$\boxed{\text{Quillen: Cyclic Homology.}}$

Sept 11.

Connes' program of noncommutative geometry -
Geometry like politics is partially ordered by rigidity:

<u>Geo.</u>

arithmetic.	alg.	hol. geo.	C^∞	topology	meas. thy.
$\mathcal{O}_X$ ~~...~~	$\mathcal{O}_X$	$\mathcal{O}_X$	sheaves $C^\infty X$	$C(X)$	$L^\infty X$
?	?, - Artin - Manin	?	?; <u>but one has examples</u>:	C^*-alg.	von Neumann alg. ↓ <u>factors</u>; $L^\infty X$ is type I.

Factors of type II, III occur via ~~an~~ eq. rel's. which are locally nice but globally bad; like a foliation. This led Connes to define the C^* alg. of a foliation.

K theory turns out to extend naturally to non-com. algs.
It turns ~~out~~ out there ~~also~~ to be the natural context for index thy:

In C^*-alg. "KK = Index thy".

(the Annals paper becomes a defn of $\cup$ in KK.)

There are also analytic proofs of ASI: Patodi, Gilkey, ABP, Getzler, Bismut:

$$\text{index } D = \operatorname{tr}(\ -\ )$$

& the trace is evaluated ~~as a~~ ~~as~~ asymptotics; this leads to $\int_X$ (diff. forms)

Connes saw how to do this pf in his C^*. alg. of foliations. Along the way he had to describe forms; this led to ~~cyclic homology~~ cyclic chains;

[Cyclic theory is the noncommutative generalization of deRham cohomology.]

DeRham cohomology ...

To avoid the topology of $C^\infty(X)$, do things algebraically.

... for f. gen. com. alg's. $/\mathbb{C}$.

$$A \mapsto \mathrm{Var}(A) \qquad (= \mathrm{Hom}(A, \mathbb{C}).)$$

$$\| $$

$\mathbb{C}[x_1, \dots x_n]/(f_1, \dots f_k)$; Var A = Zeros of $f_1, \dots f_k$.

-with its cx. topology. Want $H^*_{top}(\mathrm{Var}(A); \mathbb{C})$.

Nontrivial fact: $H^*_{top}(\mathrm{Var}\, A; \mathbb{C})$ can be computed algebraically from A:

$$\Omega_A : \quad A \xrightarrow{d} \Omega^1_A \longrightarrow \Omega^2_A \longrightarrow \cdots$$

Ω^1_A gen'd over A by dA; ~~the relation~~

Ω^*_A gen'd as ~~DG~~ com. DGA. by Ω^*_A & A.

This gives all relations. Then

$$H^{\cdot}_{dR}(A) = H^{\cdot}(\Omega_A ; d).$$

<u>Th. 1.</u> Suppose A is <u>smooth</u> ($\Leftrightarrow$ Var(A) is nonsingular and A is reduced). Then

$$H^{\cdot}_{dR}(A) \xrightarrow{\cong} H^{\cdot}(\mathrm{Var}\, A ; \mathbb{C}).$$

(alg. forms $\hookrightarrow$ smooth forms).

History : - Lefschetz knew it
- Atiyah + Hodge prvd it assuming res. of sing.
- Grothendieck worked this out after Hironaka.

<u>Th 2.</u> (Grothendieck's formulation of the Jacobian criterion for nonsingularity.)
A is smooth $\Leftrightarrow$ $\forall$ $0 \to I \to R \to A \to 0$ with $I^2 = 0$ (or I nilpotent) splits.

To extend to singular varieties, ebed them into smooth ones:

<u>Th 3.</u> if $A = R/I$ with R smooth, then

$$H\left(\varprojlim \Omega^{\cdot}_R / I^n \Omega^{\cdot}_R ; d\right) \xrightarrow{\cong} H^{\cdot}(\mathrm{Var}(A); \mathbb{C}).$$

(used eg by Robin Hartshorne).

Noncom. analogues.

1) A any assoc. alg. $/\mathbb{C}$.

ΩA : $A \longrightarrow \Omega^1 A \longrightarrow \cdots$

free (noncom) dg algebra. Result: if $\bar{A} = A/\mathbb{C}$ then

$$\Omega^n A = A \otimes \bar{A}^{\otimes n}.$$

eg $a_0\, da_1 \leftrightarrow a_0 \otimes a_1$ But

$$H^i(\Omega A; d) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

But: $\underline{\underline{b}}, \underline{\underline{B}}$: $\omega \in \Omega^n A$, $a \in A$ $\Rightarrow$

down. $b(\omega\, da) = (-1)^n (\omega a - a\omega)$.

up $B(a_0\, da_1 \cdots da_n) = \sum_{i=0}^{n} (-1)^{in} da_i \cdots da_n\, da_0 \cdots da_{i+1}$

"cyclic symmetrization of d." $0 = b^2 = bB + Bb = B^2$.

Def. The periodic cyclic homology of A, $HP_i(A)$:

$B+b$ acts on $\mathbb{Z}/2$-graded version. $n \in \mathbb{Z}/2$.

$$HP_n(A) = H_n\left(\prod_{i \geq 0} \Omega^{2i} A \rightleftarrows \prod_{i \geq 0} \Omega^{2i+1} A \right)$$

Th (Connes) If A is com, + smooth then $HP_n(A) = \bigoplus_{i \equiv n\,(2)} H^i_{dR}(A)$.

Def. A is quasi-free if any $0 \to I \to R \to A \to 0$ with $I^2 = 0$ splits.

ie HH-dim $= 1$. This replaces Th 2. For Th 3:

Th. If A is quasi-free then let

$$X(A): \quad A \underset{B}{\overset{b}{\rightleftarrows}} \Omega^1 A / [A, \Omega^1 A]$$

Then $H^{\cdot}(X(A)) \cong HP^{\cdot}(A)$.

Th. If $A = R/I$, where R is quasi-free, then

$$HP^{*}_{\cdot}(A) = H^{*}_{\cdot}\left(\varprojlim X(R/I^n)\right).$$

(Note: $b = 0$ in com. case)

This is simpler than com. thy; eg no Noetherian assumption. The trouble is ~~there are~~ fewer "nonsingular" objects. Eg. $\otimes$ of ns. no longer ns (HHDim 2 instead).

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A $\mathbb{C}$-alg $\quad \bar{A} = A/\mathbb{C}$. $(\Omega A)^*$: DG alg of noncom. diff forms on A.

$$(\Omega A)^n = \Omega^n A = \begin{cases} A \otimes \bar{A}^{\otimes n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$a_0, \dots a_n \in A \to (a_0, \dots a_n) \in \Omega^n A$.

Prop 1. $\exists!$ DG alg str. on ΩA st ($|d| = +1$) either
1) $a_0\, da_1 \cdots da_n = (a_0, \dots a_n)$. or
2) Given a DG alg Γ & alg. map $A \longrightarrow \Gamma^0$, $\exists!$ DG alg map $u_*: \Omega A \longrightarrow \Gamma$ extending u.

pf. 1) Uniqueness - via identities

$$d(a_0 da_1 \cdots da_n) = da_0 \cdots da_n$$
$$a_0\, da_1 \cdots da_n\, a_{n+1}\, da_{n+2} \cdots da_k = (-1)^n a_0 a_1\, da_2 \cdots da_k + \sum_{i=1}^{n} (-1)^{n-i} a_0\, da_1 \cdots d(a_i a_{i+1}) \cdots da_k$$

So the assumed DG alg. str. on ΩA ~~is given~~ & has
$$d(a_0, \dots, a_n) = (1, a_0, \dots a_n)$$
$$(a_0, \dots a_n)(a_{n+1}, \dots a_k) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \dots a_i a_{i+1}, \dots a_k) \quad \triangle$$

Existence: These formulas do define a differential & mult. What's missing is associativity. Do this by exhibiting the left regular rep:

Have d with $d^2 = 0$: ΩA is a complex.
$\mathrm{End}(\Omega A) = E^* =$ endomorphism algebra of this cx:

$$E^n = \prod_p \operatorname{Hom}(\Omega^p, \Omega^{n+p}).$$

$$\begin{aligned} d\omega &= [d, \omega] \qquad \text{"super bracket"} \\ &= d\omega - (-1)^{|\omega|}\omega d. \end{aligned}$$

Let $\quad \ell : A \longrightarrow E^0 : \quad \ell(a)(a_0, \ldots a_n) = (aa_0, \ldots, a_n).$

Extend to linear map $\ell_* : \Omega A \longrightarrow E$ by

$$\ell_*(a_0, \ldots a_n) = \ell a_0 \, [d, \ell a_1] \cdots [d, \ell a_n].$$

(as you ~~to~~ expect). Easy check this is wd.
By the identities above (applied to the image of ℓ in E^0), the image of ℓ_* is the DG subalgebra of E gen'd by A.

We claim ℓ_* is monic. Here's a splitting:

$$E \to \Omega A \qquad \text{by} \qquad \omega \longmapsto \omega(1).$$

Apply this to $\ell_*(a_0, \ldots a_n)$:

$$\begin{aligned} [d, \ell a_i](1, a_{i+1}, \ldots a_n) &= d(a_i, \ldots a_n) - \ell a_i \, \overbrace{d(1, a_{i+1}, \ldots a_n)}^{0} \\ &= (1, a_i, \ldots a_n) \end{aligned}$$

$$\Rightarrow \quad \ell_*(a_0, \ldots a_n)(1) = \ell a_0 (1, a_1, \ldots a_n) = (a_0, \ldots a_n).$$

2) Must have, for $u : A \to \Gamma^0$;

$$u_*(a_0\, da_1 \cdots da_n) = ua_0\, d(ua_1) \cdots d(ua_n).$$

this shows uniqueness. Conversely, use this to <u>define</u> u_*. Compat with d & mult follow from initial identities. ∎.

<u>Cor.</u> 1. $d : A \to \Omega^1 A$ is a universal derivation.
($\Omega^1 A$ is an A-bimodule via

$$a(a_0, a_1) = (aa_0, a_1) \; ; \; (a_0, a_1)a = a_0\, d(a_1 a) - a_0 a_1\, da.$$

<u>pf.</u> Given deriv. $A \xrightarrow{D} M$, get DG alg $\{0 \to A \to M \to 0\}$.
Apply univ. property. ∎.

<u>Cor 2.</u> $\exists$ ex. seq. of A-bimodules

$$0 \to \Omega^1 A \xrightarrow{j} A \otimes A \xrightarrow{m} A \to 0.$$

where

$$m(a_0 \otimes a_1) = a_0 a_1$$
$$j(a_0\, da_1) = a_0 a_1 \otimes 1 - a_0 \otimes a_1.$$

<u>pf.</u>

$$\begin{tikzcd} 0 \arrow[r] & \ker m \arrow[r] & A \otimes A \arrow[r] \arrow[l, bend left, dashed, "p"] & A \arrow[r] \arrow[l, bend left, "i"] & 0 \\ & & a \otimes 1 & a \arrow[l, mapsto] & \end{tikzcd}$$

$\Rightarrow$ $\quad p = 1 - im \; ; \; p(a_0 \otimes a_1) = a_0 \otimes a_1 - a_0 a_1 \otimes 1$

$$\operatorname{Coker} i = A \otimes A / A \otimes 1 \cong A \otimes \bar{A} = \Omega^1 A$$ ✓

Also ought to check that j is ~~the~~ a bimodule map; in fact it's the one corresponding to the <u>inner</u> derivation $A \to A \otimes A$ by

$$a \mapsto [a, 1 \otimes 1] = a \otimes 1 - 1 \otimes a.$$

Def. Let M be an A-bimodule. The <u>tensor alg.</u> of M is

$$T_A M = A \oplus M \oplus (M \otimes_A M) \oplus \cdots .$$

$$= A \oplus \bigoplus_{n \geq 1} M \left(\otimes_A M\right)^{n-1} \qquad \text{(notation)}.$$

~~Rk~~. $T_A(M)$ has univ property:
Given an alg. map $u: A \longrightarrow R$, & $v: M \longrightarrow R$ an A-bimodule map, $\exists!$ alg. hom $T_A M \longrightarrow R$ extending u & v.

