC. Make the following assumptions.

(1) In his feeble dying breath, Caesar exhaled about 10 c.c..

(2) During the more than 2000 years since, Caesar's dying breath has become well mixed with the rest of the atmosphere.

(3) (An unrealistic assumption for the purpose of this problem.) During this period, none of the molecules from Caesar's dying breath has been chemically altered or has been lost from the atmosphere (by absorption in ocean, rain, etc.)

(4) You take 15 breaths per minute, and each breath has a volume of one litre.

(a) Estimate the probability that on your next breath you will inhale at least one molecule from Caesar's dying breath.

(b) Estimate the probability that during the next minute you will inhale at least one molecule from Caesar's dying breath.

[You may make the following approximate assumptions (and others as well, if you wish): atmospheric pressure is 15 $lb/in^2$ ; one mole of gas occupies 22 litres volume at standard temperature and pressure; one mole of atmosphere weighs 30 grams; the radius of the earth is 6,400 kilometers; and Avogadro's number is $.6 \times 10^{24}$ .]