<u>Cor 3</u>. $T_A(\Omega^1 A) \xrightarrow{\cong} \Omega^\bullet A$.

<u>pf</u>. $\Omega^1 A\left(\otimes_A \Omega^1 A\right)^{n-1} = (A \otimes \bar{A})\left(\otimes_A (A \otimes \bar{A})\right)^{n-1} = \cdots = A \otimes \bar{A}^{\otimes n} = \Omega^n A$. $\square$

Next: various other univ. alg's: Cuntz alg., $A * A$; Fedosov product:

Def. Let Γ be a DG alg. The Fedosov product on Γ is defined by

$$x \circ y = xy - (-1)^{|x|} dx\, dy$$

This gives a $\mathbb{Z}/2$-graded algebra; <u>associative</u>!: a deformation of the orig. alg. Even if Γ is commutative, the new product isn't (unless Γ is even)

We'll see: $\quad A * A = \Omega A$ with the Fedosov product

Fedosov construction. Γ an DG alg.

$\Rightarrow$ superalgebra with same underlying $\mathbb{Z}/2$-grading;

$$x \circ y = xy - (-1)^{|x|}\, dx\, dy$$

(No change if one factor is closed). Check associativity

Def if A is a $\mathbb{C}$ alg., a based linear map $\rho : A \to R$ (R an alg.) is a linear map st $\rho(1) = 1$.

Then the curvature of ρ is

$$\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$$

This vanishes if a_1 or $a_2 = 1$, so

$$\bar{\omega} : \bar{A}^{\otimes 2} \longrightarrow R.$$

The universal extension of A, RA, is the univ. ex. of an alg. with a based linear map from A.

Concretely $RA = TA / (1_{TA} - 1_A)$.

& $\rho : A \to TA \to RA$ is then based.

This depends only on $(A, 1)$, of course.

But A alg $\Rightarrow$

$$\begin{tikzcd} RA \arrow[r, "\text{alg map}"] & A \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

$IA = \ker(RA \to A)$; $RA \to A$ is the "univ. extension" ~~(Kuan~~ (Cuntz) - because if

$$0 \to I \to R \underset{\rho}{\overset{}{\rightleftarrows}} A \to 0$$

is any extension & ρ is a splitting sending 1 to 1, $\exists!$

$$\begin{array}{ccccccccc} 0 & \to & IA & \to & RA & \rightleftarrows & A & \to & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \to & I & \to & R & \to & A & \to & 0. \end{array}$$

Prop. $\exists$ nat alg iso $RA \longrightarrow (\Omega^{even} A, \circ)$

$$\rho a_0 \cdot \omega(a_1, a_2) \cdots \omega(a_{2n-1}, a_{2n}) \longmapsto a_0\, da_1 \cdots da_{2n}.$$

Moreover $(IA)^n \longleftrightarrow \bigoplus_{k \geq n} \Omega^{2k} A$.

(so RA together with IA-adic filtration captures the alg A).

Pf. $A \longrightarrow (\Omega^{even} A, \circ)$ as $\Omega^0 A$ is a based linear map, with curvature

$$a_1 a_2 - a_1 \circ a_2 = a_1 a_2 - (a_1 a_2 - da_1\, da_2) = da_1\, da_2.$$

So get alg hom $\Phi: RA \longrightarrow (\Omega^{ev} A, \circ)$

st $\rho a \longmapsto a$

$\omega(a_1, a_2) \longmapsto da_1\, da_2$

so the general formula holds because da_i is closed so Fedosov $\circ$ = old prod. Onto, because obvious

$$\Phi: \Omega^{2n} A \cong A \otimes \bar{A}^{\otimes 2n} \longrightarrow RA \quad \text{is w d.}$$

To see Φ onto, notice that $\mathrm{Im}\,\Phi$ is closed under left mult. by $p(a)$, $a \in A$, and contains $\Phi.1$. Since $p(a)$ generate $R(A)$ we have a left ideal, containing 1: $\triangle$.

/ Comp. of ideals: Say $F^n = \bigoplus_{k \geq n} \Omega^{2k} A$.

$\omega(a_1, a_2) \in IA$; clearly $F^n \subseteq (IA)^n$ (omit Φ)

clearly $F^1 = IA$

& $F^p F^q \subseteq F^{p+q}$

so $(IA)^n = (F^1)^n \subseteq F^n$ $\triangle$.

Cuntz alg. $A * A = Q(A)$, free product = coproduct.

$*(\iota, 1) \Rightarrow Q(A) \longrightarrow A$. "folding": so $A \rightrightarrows QA$ split monos.

$*$ automorphism $\gamma : QA \circlearrowleft$ of order 2, interchanging axes. ι, ι^γ.

A superalg. $\Longleftrightarrow$ alg with aut. of order 2.

So QA is naturally super. Indeed, it's the enveloping superalg. of A: adjoint to forgetting $\mathbb{Z}/2$-grading.

[Let $q(A) = \ker(QA \longrightarrow A)$.

Let $p, q : A \longrightarrow QA$ be the even & odd parts of the first embedding ι:

$$\begin{cases} \iota a = pa + qa \\ \iota^\gamma a = pa - qa \end{cases}$$

$p(a_1, a_2) = pa_1\, pa_2 + qa_1\, qa_2$

$q(a_1, a_2) = pa_1\, qa_2 + qa_1\, pa_2$.

so ~~q~~ qA is the ideal gen'd by $\{qa : a \in A\}$.

Th. $\exists$ canon. superalg iso $QA \xrightarrow{\cong} (\Omega A, \circ)$. given by

$$p(a_0)\, q(a_1) \cdots q(a_n) \longmapsto a_0\, da_1 \cdots da_n .$$

Under this iso, $(qA)^n \longleftrightarrow \bigoplus_{k \geq n} \Omega^k A$.

Pf. $A \longrightarrow (\Omega A, \circ)$ $\qquad a \longmapsto a + da$.

This is a hom.:

$$(a_1 + da_1)\circ(a_2 + da_2) = a_1 \circ a_2 + a_1\, da_2 + (da_1) a_2 + da_1\, da_2$$
$$= a_1 a_2 + d(a_1 a_2). \quad ✓.$$

So get superalg hom $\Phi : QA \longrightarrow \Omega A, \circ$.

$$\iota a \longmapsto a + da \;; \quad \check{\iota} a \longmapsto a - da$$
$$\Rightarrow \quad pa \longmapsto a \;, \quad qa \longmapsto da.$$

so $\Phi(pa_0 \cdot qa_1 \cdots qa_n) = a_0\, da_1 \cdots da_n$ (as before)

Continue as before. $\square$

S, T alg's; $x \in S$, $y \in T$. $S * T$ = coproduct.

Let $J = \langle\{[x, y] : x \in S, y \in T\}\rangle \subset S * T$; $S*T/J \cong S \otimes T$.

Set $\Gamma = \bigoplus_{n \geq 0} \Omega^n S \otimes \Omega^n T$ with product:

$$(\xi_1 \otimes \eta_1)\circ(\xi_2 \otimes \eta_2) = \xi_1 \xi_2 \otimes \eta_1 \eta_2 - (-1)^{|\xi_1|} \xi_1 d\xi_2 \otimes d\eta_1 \cdot \eta_2 .$$

Prop $\exists$ alg iso. $S * T \xrightarrow{\cong} \Gamma$. given by

$x_0 y_0 [x_1, y_1] \cdots [x_n, y_n] \mapsto x_0 dx_1 \cdots dx_n \otimes y_0 dy_1 \cdots dy_n$.

Under this iso, $J^n = \bigoplus_{k \geq n} \Omega^k S \otimes \Omega^k T$.

(The work is in checking associativity of the product)

The map: $S \to \Gamma \leftarrow T$
$x \mapsto x \otimes 1, \; 1 \otimes y \leftarrow y$

Then compute $[x, y] \mapsto dx \otimes dy$.

Sept.

Hochschild cohomology:

M an A-bimodule. $A^e = A \otimes A^{op}$: "enveloping alg."
so M is an A^e-module. So free: $A \otimes V \otimes A$.

A is an A-bimodule, there is a standard (normalized) resol:

$$0 \leftarrow A \xleftarrow{b'} A \otimes A \xleftarrow{b'} A \otimes \bar{A} \otimes A \longleftarrow \cdots$$

$$b'(a_0, \ldots a_{n+1}) = \sum_{i=0}^{n} (-1)^i (a_0, \ldots a_i a_{i+1}, \ldots a_{n+1}).$$

In terms of forms:- $A \otimes \bar{A}^{\otimes n} \otimes A = \Omega^n A \otimes A$
$(a_0, \ldots a_{n+1}) \longleftrightarrow a_0 \, da_1 \cdots da_n \otimes a_{n+1}$.

$$b'(a_0 \cdots \otimes a_{n+1}) = a_0 a_1 \, da_2 \cdots da_n \otimes a_{n+1} + \sum (-1)^i a_0 \cdots d(a_i a_{i+1}) \cdots a_{n+1} + (-1)^n a_0 \, da_1 \cdots da_{n-1} \otimes a_n a_{n+1}$$

Lots of cancellation; get

$$(-1)^{n-1} a_0\, da_1 \cdots da_{n-1} \cdot a_n \otimes a_{n+1}$$
$$+ (-1)^n a_0\, da_1 \cdots da_{n-1} \otimes a_n a_{n+1}$$

(*) ie $b'(\omega\, da \otimes a') = (-1)^{|\omega|} (\omega a \otimes a' - \omega \otimes aa')$
$= \underline{(-1)^{|\omega|} \omega (a \otimes 1 - 1 \otimes a) a'}$.

So:

The standard A^e- res. of A is

$$(\Omega A \otimes A,\ b') \xrightarrow[m]{\text{augmentation}} A$$

One virtue is that it is easier to verify that $b'^2 = 0$ using (*) than using the orig. def.

There are two standard contracting htpies: right- or left- A-linear. In form notation:

$d \otimes 1$ on $\Omega A \otimes A$. Then

$$b'(d \otimes 1) + (d \otimes 1) b' = 1 - im \qquad i(a) = 1 \otimes a.$$

b' is of course a bimodule map. $(d \otimes 1)$ is A linear. So to check this we need only compute

$$\omega\, da \otimes 1 \overset{b'}{\longmapsto} (-1)^{|\omega|} (\omega a \otimes 1 - \omega \otimes a) \overset{d\otimes 1}{\longmapsto} (-1)^{|\omega|} (d(\omega a) \otimes 1 - d\omega \otimes a)$$

$d\otimes 1 \searrow$

$$d\omega\, da \otimes 1 \mapsto (-1)^{|\omega|+1} (d\omega \cdot a \otimes 1 - d\omega \otimes a)$$

using derivation property

Exercise: Write down the other homotopy.

Exactness also follows from: bimodule seq.:

$$0 \to \Omega^1 A \xrightarrow{j} A \otimes A \xrightarrow{m} A \to 0.$$
$$j(a_0\, da_1) = a_0(a_1 \otimes 1 - 1 \otimes a_1).$$

This is split as rt. or left mod. seq. Tensor with $\Omega^n A$ on left, & use

$$\Omega^n A \otimes_A \Omega^1 A \cong \Omega^{n+1} A:$$

$$0 \to \Omega^{n+1} A \xrightarrow{j} \Omega^n A \otimes A \xrightarrow{m} \Omega^n A \to 0.$$
$$j(\omega\, da) = \omega(a \otimes 1 - 1 \otimes a).$$

These then splice; $b' = (-1)^n jm$. This also implies:

$$\Omega^{n+1} A \otimes A \xrightarrow{b'} \Omega^n A \otimes A \to \Omega^n A \to 0.$$

is exact; a presentation of $\Omega^n A$ by free bimodules.

Hochschild. Basic functors on A-bimodules.

$M^\natural = \mathrm{Hom}_{A^e}(A, M) =: \{ m \in M : am = ma \ \forall a \}$. "center of M"

$M_\natural = A \otimes_{A^e} M = M/[A,M]$ — subspace spanned by $am - ma$.

$a \otimes m \mapsto am \equiv ma$ "Commutator quotient of M"

$$M^\natural \to M \to M_\natural.$$

$H^i(A; M) =:$ ith derived functors of $M \mapsto M^\natural$

ie $\mathrm{Ext}^i_{A^e}(A, M)$.

$$= H\big(\underbrace{\mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes *} \otimes A, M)}_{C^*(A;M)}\big).$$

so $C^n(A;M) = \mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes n} \otimes A, M) = \mathrm{Hom}(\bar{A}^{\otimes n}, M)$

$(\delta f)(a_1, \dots a_{n+1}) = a_1 f(a_2, \dots a_{n+1})$

$+ \sum_{i=1}^{n} (-1)^i f(a_1, \dots a_i a_{i+1}, \dots a_{n+1}) + (-1)^{n+1} f(a_1, \dots a_n)\, a_{n+1}$

eg $(\delta m)(a) = am - ma$: zero cocycle = elt of center: ↯ is left ex.

$\delta f(a_1, a_2) = a_1 f(a_2) - f(a_1 a_2) + f(a_1) a_2$:

so a <u>1-cocycle is a derivation to M.</u>

<u>Cupproduct</u>: $f \in C^p(A, M)$, $g \in C^q(A;N)$.

$f \cup g \in C^{p+q}(A; M \otimes_A N)$ by

$(f \cup g)(a_1, \dots a_{p+q}) = f(a_1, \dots a_p) \otimes g(a_{p+1}, \dots a_{p+q})$.

This is compat. with δ.

$d: A \to \Omega^1 A$ is a derivation : ie a 1-cocycle:
Get $d^n \in C^n(A; (\Omega^1 A)^{\otimes_A n})$. Push by
mult. to $\Omega^n A$: get $d^{\cup n} \in C^n(A; \Omega^n A)$, by

$d^{\cup n}(a_1, \dots a_n) = da_1 \cdots da_n$

<u>Prop</u>. $d^{\cup n}$ is the universal n-cocycle on A:

$f \in C^n(A;M)$ with $\delta f = 0 \Rightarrow f_* : \Omega^n A \to M$.
st $da_1 \cdots da_n \mapsto f(a_1, \dots a_n)$.

pf: $C^n(A;M) = \mathrm{Hom}_{A^e}(\Omega^n A \otimes A; M)$.

so any n-cochain to M induces $f_* : \Omega^n A \otimes A \longrightarrow M$.
s.t. $f_*(da_1 \cdots da_n \otimes 1) = f(a_1, \dots a_n)$.

Go back to the presentation above:

$$\Omega^{n+1}A \otimes A \longrightarrow \Omega^n A \otimes A \longrightarrow \Omega^n A \longrightarrow 0$$

Hom it into M:

$$0 \to \mathrm{Hom}_{A^e}(\Omega^n A, M) \to C^n(A;M) \xrightarrow{\delta} \boxed{\cancel{\mathrm{Hom}_{A^e}(\Omega^{n+1}A \otimes A; M)}}\ C^{n+1}(A;M).$$

Low dimensions. $H^0(A;M) = M^A$.

$$H^1(A;M) = \frac{\text{derivations}}{\text{inner deriv.}} \quad a \mapsto [a, m].$$

$$H^2(A;M) = \left\{ \begin{array}{l} \text{isom. classes of square-zero extensions} \\ \text{of } A \text{ ~~with~~ by } M. \end{array} \right\}.$$

pf — pick linear section of $R \to A$, carrying $1 \mapsto 1$.
Take the curvature; it's a 2-cocycle with values in M.

Connection with univ. extension RA: then

$$0 \longrightarrow IA/(IA)^2 \longrightarrow RA/IA^2 \longrightarrow A \to 0$$

is the univ. square-zero extension (with section).
And we showed that $RA/(IA)^2 = A \oplus \Omega^2 A$ with $\circ$.
The corresp. curvature was $a_1 a_2 - a_1 \circ a_2 = da_1 da_2$:

ie the univ. curvature is dud, the universal cocycle.

o Sept. forms = normalized chains.

$$\Omega^n A = A \otimes \bar{A}^{\otimes n} \qquad a_0\, da_1 \cdots da_n \leftrightarrow (a_0, \ldots a_n).$$

Can't cyclicly permute rhs; use the "Karoubi operator": What do we have? -

① $d : a_0\, da_1 \cdots da_n \mapsto da_0 \cdots da_n$; $(a_0, \ldots a_n) \mapsto (1, a_0, \ldots a_n)$.

so $$d\Omega^n A = \bar{A}^{\otimes(n+1)}.$$

$$\begin{array}{ccccccccc} 0 \to & \bar{A}^{\otimes n} & \to & A \otimes \bar{A}^{\otimes n} & \longrightarrow & \bar{A}^{\otimes(n+1)} & \to 0 \\ & \| & & \| & & \| & \\ n \geq 1: \quad 0 \to & d\Omega^{n-1}A & \longrightarrow & \Omega^n A & \xrightarrow{d} & d\Omega^n A & \to 0 \end{array}$$

$$n = 0: \quad 0 \to \mathbb{C} \longrightarrow A \longrightarrow dA \longrightarrow 0.$$

Splicing gives the deRham complex, which is thus exact:

$$H(\Omega A; d) = \mathbb{C}.$$

② b :-

$$\Omega^n A \otimes \bar{A} \cong \Omega^{n+1}$$
$$\omega \otimes a \leftrightarrow \omega\, da.$$

$$b(\omega\, da) = (-1)^{|\omega|}(\omega a - a\omega)$$

check: $b^2 = 0$, & that in chain notation

$$b(a_0, \ldots a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots a_n) + (-1)^n (a_n a_0, a_1, \ldots a_{n-1})$$

Recall the Hochschild operator

$$b' : \Omega^i A \otimes A \longrightarrow \Omega^{i-1} A \otimes A \quad : \text{standard res.}$$

$$b'(\omega\, da \otimes a') = (-1)^{|\omega|} \omega (a \otimes 1 - 1 \otimes a) a'$$

Defn. The Hochschild homology $H_n(A;M) = L_n(\ )_\natural (M)$

Put $\quad HH_n(A) = H_n(A;A) = H((\text{st. res})_\natural)$.

if X is a left A-module, $X \otimes A$ is a bimodule; and

$$(X \otimes A)_\natural \cong X$$
$$x \otimes a \longleftrightarrow ax.$$

So

$$HH_\cdot(A) = H\{\Omega A;\ b\}, \quad \text{since} \quad b'_\natural \longleftrightarrow b \ :$$

Since $\quad \omega a \otimes 1 - \omega \otimes a \longleftrightarrow \omega a - a\omega$.

) Karoubi operator κ (close to λ of Connes)

$$1 - \kappa = db + bd :$$

$$\begin{aligned} db(\omega\, da) &= d(-1)^{|\omega|}(\omega a - a\omega) \\ &= (-1)^{|\omega|}(d\omega a + (-1)^{|\omega|} \omega\, da \\ &\qquad - da\, \omega - a\, d\omega) \end{aligned}$$

$$bd(\omega\, da) = b\, d\omega\, da = (-1)^{|\omega|+1}(d\omega\, a - a\, d\omega)$$

$$\Rightarrow (db + bd)(\omega\, da) = \omega\, da - (-1)^{|\omega|} da \cdot \omega.$$

so $\kappa(\omega\, da) = (-1)^{|\omega|}\, da \cdot \omega$.

$\kappa = 1$ on $\Omega^n A$ $n \le 0$.

ie $\kappa(a_0\, da_1 \cdots da_n) = (-1)^{n-1} da_n\, a_0\, da_1 \cdots da_{n-1}$.

$= (-1)^n a_n\, da_0\, da_1 \cdots da_{n-1} + (-1)^{n-1} d(a_n a_0)\, da_1 \cdots da_{n-1}$.

or $\kappa(a_0, \dots a_n) = (-1)^n (a_n, a_0, \dots a_{n-1})$
$+ (-1)^{n-1} (a_n a_0, a_1, \dots a_{n-1})$.

First term is Connes' $\underline{\lambda}$, cyclic permutation.
The second term takes care of the normalization:
try putting $a_n = 1$.

Properties: $1 - \kappa = db + bd$:

~~So~~ κ is homotopic to 1 wrt either b or d.
In particular
$$d\kappa = \kappa d \quad ; \quad b\kappa = \kappa b.$$

$\kappa(da_0 \cdots da_n) = (-1)^n da_n\, da_0 \cdots da_{n-1}$:

"$\kappa = \lambda$" on $d\Omega^n = \bar{A}^{\otimes(n+1)}$.

so $\kappa^{n+1} d = d$ on Ω^n.

or $d(\kappa^{n+1} - 1) = 0$ on Ω^n.

Apply this to $0 \to d\Omega^{n-1}A \to \Omega^n A \to d\Omega^n \to 0$;

$$(\kappa^{n+1}-1)\,\Omega^{n} \subseteq d\,\Omega^{n-1}$$

& $d\Omega^{n-1}$ is killed by $(\kappa^n - 1)$: so:

$$(\kappa^n - 1)(\kappa^{n+1} - 1) = 0 \quad \text{on } \Omega^n.$$

This replaces "λ is of finite order," and it implies

κ is invertible.

$$\kappa^j(a_0\,da_1 \cdots da_n) = (-1)^{j(n-1)}\, da_{n-j+1} \cdots da_n\, a_0\, da_1 \cdots da_{n-j}$$

for $0 \le j \le n$;

$$\kappa^n(a_0\,da_1 \cdots da_n) = da_1 \cdots da_n\, a_0$$

$$= a_0\,da_1 \cdots da_n + [da_1 \cdots da_n, a_0]$$

$$= a_0\,da_1 \cdots da_n + (-1)^n\, b(da_1 \cdots da_n\, da_0)$$

$$\cancel{= \kappa^{-1}(da_0 \cdots da_n)};$$

$$= a_0\,da_1 \cdots da_n + b\kappa^n(da_0 \cdots da_n).$$

ie $\kappa^n = 1 + b\kappa^n d$ on Ω^n.

Since $\kappa^n = \kappa^{-1}$ on $d\Omega^n$, this says

$$\boxed{\kappa^n = 1 + b\kappa^{-1}d \quad \text{on } \Omega^n.}$$

$$\kappa^{n+1} = \kappa + \kappa b \kappa^{-1} d = \kappa + bd = 1 - db$$

so $\boxed{\kappa^{n+1} = 1 - db \qquad \text{on } \Omega^n.}$

(Putting 1's over, these two formulas give another pf of $(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$ on Ω^n)

4) Connes' B : op. of deg. $+1$ on Ω :

$$B = \sum_{j=0}^{n} \kappa^j d \qquad \text{on } \Omega^n.$$

$$B(a_0\, da_1 \cdots da_n) = \sum_{j=0}^{n} (-1)^{jn}\, da_j \cdots da_n\, da_0 \cdots da_{j-1}.$$

Properties. $B\kappa = \kappa B$.

Since $\kappa^{n+1} d = d$ on Ω^n, $\kappa B = B$.

$$Bd = dB = B^2 = 0.$$

Using 2 boxes above, compute

$$\kappa^{n(n+1)} - 1 = \sum_{j=0}^{n} \kappa^{nj}(\kappa^n - 1) \qquad \text{(geo series)}$$

$$= \sum_{j=0}^{n} \kappa^{nj}(b\kappa^{-1}d)$$

$$= \sum b\,\kappa^{nj-1} d = \sum b\,\kappa^{-j-1} d = bB.$$

$$\kappa^{n(n+1)} - 1 = \sum_{j=0}^{n-1} \kappa^{(n+1)j}(\kappa^{n+1} - 1)$$

$$= -\sum_{j=0}^{n-1} \kappa^{nj+j}\, db = -\sum_j \kappa^j db = -Bb.$$

so $$\boxed{bB + Bb = 0}$$

$$\kappa^{n(n+1)} = 1 - Bb$$

$$(Bb)^2 = Bb(-bB) = 0.$$

so $1 - Bb$ is unipotent; κ is "quasi-unipotent."

Sketch of next:

$$\Omega^0 \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 \underset{d}{\overset{b}{\rightleftarrows}} \Omega^2 \quad \ldots$$

Pretend b = adjoint of d. Do the theory of harmonic forms:.

Laplacian $\quad bd + db = 1 - \kappa$.

On nonzero eigenspaces of Δ, have invert op $\simeq 0$.

Roots of $(\kappa^n - 1)(\kappa^{n+1} - 1)$ are distinct except for 1, which has order 2.

$$\Omega = \underset{\substack{\uparrow \\ \text{harmonic forms.}}}{\mathrm{Ker}\,(1-\kappa)^2} \oplus \bigoplus_{1 \neq \zeta \text{ root of } 1} \mathrm{Ker}\,(\kappa - \zeta)$$

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On $\Omega = \Omega A$: $\qquad db + bd = 1 - \kappa$

On Ω^n, $\qquad (\kappa^n - 1)(\kappa^{n+1} - 1) = 0$

An op. on a v.s. which satisfies a poly. gives rise to a decomp. into "gen. eigenspaces" corresp. to distinct roots.

$$(x^n - 1)(x^{n+1} - 1) = \prod_{\zeta^n = 1} (x - \zeta) \prod_{\zeta^{n+1} = 1} (x - \zeta)$$

so 1 is a double root, the rest simple.

$$\Omega^n = \ker (1 - \kappa)^2 \oplus \bigoplus_{\zeta \neq 1} \ker (\kappa - \zeta).$$

If we let ζ range over all roots of 1 other than 1, this is a decomp. of $\Omega^\cdot$. It's preserved by any operator commuting with κ: eg. d, b, B.

$1 - \kappa$ is invertible on $\bigoplus_{\zeta \neq 1}$, & $1 - \kappa$ kills 1st factor:

$$\Omega = \ker (1 - \kappa)^2 \oplus \operatorname{Im} (1 - \kappa)^2.$$

(This fits with the Krull-Schmidt thm; $\ker (1 - \kappa)^m$ & $\operatorname{Im} (1 - \kappa)^m$ stabilize for $m \geq 2$.)

Let P project to $\ker (1 - \kappa)^2$, $P^\perp = 1 - P$.
Let $G = (1 - \kappa)^{-1}$ on $\operatorname{Im} (1 - \kappa)^2$:

spectral projection of 1. $\qquad$ Green's operator for $1 - \kappa$.

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 \\ 0 & (1 - \kappa)^{-1} \end{pmatrix}.$$

$P^2 = P$; P & G commute with any op. commuting with κ.
(so they commute with κ, & with each other.)

$P^\perp = G(1-\kappa)$ $PG = 0$.

=

Reminder on finite order operators:

T on V, say $T^m = 1$. then spectral proj at 1 is

$$P_T = \frac{1}{m}\sum_{i=0}^{m-1} T^i$$

&

$$G_T = \frac{1}{m}\sum_{i=0}^{m-1}\left(\frac{m-1}{2} - i\right)T^i$$

Verify: $TP_T = P_T$ $P_T^2 = P_T$.

$(1-T)G_T = 1 - P_T$ $P_T G_T = 0$.

(If you want to satisfy $(1-T)G_T = 1 - P_T$, you can have various coeffs in the sum; only the one given makes $P_T G_T = 0$.). So:

$$V = V^T \oplus (1-T)V \quad ; \quad V^T = P_T V.$$

& then $G = \begin{pmatrix} 0 & 0 \\ 0 & (1-T)^{-1} \end{pmatrix}$.

Also note: m could be any pos. int. s.t. $T^m = 0$, not nec. the order; the op's P_T, G_T don't change.

=

Apply this to $d\Omega^n = \bar{A}^{\otimes(n+1)}$

then

$$\kappa \longleftrightarrow \lambda.$$
$$P \longleftrightarrow P_\lambda \quad \text{(in } P_T \text{ notation)}$$
$$G \longleftrightarrow G_\lambda.$$

So on $d\Omega$ we have explicit formulas for P, G in terms of T. So on Ω^n,

$$Pd = \frac{1}{n+1}\sum_{i=0}^{n} \kappa^i d = \frac{1}{n+1} B.$$

$$Gd = \frac{1}{n+1}\sum_{i=0}^{n} \left(\tfrac{n}{2} - i\right) \kappa^i d.$$

– a good interpretation of B.

Aim for P: using Gd.
$$1 - P = P^{\perp} = G(1-\kappa)$$

$$= G(db + bd) = (Gd)b + b(Gd) \implies$$

$$\boxed{P = 1 - (Gd)b - b(Gd)}$$

Similarly:

$$G = G^2(1-\kappa) = (G^2d)b + b(G^2d)$$

& thus leads to an explicit formula.

Consequences for homology.

$$P^{\perp} = (Gd)b + b(Gd) = d(Gb) + (Gb)d. \qquad \text{So:}$$

On $P^{\perp}\Omega$, $1 \sim 0$ wrt d or b:

Prop. $P^{\perp}\Omega$ is contractible as complex, wrt d or b:

$$H(P\Omega; b) = H(\Omega; b) = HH(A).$$
$$H(P\Omega; d) = H(\Omega; d) = \mathbb{C}.$$

Now for B: $\quad B = \begin{cases} 0 & \text{on } P^\perp\Omega \\ Nd & \text{on } P\Omega. \end{cases}$

where $\quad N\omega = |\omega|\omega.$

So

$$H(\Omega; B) = \mathbb{C} \oplus P^\perp\Omega.$$

Concrete description of $P\Omega$:

Prop. Let $\omega \in \Omega$. $P\omega = \omega \iff d\omega$ & $db\omega$ are κ-fixed.

(if ω is κ-fixed then rhs is clear, & ω is in 1-eigensp. The rhs is the weakening giving gen eigenspace.)

pf. On $d\Omega$, $\mathrm{Im}\, P$ = subspace of elts fixed by κ.

$\Rightarrow$: $P\omega = \omega \Rightarrow Pd\omega = d\omega \Rightarrow \kappa d\omega = d\omega$

$P\omega = \omega \Rightarrow Pdb\omega = db\omega \Rightarrow \kappa(db\omega) = db\omega.$

$\Leftarrow$: $P\omega = \omega - G\underbrace{db\omega} - bG\underbrace{d\omega}.$

if they are killed by G.   are κ-fixed, $\triangle$.

Defn An augmented algebra is an alg. A equipped with a hom. $\varepsilon: A \to \mathbb{C}$; $\quad A = \mathbb{C} \oplus \ker \varepsilon$;

A is got from "nonunital alg. $\ker\varepsilon$" by adjoining a 1.

Since $\quad \ker\varepsilon \xrightarrow{\cong} A/\mathbb{C} = \bar{A}$.

Use $\bar{A}$ to denote $\ker\varepsilon \subset A$.

Think: $\bar{A}$, before a v.s., so has str. of nonunital alg.

A aug. $\Rightarrow$ $\Omega A \longrightarrow \Omega\mathbb{C} = \mathbb{C}$, augmentation.
Again, $\bar{\Omega} = \ker(\Omega \to \mathbb{C})$; it's the univ. nonunital DG alg. gen'd by nonunital alg $\bar{A}$. This is Connes approach.

The aug. splits $0 \to \mathbb{C} \to A \longrightarrow \bar{A} \longrightarrow 0$
so

$$\Omega^n A \cong \bar{A}^{\otimes(n+1)} \oplus \bar{A}^{\otimes n}$$

$$\begin{array}{lll} a_0\, da_1 \cdots da_n \longleftrightarrow (a_0, \dots a_n) & 0 & \\ 1\, da_1 \cdots da_n \longleftrightarrow \quad 0 & (a_1, \dots a_n) & a_i \in \bar{A} \end{array}$$

write $1 \otimes (a_1, \dots a_n)$ for elt. of $\bar{A}^{\otimes n}$. Then:

$$b(a_0\, da_1 \cdots da_n) = b(a_0, \dots a_n)$$

$$b(da_1 \cdots da_n) = (1-\lambda)(a_1, \dots a_n) - 1 \otimes b'(a_1, \dots a_n)$$

$$d(a_0\, da_1 \cdots da_n) = 1 \otimes (a_0, \dots a_n).$$

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On ΩA we have: d, b, κ, B, P, G.

For nonunital a, on $\bigoplus_{n\geq 0} a^{\otimes(n+1)}$ we have:

$$b(a_0,\dots a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0,\dots a_ia_{i+1},\dots a_n) + (-1)^n(a_na_0, a_1,\dots a_{n-1})$$

$$b'(a_0,\dots a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0,\dots a_ia_{i+1},\dots a_n).$$

$$\lambda(a_0,\dots a_n) = (-1)^n(a_n, a_0,\dots a_{n-1}).$$

$$N_\lambda(a_0,\dots a_n) = \sum_{i=0}^{n}\lambda^i(a_0,\dots a_n).$$

Tsygan's identities (or at least his proof):
There is a double complex:

$$\begin{tikzcd}
a^{\otimes(n+1)} \arrow[d, "b"'] & a^{\otimes(n+1)} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & a^{\otimes(n+1)} \arrow[l, "N_\lambda"'] \arrow[d, "b"] \\
a^{\otimes n} \arrow[d, "b"'] & a^{\otimes n} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & a^{\otimes n} \arrow[l, "N_\lambda"'] \arrow[d, "b"] \\
a^{\otimes(n-1)} & a^{\otimes(n-1)} \arrow[l, "1-\lambda"'] & a^{\otimes(n-1)} \arrow[l, "N_\lambda"']
\end{tikzcd}$$

(with anticommuting squares)

Our approach leads to another proof:

$$A = \bar{A}\oplus\mathbb{C} \Rightarrow \Omega^n A = A\otimes\bar{A}^{\otimes n} = \bar{A}^{\otimes(n+1)}\oplus\mathbb{C}\otimes\bar{A}^{\otimes n}.$$

$\underline{a_i\in\bar{A}}$ $\quad a_0\,da_1\cdots da_n \leftrightarrow (a_0,\dots a_n)$

$da_1\cdots da_n \leftrightarrow 1\otimes(a_1,\dots a_n).$

Aim to compute d, b, κ, B, P, G in terms of $b, b', \lambda, N_\lambda$.

$$b(a_0\, da_1 \cdots da_n) = a_0 a_1\, da_2 \cdots da_n + \sum_{i=1}^{n-1} (-1)^i a_0\, da_1 \cdots d(a_i a_{i+1}) \cdots da_n$$
$$+ (-1)^n a_n a_0\, da_1 \cdots da_{n-1}.$$

Taking all $a_i \in \bar{A}$, this translates to the above formula for b.
If $a_0 = 1$, $a_i \in \bar{A}$ for $i > 0$:

$$b(da_1 \cdots da_n) = a_1\, da_2 \cdots da_n + \sum_{i=1}^{n-1} (-1)^{i-1} da_1 \cdots d(a_i a_{i+1}) \cdots da_n$$
$$- (-1)^{n-1} a_n\, da_1 \cdots da_{n-1}$$

ie $$1 \otimes (a_1, \ldots a_n) \longmapsto (a_1, a_2, \ldots a_n) + \sum (-1)^{i-1} (a_1, \ldots a_i a_{i+1}, \ldots a_n)$$
$$- \lambda (a_n, a_1, \ldots a_{n-1})$$

ie $$b(1 \otimes a) = (1-\lambda)(a) - 1 \otimes b'a.$$

Use matrix notation: $$b \leftrightarrow \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}.$$

Also $d: (a_0, \ldots a_n) \longmapsto 1 \otimes (a_0, \ldots a_n)$; $d \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\quad 1 \otimes (a_1, \ldots a_n) \longmapsto 0.$

So: $$1 - \kappa = bd - db = \begin{pmatrix} 1-\lambda & 0 \\ b-b' & 1-\lambda \end{pmatrix}$$

cross-over term.

ie $$\kappa = \begin{pmatrix} \lambda & 0 \\ b' - b & \lambda \end{pmatrix}.$$

Recall on Ω^n: $Pd = \frac{1}{n+1} B$, $Gd = \frac{1}{n+1} \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \kappa^j d.$

so $$Pd \leftrightarrow \begin{pmatrix} 0 & 0 \\ P_\lambda & 0 \end{pmatrix}; \qquad Gd \leftrightarrow \begin{pmatrix} 0 & 0 \\ G_\lambda & 0 \end{pmatrix} \quad \left(\text{indep. of } n\right)$$

& $$B \leftrightarrow \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

Recall $$1 - P = G(1-\kappa) = (Gd)b + b(Gd).$$

so you find: $$1 - P \leftrightarrow \begin{pmatrix} 0 & 0 \\ G_\lambda b & P_\lambda^\perp \end{pmatrix} + \begin{pmatrix} P_\lambda^\perp & 0 \\ -b' G_\lambda & 0 \end{pmatrix}$$

so $$P \leftrightarrow \begin{pmatrix} P_\lambda & 0 \\ b' G_\lambda - G_\lambda b & P_\lambda \end{pmatrix}.$$

Recall: $\omega \in P\Omega A \iff d\omega$ & $db\omega$ are κ-fixed.
This means: using $\leftrightarrow$'s $\begin{pmatrix} x \\ y \end{pmatrix} \in P\Omega A \iff x$ & $bx + (1-\lambda)y$ are λ-invariant.

Why Tsygan's identities?: compute

$$b^2 \leftrightarrow \begin{pmatrix} b^2 & b(1-\lambda) - (1-\lambda)b' \\ 0 & b'^2 \end{pmatrix} = 0.$$

Anticommutativity comes from $bB + Bb = 0$.

[Application to cyclic theory of algebras:
Mixed complexes & ~~elementary~~ abstract cyclic formalism.
- from viewpt. of $\mathbb{Z}/2$ - graded complexes.]

Def. A <u>mixed complex</u> is a $\mathbb{Z}$-graded v.s. M. with operators b, B; $|b| = -1$, $|B| = +1$ st $b^2 = B^2 = 0$ & $Bb + bB = 0$ (ie $(b+B)^2 = 0$).

We will always assume $M_n = 0$ for $n < 0$.
This breaks the symmetry between b, B.
b is the primary differential, B is an operator on the chain cx $(M_\cdot, b)$: DG $E[B]$-module.

A mixed cx $\Rightarrow$ $\mathbb{Z}/2$ - graded complex

$$\bigoplus M_{2n} \underset{b+B}{\overset{b+B}{\rightleftarrows}} \bigoplus M_{2n+1}$$

Define $H_i(M)$ as the homology of this cx; $i \in \underline{\mathbb{Z}/2}$.

Def A map of mixed complexes is a <u>quasi-iso</u> if it induces iso on $H_\cdot(-; \underline{b})$.

NB H_i is <u>not</u> a q'ism invariant. But we want to study homological functors which are These will be "cyclic theories" eg. $H(\,; b)$.
[also differential $\mathrm{Tor}^{E[B]}(\mathbb{C}, -)$]

Natural filtrations:

A decreasing filtration: $F^n M = bM_{n+1} \oplus M_{n+1} \oplus \cdots \subset M$

$F^{-1} M = M.$

An increasing filtration: $G_n M = M_0 \oplus \cdots \oplus M_{n-1} \oplus BM_{n-1}$

$$F^n/F^{n+1} = \left[\overset{n}{bM_{n+1}} \underset{B=0}{\overset{b}{\rightleftarrows}} \overset{n+1}{M_{n+1}/bM_{n+2}} \right]$$

$$\& \quad H_i(F^n/F^{n+1}) = \begin{cases} 0 & i = n + 2\mathbb{Z} \\ H_{n+1}(M; b) & i = (n+1) + 2\mathbb{Z} \end{cases}$$

– a q'ism invariant.

$$0 \to F^n/F^{n+1} \to M/F^{n+1} \to M/F^n \to 0.$$

so by 5-lem + ind, $H_i(M/F^{n+1})$ is a q'ism invariant.

Milnor seq. $\Rightarrow$ $H(\varprojlim M/F^{n})$ is a q'ism invariant.

$\hat{M} = \varprojlim (M/F^n)$ (as $\mathbb{Z}/2$-complexes).

$$H_i(G_n/G_{n-1}) = \begin{cases} 0 & i = n + 2\mathbb{Z} \\ H_{n-1}(M; B) & i = n - 1 + 2\mathbb{Z}. \end{cases}$$

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$$F^n/F^{n+1}: \qquad \overset{n}{bM_{n+1}} \underset{B=0}{\overset{b}{\leftrightarrows}} \overset{n+1}{M_{n+1}/bM_{n+2}}$$

$$G_n/G_{n-1}: \quad M_{n-1}/bM_{n-2} \underset{B}{\overset{b=0}{\rightleftarrows}} BM_{n-1}$$

<u>Def.</u> $HC_n(M) = H_{n+2\mathbb{Z}}(M/F^n M)$. C: cyclic

$HD_n(M) = H_{n+2\mathbb{Z}}(M/F^{n+1}M)$ D: de Rham.

There are q'ism invariants; 6-term seq $\Rightarrow$

$$0 \to HD_n(M) \to HC_n(M) \to H_{n+1}(M;b) \to$$
$$HC_{n+1}(M) \to HD_{n-1}(M) \to 0.$$

Splice out HD: <u>Connes Exact Sequence</u>:

$$\cdots \to HC_n(M) \xrightarrow{B} H_{n+1}(M;b) \xrightarrow{I} HC_{n+1}(M) \xrightarrow{S}$$
$$HC_{n-1}(M) \to \cdots$$

S, I, B standard names. Further:

$$HD_n(M) = \operatorname{Im}\left(S: HC_{n+2}(M) \to HC_n(M)\right)$$
$$= \ker\left(B: HC_n(M) \to H_{n+1}(M;b)\right).$$

$S: HC_{n+2}(M) \to HC_n(M)$ is the map induced by

$$M/F^{n+2}M \longrightarrow M/F^n M.$$

Special case: Assume $H(M;B)=0$.

Lemma. $HC_n(M) = H_n(M/BM, b)$.

$HD_n(M) = H_n(M/bM, B)$.

pf. $H(M;B)=0 \Rightarrow H_i(G_n M) = 0 \; (\forall i)$. ~~by~~

since $H_i(G_n/G_{n-1}) = \begin{cases} 0 & i = n+2\mathbb{Z} \\ H_{n-1}(M;B) & i = n-1+2\mathbb{Z}. \end{cases}$

acyclic: $\rightarrow$

$$\begin{array}{lcl} G_n & & \cdots \oplus M_{n-1} \oplus BM_n \\ \downarrow & & \\ M/F^n : & & \cdots \oplus M_{n-1} \oplus M_n \oplus M_{n+1}/bM_{n+2}. \\ \downarrow & & \\ M/(F^{n+1}+G_n) : & & M_n/BM_{n-1} \underset{B}{\overset{b}{\leftrightarrows}} M_{n+1}/bM_{n+2} \\ \downarrow & & \\ 0 & & \end{array}$$

So $HC_{n+1}(M) = H_{n+1+2\mathbb{Z}}(M/F^{n+1}) \overset{\text{by les. + acyclicity.}}{=} H_{n+1+2\mathbb{Z}}(M/G_n+F^{n+1})$

$$= \frac{b^{-1}(BM_{n-1})/bM_{n+2}}{(BM_n + bM_{n+2})/bM_{n+2}} \cong \frac{b^{-1}(BM_{n-1})}{BM_n + bM_{n+2}}.$$

& $HD_n(M) = H_{n+2\mathbb{Z}}(M/F^{n+1}) = H_{n+2\mathbb{Z}}(M/G_n+F^{n+1})$

$$= \frac{B^{-1}bM_{n+2}}{BM_{n-1}+bM_{n+1}}.$$

//$\mathcal{b}$.

Traditional approach. (Kassel; Burghelea; Quillen-Loday).

To M associate a first quadrant bicomplex

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\ \downarrow b & & \downarrow b & & \\ M_1 & \xleftarrow{B} & M_0 & & \\ \downarrow b & & & & \\ M_0 & & & & \end{array}$$

Let $B(M)$ be the assoc. total complex:

$$(BM)_n = M_n \oplus M_{n-2} \oplus \cdots$$

with differential $b + B$.

<u>Claim</u>: $HC_n(M) = H_n(BM)$.

S shifts SW 1 step, & kills left column:

$$0 \to (M, b) \to BM \to \Sigma_{1,1} BM \to 0$$

$\Rightarrow$ Connes exact sequence.

<u>Claim</u>: $HD_n(M) = \mathrm{Im}\left(S : H_{n+2}(BM) \to H_n(BM)\right)$.

Check of 1^{st} claim:

$$Z_n(\mathcal{B}M) = \{(x_n, x_{n-2}, \dots) : bx_n + Bx_{n-2} = 0,\ bx_{n-2} + Bx_{n-4} = 0, \dots\}$$

$$B_n(\mathcal{B}M) = \{(by_{n+1} + By_{n-1},\ by_{n-1} + By_{n-3}, \dots)\}$$

$$M/F^n: \qquad \cdots \oplus M_{n-1} \oplus M_n/bM_{n+1}$$

$$Z_n(M/F^nM) = \{(x_n + bM_{n+1}, x_{n-1}, \dots) : bx_n + Bx_{n-2} = 0 \dots\}$$

& $B_n(M/F^nM) =$ same as $B_n(\mathcal{B}M)/bM_{n+1}$ $\triangleleft$.

30 Sept.

Recollection: (M, b, B) mixed complex

$$\mathcal{B}M: \quad \mathrm{tot}\begin{pmatrix} M_2 \leftarrow M_1 \leftarrow M_0 \\ \downarrow \quad\quad \downarrow \\ M_1 \leftarrow M_0 \\ \downarrow \\ M_0 \end{pmatrix} \qquad \begin{array}{l} F^n M \\ = bM_{n+1} \oplus M_{n+2} \oplus \cdots \end{array}$$

$$H^c_n M = H^*_n(\mathcal{B}M) \cong H_{n+2\mathbb{Z}}(M/F^n M, b+B).$$

Connes exact seq.

$$\cdots \to H^c_{n+2} \xrightarrow{S} H^c_n \xrightarrow{B} H^b_{n+1} \xrightarrow{I} H^c_{n+1} \longrightarrow \cdots$$

Periodic cyclic homology: extend $\mathcal{B}$ to $\hat{\mathcal{B}}$:

$$\begin{array}{ccccccc} & \leftarrow M_2 & \leftarrow & M_1 & \leftarrow & M_0 & -- \\ \cdots & \downarrow & & \downarrow & & & \\ & \leftarrow M_1 & \leftarrow & M_0 & & & \\ & \downarrow & & & & & \\ & M_0 & & & & & \\ \cdots & p=-1 & & p=0 & & & \end{array}$$

form product tot: $(\hat{\mathcal{B}}M)_n = \prod_{p \in \mathbb{Z}} M_{n-2p}$

& define $H^p_{n+2\mathbb{Z}}(M) = H_n(\hat{\mathcal{B}}M)$.

Alt.: let $\hat{M} = \varprojlim (M/F^n M)^{\leftarrow}$ ← $\mathbb{Z}/2$-grade.

$$\cong \prod_{n \in 2\mathbb{Z}} M_n \oplus \prod_{n \in 1+2\mathbb{Z}} M_n$$

Then $H^p_n M = H_n(\hat{M})$.

$$0 \longrightarrow R^1 \varprojlim_n H_{i+1}(M/F^n M) \longrightarrow H_i(\hat{M}) \to \varprojlim_n H_i(M/F^n) \to 0$$

ie $$0 \longrightarrow R^1 \varprojlim_{n \in i} H^c_{n+1} M \longrightarrow H^P_i M \longrightarrow \varprojlim_{n\in i} H^c_n M \longrightarrow 0.$$

so $H^P_i M$ is a quasi-iso invariant of M.

<u>Negative cyclic homology:</u>

$$\hat{\mathcal{B}}^- = \mathrm{tot} \left(\begin{array}{cc} \leftarrow M_2 & \leftarrow M_1 \\ \downarrow & \\ \leftarrow M_1 & \leftarrow M_0 \\ \downarrow & \\ \leftarrow M_0 & \\ \cdots & \\ p=-1 & p=0 \end{array} \right).$$

$$H^{c-}_n(M) = H_n(\hat{\mathcal{B}}^- M).$$

$\hat{\mathcal{B}}^-$ contains itself with a shift so there's a Connes exact sequence:

$\hat{\mathcal{B}} / \hat{\mathcal{B}}^- \cong$ shift of $\mathcal{B}$; so get LES.

Exer. Filter $\hat{M}$: $F^n \hat{M} = bM_{n+1} \times \prod_{k \geq n+1} M_k$ (with $\mathbb{Z}_2$-grading).

Show $H_{n+1+2\mathbb{Z}}(F^n \hat{M}) = H^{c-}_{n+1} M$

$H_{n+2\mathbb{Z}}(F^n \hat{M}) = \ker\left(H^{c-}_n M \to H^b_n M\right).$

Cyclic Homology of Algebras.

Program. Construct various cyclic theories of A, constructed as above using ΩA for M.

Morita invariance –
- Dir prd. thm $HC(A\times B) = HC(A)\times HC(B)$
- Matrix thm $HC(M_n A) = HC(A)$.

Method due to Lars Kadison's thesis: use relative cyclic thy.
Given subalg. $S \subseteq A$. $\Rightarrow HC(A,S)$.

Then discuss "reduced cyclic theory" $\overline{HC}$
Doesn't have Morita invariance; but better suited to $\otimes$; have "Connes' Lemma" & "Connes-Karoubi th."

S. A alg. ΩA ; $\Omega^n A = A \otimes \bar{A}^{\otimes n}$.
d, b, κ, B, P, G.
$(\Omega A, b, B)$ is a mixed complex.

$\Rightarrow$ $HH_n(A) = H^b_n(\Omega A)$ $\qquad HC^-_n(A) = H^{c-}_n(\Omega A)$.
$HC_n(A) = H^c_n(\Omega A)$

$$HP_n(A) = H^P_n(\Omega A) = H_n\left\{\prod_i \Omega^{2i}A \underset{b+B}{\overset{b+B}{\rightleftarrows}} \prod_i \Omega^{2i+1}A\right\}$$

This last is the most important; it's the noncom. analogue of deRham; HC_n arises from a "Hodge filtration"

Recall $\Omega A \simeq P\Omega A \oplus P^{\perp}\Omega A$.

gen. eigenspace for value 1 of κ.

and $H(P^{\perp}\Omega A; b) = 0$: so $P\Omega A \hookrightarrow \Omega A$ is q.iso.

so the cyclic theories can be computed using $P\Omega A$.

<u>Relative differential forms.</u> $S \subseteq A$ subalg.

(More generally can take hom. $f: S \to A$; but $\operatorname{Im} f \hookrightarrow A$ will give the same results.)

Put $\Omega^n_S A = A \otimes_S (A/S)^{\otimes_S n}$.

<u>Prop.</u> $\exists!$ DG alg str. on $\Omega^{\cdot}_S A$ st $\left[\begin{matrix}\text{(this follows)} \\ dS = 0 \ \&\end{matrix}\right]$

(1) $a_0 da_1 \cdots da_n = (a_0, \ldots a_n)$ <u>OR</u>

(2) Given DG alg Γ & hom. $u: A \to \Gamma^0$,

st $duS = 0$, $\exists!$ DG alg. hom $u_*: \Omega_S A \to \Gamma$ extending u.

(The hard part of the pf. with $S = \mathbb{C}$ was to show ΩA with explicit prd. is associative. Knowing this, associativity of $\Omega_S A$ follows.)

<u>Properties:</u> 1. $\Omega_S A \cong \Omega A / (dS)$.

2. $d: A \to \Omega^1_S A$ is universal ~~S-bimod~~ derivation killing S.

3. Nat ex. seq. of bimodules: $j(a_0 da_1) = a_0 a_1 \otimes 1 - a_0 \otimes a_1$

$$0 \to \Omega^1_S A \xrightarrow{j} A \otimes_S A \xrightarrow{m} A \to 0.$$

Claim. d, b, κ, B, P, G on ΩA descend to

$$\Omega_S A / [\Omega_S A, S]$$

(Not to $\Omega_S A$, since $b(\omega ds) = \pm [\omega, s]$)

Lets write $M \otimes_S = M / [M, S]$, M a bimodule.
$= M_\natural$

Get mixed complex $(\Omega_S A \otimes_S, b, B)$ which leads to rel cyclic theories

$$HH(A, S), \quad HC(A, S), \quad HP(A, S), \quad HC^-(A, S).$$

For example:

Prop. $S \hookrightarrow S \otimes A$, S, A $\mathbb{C}$-algebras. Then

$$\Omega_S(S \otimes A) = S \otimes \Omega A$$

$$\Omega_S(S \otimes A) \otimes_S = S_\natural \otimes \Omega A$$

Eg: $S = M_n \mathbb{C}$; $S \otimes A = M_n A$;

$$\Omega_{M_n \mathbb{C}}(M_n A) \cong M_n(\Omega A)$$

$$\Omega_{M_n \mathbb{C}}(M_n A) \otimes_{M_n \mathbb{C}} \cong \Omega A$$

using $(M_n \mathbb{C})_\natural \xrightarrow[\cong]{tr} \mathbb{C}$.

20 Oct

Example 1. S com. $\mathbb{C}$-alg, and A an "algebra over S": i.e. $S \longrightarrow A$ whose image is in the center of A.
Then $S \longrightarrow \Omega_S A$ maps to the center, again:
$$s\,da = d(sa) = d(as) = da \cdot s;$$
$\Omega_S A$ is the univ. DG alg. over S containing A.

Example 2. A any alg; $S \otimes A$ an S-algebra. Then as in prev lecture, $\Omega_S(S \otimes A) \cong S \otimes \Omega A$.

Relative standard resol. of the A-bimodule A:-

Abs case: $$\cdots \longrightarrow A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{m} A \longrightarrow 0$$

Rel case: $$\longrightarrow A \otimes A/S \otimes A \longrightarrow A \otimes_S A \longrightarrow A \longrightarrow 0$$

– a quotient complex. Namely,
$$\left(\Omega^{\cdot} A \otimes A \longrightarrow A\right) \longrightarrow \left(\Omega^{\cdot}_S A \otimes_S A \longrightarrow A\right).$$

with d st $$b'(\omega\, da \otimes a') = (-1)^{|\omega|}\omega(a \otimes 1 - 1 \otimes a)\,a'.$$

The same homotopy operator shows exactness.
Apply $(\ )_\natural = (\ )_{\otimes A}$; use
$$(X \otimes_S A) \otimes_{A} = X \otimes_S$$
for ${}_A X_S$; generalizes $(X \otimes A) \otimes_A = X$). Then
$$(\Omega_S A \otimes A) \otimes_A \cong (\Omega_S A) \otimes_S; \quad b' \longleftrightarrow b.$$

So all the operators b, d, κ, B, G, P descend

to $(\Omega_S A)\otimes_S$: get a mixed complex.

Ex. In case $S \to A$ is central, then $\Omega_S A$ has b, B.

And again, $P(\Omega_S A \otimes_S) \subset \Omega_S A \otimes_S$
is a quasi-isom. sub-mixed complex.

Kadison's Thm. Assume $A \otimes_S A$ is a projective A-bimodule.
Then $\Omega A \longrightarrow \Omega_S A \otimes_S$ is a quasi-iso.
of mixed cxs.

What we need to do is show that the objects in the relative standard resol. are projective.
So the hypothesis is certainly necessary!

How about $\Omega_S^1 A \otimes_S A$. Recall

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 A \arrow[r] \arrow[d] & A \otimes A \arrow[r] \arrow[d, shift right] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_S^1 A \arrow[r] & A \otimes_S A \arrow[r] \arrow[u, dashed, shift right] & A \arrow[r] & 0
\end{tikzcd}$$

Lifting exists by projectivity. It will induce a section of $\Omega^1 A \twoheadrightarrow \Omega_S^1 A$: so $\Omega_S^1 A$ is a bimodule summand of $\Omega^1 A$

Then : $\Omega_S^n A \otimes_S A = \overbrace{\Omega_S^1 A \otimes_A \cdots \otimes_A \Omega_S^1 A}^{n} \otimes_A (A \otimes_S A)$

is a dir. summand of $\Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A \otimes_A (A \otimes A)$

which is $\Omega^n A \otimes A$, which is a free A-bimodule //

Example: separability.

Def: S is separable iff it is a projective S-bimodule.

Over an alg. cl. field, such S is a finite product of matrix algebras.

S is proj. $\Leftrightarrow$ $0 \to \Omega^1 S \to S \otimes S \to S \to 0$ splits.

So an equiv. def. is: S is separable iff $\nearrow$ splits.

Form $A \otimes_S - \otimes_S A$: get split cx seq of A-bimodules

$$0 \to A \otimes_S \Omega^1 S \otimes_S A \longrightarrow A \otimes A \longrightarrow A \otimes_S A \longrightarrow 0.$$

so $A \otimes_S A$ is a projective A-bimodule.

Corollary 1. $A = A_1 \times A_2$. Then $\Omega A \to \Omega A_1 \times \Omega A_2$.
This is a quasi-iso of mixed complexes.

pf: Take $S = \mathbb{C} \times \mathbb{C} \subset A_1 \times A_2 = A$.
S is separable, & $\Omega_S A = \Omega A_1 \times \Omega A_2$:

S is central, so this is also $(\Omega_S A) \otimes_S$. $\triangle$.

Corollary 2. If S is separable then $\Omega(S \otimes A) \to S_\natural \otimes \Omega A$
is a quasi-iso. of mixed complexes.

so $HC(S \otimes A) \cong S_\natural \otimes HC(A)$, etc.

pf. $\Omega_S(S\otimes A) = S\otimes \Omega A$.

$\Omega A \xrightarrow{} \Omega_S(S\otimes A)\otimes_S = S_\natural \otimes \Omega A$.
q'ism by Kadison. ∎.

Case: $S = M_n\,\mathbb{C}$; then $S\otimes A = M_n A$.
$S_\natural \xrightarrow{\sim} \mathbb{C}$ by trace.

so $\Omega(M_n A) \xrightarrow{\simeq} \Omega A$ by trace.

& $HC(M_n A) \xrightarrow{\simeq} HC(A)$ etc.

Sept 4.

$\Omega\mathbb{C} = \mathbb{C}$; $\bar{\Omega} A = \Omega A / \Omega\mathbb{C}$: a mixed complex.
Then $\bar{H}H_n(A) = H_n(\bar{\Omega}A; b)$, etc.
Connes ex. seq: $\cdots \bar{H}C_{n+2} \xrightarrow{S} \bar{H}C_n \xrightarrow{B} \bar{H}H_{n+1} \xrightarrow{I} \bar{H}C_{n+1} \cdots$
$\mathcal{B}$ is exact so $0 \to \mathcal{B}\mathbb{C} \to \mathcal{B}\Omega A \to \mathcal{B}\bar{\Omega}A \to 0$;
$\Rightarrow$ $\cdots \to HC_n(\mathbb{C}) \to HC_n A \to \bar{H}C_n A \to HC_{n-1}\mathbb{C} \to \cdots$

$HC_n(\mathbb{C}) = \begin{cases} \mathbb{C} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ Usually $HC(\mathbb{C}) \hookrightarrow HC(A)$.

Reason for studying it: $H(\bar{\Omega}A; d) = 0$ (not $\mathbb{C}$)

Important example: augmented alg; $A = \mathbb{C}\oplus \bar{A}$.
ΩA augmented by $\Omega A \to \Omega\mathbb{C} = \mathbb{C}$; $\ker \cong \bar{\Omega}A$

Connes-Tsygan bicomplex:

$$\begin{tikzcd}
\cdots \bar{A}^{\otimes 3} \arrow[d, "b"] & \bar{A}^{\otimes 3} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \bar{A}^{\otimes 3} \arrow[l, "N_\lambda"'] \arrow[d] & \cdots \\
\cdots \bar{A}^{\otimes 2} \arrow[d, "b"] & \bar{A}^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \bar{A}^{\otimes 2} \arrow[l, "N_\lambda"'] \arrow[d] & \cdots \\
\cdots \bar{A} & \bar{A} \arrow[l, "1-\lambda"'] & \bar{A} \arrow[l, "N_\lambda"'] & \cdots \\
0 & 1 & 2 &
\end{tikzcd}$$

$S \leftarrow$ 2 steps

$S^* \rightarrow$

$\mathcal{C}(\bar{A})$ = total cx.

vs: for any mixed complex:

$$\begin{tikzcd}
M_2 \arrow[d, "b"] & M_1 \arrow[l, "B"'] \arrow[d, "b"] & M_0 \arrow[l, "B"'] & \cdots \\
M_1 \arrow[d, "b"] & M_0 \arrow[l, "B"'] & & \\
M_0 & & & \\
0 & 1 & 2 &
\end{tikzcd}$$

$S \swarrow$

$S^* \nearrow$

$B(M)$ = tot cx.

Then:

$$\mathcal{B}(\bar{\Omega}A) \cong \mathcal{C}(\bar{A})$$

in a way reflecting periodicity S, S^*.

Recall $\Omega^n A \cong \bar{A}^{\otimes(n+1)} \oplus \bar{A}^{\otimes n}$

$$b \longleftrightarrow \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}$$

This shows that tot of 1^{st} two col's of $\mathcal{C}(\bar{A})$ is isom to ΩA.

$$B \longleftrightarrow \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

You see that both $\mathcal{B}(\bar\Omega A)$ and $\mathcal{C}(\bar A)$ are

$$\bigoplus_{p\geq 0} (S^*)^p \,\bar\Omega A \qquad \text{with diff.} \qquad b + SB.$$

$\triangle$

<u>Conclusion</u>: $\overline{HC}(A)$ can be computed using $\mathcal{C}$.

"Problem": if $\bar A$ in fact has an identity, what is the relation between $\overline{HC}(A)$ & $HC(\bar A)$?
ie $\bar\Omega(\mathbb{C}\oplus\bar A)$ vs $\Omega\bar A$.

Claim: there's a quasi-iso of mixed complexes

$$\bar\Omega(\mathbb{C}\oplus\bar A) \longrightarrow \Omega\bar A :$$

$$\mathbb{C}\oplus\bar A \xrightarrow{\cong} \mathbb{C}\times\bar A$$

(2nd factor uses $\mathbb{C}\to\bar A$) So the direct product th. $\Rightarrow$

$$\begin{tikzcd}
\mathbb{C} & \mathbb{C} \\
\Omega(\mathbb{C}\oplus\bar A) \arrow[u] \arrow[r, "\cong"] & \Omega\mathbb{C}\times\Omega\bar A \arrow[u] \\
\bar\Omega(\mathbb{C}\oplus\bar A) \arrow[u] \arrow[r] & \Omega\bar A \arrow[u]
\end{tikzcd}$$

$\triangle$.

$=$ $d\,|\,\bar\Omega A$ is exact: so !-

$$\begin{array}{ccccc}
\mathbb{C} & = & \mathbb{C} & \oplus & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega A & = & P\Omega A & \oplus & P^{\perp}\Omega A \\
\downarrow & & & \downarrow & \\
\bar\Omega A & = & P\bar\Omega A & \oplus & \underbrace{P^{\perp}\Omega A}_{b,\ d \text{ exact here.}}
\end{array}$$

$$\bar{\Omega} A \quad = \quad P\bar{\Omega} A \quad \oplus \quad P^{\perp}\Omega A$$

$$\begin{array}{ccc} d \text{ exact} & d \text{ exact} & b, d \text{ exact} \\ & B = Nd \text{ exact} & B = 0. \end{array}$$

So to compute $\overline{HC}$ you can use $P\bar{\Omega}A$, on which B is ex.

This is a strong form of Connes' Lemma (which is fundamental from his pt of view). :

On $\bar{\Omega}A$, $H(\ker B / \operatorname{Im} B \,;\, b) = 0$. clean.

Recall that if M is a mixed cx with B exact, then

(1)
$$H_n^c(M) = H_n(M/BM \,;\, b).$$

and
$$\ker\left(B : H_n^c M \to H_{n+1}^b M\right) = \operatorname{im}\left(S : H_{n+2}^c M \to H_n^c M\right)$$

(2)
$$= H\left(M/bM ; B\right).$$

Defn. The reduced cyclic complex is

$$\overline{CC}(A) = \Omega A / \ker B \quad , \text{ with } b.$$

Compute :
$$\overline{CC}_n(A) = A \otimes \bar{A}^{\otimes n} / \ker B$$

$$\cong \bar{A}^{\otimes(n+1)} / (1-\lambda)\bar{A}^{\otimes(n+1)}$$

via $a_0\, da_1 \cdots da_n \to da_0\, da_1 \cdots da_n$.

$$\bar{C}C(A) = \Omega A / \ker B = \bar{\Omega} A / \ker B$$

$$= \underbrace{P\bar{\Omega} A / \ker B}_{= \operatorname{Im} B} \oplus \underbrace{P^{\perp}\bar{\Omega} A / \ker B}_{= 0}$$

$$= P\bar{\Omega}A / BP\bar{\Omega}A = M / BM$$

with $M = P\bar{\Omega}A$: so using (1),

<u>Prop.</u> $H_n(\bar{C}C(A)) \cong \overline{HC}_n(A)$.

<u>Connes - Karoubi Thm.</u>

$\Omega A, d$ is acyclic. An integral should kill commutators; so we're interested in:

<u>Th.</u> $H_n(\Omega A / \mathbb{C} + [\Omega A, \Omega A], d) \cong \operatorname{Ker}(B : \overline{HC}_n(A) \to \overline{HH}_{n+1}(A))$.

(Note: LHS doesn't involve b or B at all)

<u>pf.</u> Ω is gen'd as alg by A, dA, &

$$[\Omega, \Omega] = [\Omega, A] + [\Omega, dA].$$

$[m, xy] = [mx, y] + [ym, x]$. implies this.

$b(\omega da) = \pm[\omega, a]$, so $[\Omega, A] = b\Omega$.

$(1-\kappa)(\omega da) = [\omega, da]$, so $[\Omega, dA] = (1-\kappa)\Omega$.

so $\quad \Omega/[\Omega,\Omega] = \Omega/\left(b\Omega + (1-\kappa)\Omega\right)$

& $\quad \Omega/(\mathbb{C} + [\Omega,\Omega]) = \bar{\Omega}/(b\bar{\Omega} + (1-\kappa)\bar{\Omega})$.

Recall that on Ω^n, $\qquad \kappa^n = 1 + b\kappa^{-1}d$

so $\qquad \kappa^n = 1$ on $\Omega^n/b\Omega^n$.

When κ is of finite order, κ-invariants $= \operatorname{Im} P$.

So on $\bar{\Omega}/b\bar{\Omega}$,

$$P(\bar{\Omega}/b\bar{\Omega}) \cong \bar{\Omega}/\left(b\bar{\Omega} + (1-\kappa)\bar{\Omega}\right)$$

$$\|$$

Since P is exact, $\qquad P\bar{\Omega}/Pb\bar{\Omega}$.

But on $P\bar{\Omega}$ $\quad d$ & B are proportional: so we can use (2) above to see:

$$H(\Omega/(\mathbb{C}+[\Omega,\Omega])) \cong H(P\bar{\Omega}/Pb\bar{\Omega}\,;\, B)$$

$$\cong \operatorname{Ker}\left(B : H^c_n P\bar{\Omega} \longrightarrow H^b_{n+1} P\bar{\Omega}\right)$$

$$= \operatorname{Ker}\left(B : \overline{HC}_n(A) \longrightarrow \overline{HH}_{n+1}(A)\right) \qquad \Delta.$$

Augment the Connes Tsygan double cx:

$$\begin{tikzcd}
0 & A^{\otimes 3}/1-\lambda \arrow[l] \arrow[d, "b"] & \bar{A}^{\otimes 3} \arrow[l] \arrow[d] & \arrow[l, "1-\lambda"'] \\
0 & A^{\otimes 2}/1-\lambda \arrow[l] \arrow[d] & \bar{A}^{\otimes 2} \arrow[l] \arrow[d] & \arrow[l, "1-\lambda"'] \\
0 & \bar{A} \arrow[l] & \bar{A} \arrow[l] & \arrow[l, "1-\lambda"']
\end{tikzcd}$$

Each row is a resol; so the $\bar{C}C(A)$ is $\cong$ the traditional cyclic complex $CC(\bar{A})$.

~~$\bar{A}$ nonunital~~ :

7 Sept.

Defn. $X(R)$ is the $\mathbb{Z}/2$-graded complex

$$R \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 R_\natural$$

where $\Omega^1 R_\natural = \Omega^1 R / [R, \Omega^1 R]$, $b(x\,dy) = [x,y]$.

Recall: ΩR is a mixed complex with decreasing filt.

$$F^n \Omega R = b\Omega^{n+1} R \oplus \Omega^{n+1} R \oplus \cdots$$

$\Rightarrow$ tower of $\mathbb{Z}/2$-graded cx's.

$$\Omega R / F^n \Omega R \quad , \quad \text{diff. } b+B.$$

with $H_{n+2\mathbb{Z}}(\quad) = HC_n(R)$

& $$H_{n+1+2\mathbb{Z}}(\Omega R/F^{n+1}\Omega R) = \ker(HC_{n-1}R \xrightarrow{B} HH_n R)$$

which is the "homotopy-invariant part of $HC_{n-1}R$," as we'll see.

Then

$$\hat{\Omega}R = \lim(\Omega R/F^n\Omega R) \simeq \prod \Omega^n R.$$

& $$HP_i(R) = H_i(\hat{\Omega}R).$$

Eg: $\Omega R/F^0\Omega R$: $\quad R/[R,R] \rightleftarrows 0$.

$\Omega R/F^1\Omega R$: $\quad R \underset{B=d}{\overset{b}{\rightleftarrows}} \Omega^1 R/b\Omega^2 R$

but $\quad b\Omega^{n+1}R = [\Omega^n R, R]$

since $\quad b(\omega dx) = (-1)^{|\omega|}[\omega, x]$.

So we get exactly $\underline{X(R)}$.

Recall
$$H_i(F^n/F^{n+1}) = \begin{cases} 0 & i = n+2\mathbb{Z} \\ HH_{n+1}R. & i = n+1+2\mathbb{Z} \end{cases}$$

so if $HH_n(R) = 0$ for $n \geq 2$ then

$\Omega R/F^n\Omega R$ is quasi-iso to $\Omega R/F^1\Omega R = X(R)$.

so ~~...~~ also $\hat{\Omega}R \xrightarrow{\sim} X(R)$.

- The first order approximation is exact.
This happens for free or "quasifree" R.

Understand X complex via "dual viewpoint":

As in: $R/[R,R]$ is universal for traces:

<u>Defn</u>. M an R-bimodule. A <u>trace</u> on M with values in a v.s. V is a $\mathbb{C}$ linear map $\tau : M \to V$ st
$$\tau(mr) = \tau(rm) \quad \forall r \in R,\ m \in M.$$

Clearly $$M \to M/[R,M]$$ is universal.

<u>Prop</u>. Given a $\mathbb{C}$-bilinear map $f : R \times R \to V$,
st $$(bf)(x,y,z) = f(xy,z) - f(x,yz) + f(zx,y) = 0$$

$\exists!$ linear $f_* : \Omega^1 R_\natural \to V$

st $$f_*(x\,dy) = f(x,y).$$

<u>pf</u>. Assume $bf = 0$. Then $f(x,1) = 0$.

$$\begin{array}{ccccccc} \Omega^2 R & \xrightarrow{b} & \Omega^1 R & \longrightarrow & \Omega^1 R_\natural & \longrightarrow & 0 \\ \| & & \| & & & & \\ R \otimes \bar{R}^{\otimes 2} & \longrightarrow & R \otimes \bar{R} & & & & \end{array}$$

$$x\,dy\,dz \mapsto xy\,dz - x\,d(yz) + zx\,dy. \qquad \Delta$$

<u>Example of a Hochschild 1-cocycle</u>: traces under homotopies:

Consider 1-param. family of hom's $u_t : R \to R'$.
& trace $\tau' : R' \to \mathbb{C}$ (eg $R \to R'[t]$).

Then $\tau' u_t$ is a family of traces on R.

Set $$f(x,y) = \tau'(\dot{u}_0 x \cdot \dot{u}_0 y)$$

is a Hochschild 1-cocycle, and

$$f(1,y) = \partial_t\left(\tau' u_t\, y\right)\Big|_{t=0}$$

To check the cocycle ~~property~~, compute $f(x, yz)$

This is a restriction on families of traces which can arise as $\tau' u_t$.

Homology of $X(R)$.

$\operatorname{Im} b = [R,R]$.

$$H_0 X(R) = \ker\left(d : R_\natural \longrightarrow \Omega^1 R_\natural\right).$$

Dualize:

$$\begin{array}{ccc} R^* & \underset{d^t}{\overset{b^t}{\rightleftarrows}} & (\Omega^1 R_\natural)^* \\ \| & & \| \\ \{g : R \to \mathbb{C}\} & & \{f(x,y) : bf = 0\} \end{array}$$

& $$(b^t g)(x,y) = g[x,y]$$

[This is a 1-cocycle, via a case of the "Circular Bracket Identity":

$$[x, y_1 \cdots y_n] = \sum_{j=1}^{n} [y_{j+1} \cdots y_n x y_1 \cdots y_{j-1}, y_j]$$

Similarly, if $bf = 0$ then

$$f(x, y_1 \cdots y_n) = \sum f(y_{j+1} \cdots y_n x y_1 \cdots y_{j-1}, y_j) \quad]$$

& so $\quad b^t g = 0 \iff g$ is a trace.

$$(d^t f)(y) = f(1, y).$$

so $\quad H_0(X(R)^*) = $ traces on R / "principal traces,"

where a principal trace is one of the form

$\tau' d$, where τ' is a trace on $\Omega' R$.

<u>Example</u>. $\quad u_t(x) = (x, t\,dx)$, $u_t : R \longrightarrow R \oplus \underbrace{\Omega' R}_{\text{square-zero.}}$

If τ' is a trace on $\Omega' R$, then extending it to be zero on R gives a trace on $R \oplus \Omega' R$.
Then

$$\tau' u_t(x) = \tau'(t\,dx) = t\,\tau' dx$$

This interpolate between the 0 trace & $\tau' d$.
That is, the ~~tr~~ principal traces can be deformed to zero via homomorphisms of algebras

Sept 9

$X(R)$: $\quad R \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 R_{\natural} \qquad b(x\,dy) = [x,y].$

Additionally: $xy : R \times R \longrightarrow R$

$x\,dy : R \times R \longrightarrow \Omega^1 R_{\natural}.$

Recall the universal extension

$$A \xrightarrow{\rho} RA = T(A)/(1_{TA} - 1_A)$$

Noncanon. isom. to $T(\bar{A})$: in part., free.

ρ is universal among linear maps to alg's with $1 \longmapsto 1$.

Since RA is free, $\exists$ (canon) isom. of RA bimodules

$$\begin{array}{ccc} RA \otimes \bar{A} \otimes RA & \xrightarrow{\cong} & \Omega^1(RA) \\ x \otimes a \otimes y & \longmapsto & x\, d(\rho a)\, y. \end{array}$$

so

$$\begin{array}{ccc} RA \otimes \bar{A} & \xrightarrow{\cong} & \Omega^1(RA)_{\natural} \\ x \otimes a & \longmapsto & x\, d(\rho a). \end{array}$$

Recall the Fedosov description of RA as $\Omega^+ A$ even with product

$$x \circ y = xy - dx\, dy.$$

Then ρ is $A = \Omega^0 A \subset \Omega^+ A$. So

(*) $\quad \Omega^1(RA)_{\natural} \xleftarrow{\cong} RA \otimes \bar{A} \qquad x\,\delta a \longleftarrow x \otimes a$

We have two d's: in $X(RA)$ & in ΩA.

Notation || Write β, δ for the operators in $X(RA)$.

$$x\,\delta a \longleftarrow\!\shortmid \quad x \otimes a \quad \longmapsto \quad x\,da$$

$$\Omega^1(RA)_q \xleftarrow{\cong} RA \otimes \bar{A} \xrightarrow{\cong} \Omega^- A$$

And $RA \cong \Omega^+ A$; so $X(RA) \cong \Omega^{\pm} A$, with some operators we proceed to make explicit.

1) Product in RA = Fedosov product in $\Omega^+ A$.

2) Cocycle id: $x\,\delta(yz) = (x \circ y)\,\delta z + (z \circ x)\,\delta y \qquad x, y \in RA.$

3) $$x\,\delta a = x\,da \qquad a \in A.$$

4) Calculate $\beta : \Omega^- A \longrightarrow \Omega^+ A$:

$$\beta(x\,da) = \beta(x\,\delta a) = [x, a]_\circ = x \circ a - a \circ x$$

$$= xa - \cancel{dx}\,da - ax + da\,dx$$

$$= [x, a] - dx\,da - \kappa(dx\,da)$$

$$= b(x\,da) - (1+\kappa)\,dx\,da$$

$$= (b - (1+\kappa)d)\,x\,da.$$

$$\beta = b - (1+\kappa)d.$$

5) Compute $x\,\delta y$, $x, y \in \Omega^+ A$. $|y| = 2n$

$$x\,\delta y = -\sum_{j=0}^{n-1} \kappa^{2j}\, b(x \circ y) + \sum_{j=0}^{2n-1} \kappa^j\, d(x \circ y) + \kappa^{2n}(x\,dy)$$

Pf by ind. on n:

$n=0$ $y = a$. $x\delta a = xda$ ✓.

Ind: By linearity wma $y = z\,da_1\,da_2$ $z \in \Omega^{2n-2}A$. Cocycle $\Rightarrow$

(*) $x\,\delta(z\,da_1\,da_2) = (x \circ z)\,\delta(da_1\,da_2) + ((da_1\,da_2)\circ x)\,\delta z$ — same as.

Write $x' = x \circ z$; look at 1st term:

$x'\delta(da_1\,da_2) = x'\delta(a_1a_2 - a_1 \circ a_2)$; use cocycle again

$= x'\delta(a_1a_2) - (x' \circ a_1)\,\delta a_2 - (a_2 \circ x')\,\delta a_1$

$= x'\,d(a_1a_2) - (x' \circ a_1)\,da_2 - (a_2 \circ x')\,da_1$

$= x'(da_1 \cdot a_2 + a_1\,da_2) - (x'a_1\,da_2 - dx'\,da_1\,da_2)$

$- (a_2x'\,da_1 - da_2\,dx'\,da_1)$

$= [x'\,da_1, a_2] + (1+\kappa)(dx'\,da_1\,da_2)$

$= -b(x'\,da_1\,da_2) + (1+\kappa)\,d(x'\,da_1\,da_2)$

$x'\,da_1\,da_2 = (x \circ z)\,da_1\,da_2 = (x \circ z) \circ da_1\,da_2$

$= x \circ (z \circ da_1\,da_2) = x \circ (z\,da_1\,da_2)$ so

$\ldots = -b(x \circ z\,da_1\,da_2) + (1+\kappa)\,d(x \circ z\,da_1\,da_2)$.

$$y = z\,da_1\,da_2.$$

Now use ind. hyp. to evaluate 2nd term in ⊛.

$$(da_1\,da_2\,x)\,\delta z = -\sum_{j=0}^{n-2} \kappa^{2j}\, b(da_1\,da_2\,x \circ z)$$

$$+\sum_{j=0}^{n-3} \kappa^j d\,\underbrace{(da_1\,da_2\,x \circ z)}_{=\kappa^2(x \circ z\, da_1\, da_2)} + \kappa^{2n-2}(da_1\,da_2\,x \cdot dz).$$

$$= -\sum_{j=1}^{n-1} \kappa^{2j}\, b(x \circ y) + \sum_{j=2}^{2n-1} \kappa^j d(x \circ y) + \kappa^{2n}(x\,dy)$$

Substitute in previous page. //∆.

Notice that $d(x \circ y) = d(xy) = dx\,y + y\,dx$.
Put this into the formula:

$$x\,\delta y = -\left(\sum_{j=0}^{n-1} \kappa^{2j}\right) b(x \circ y) + \left(\sum_{j=0}^{2n-1} \kappa^j\right) dx \cdot y$$

$$+\left(\sum_{j=0}^{2n} \kappa^j\right) x\,dy.$$

Esp: $x = 1$:

$$\delta y = -\left(\sum_{j=0}^{n-1} \kappa^{2j}\right) b\,y + \sum_{j=0}^{2n} \kappa^j\, dy$$

$$= \left(-N_{\kappa^2}\, b + B\right) y.$$

Summary: $X(RA) \simeq \Omega A$ as $\mathbb{Z}/2$-gr. v.s.

prd in RA $\quad x \cdot y \longleftrightarrow x \circ y$

$x\,\delta y \longleftrightarrow$ formula above.

$\beta \longleftrightarrow b - (1+\kappa)d : \Omega^- \to \Omega^+$

$\delta \longleftrightarrow -N_{\kappa^2} b + B : \Omega^+ \to \Omega^-$.

So far alg. str. of A isn't active. It enters via the ideal I in RA & its I-adic filt.

Note: β isn't far from $b+B$; $b+sB$ is possible by scaling, for any s. To compensate for forgetting the rest of κ^j, we put N_{κ^2} in δ. Under the spectral decomp., on $P\Omega A$ you see $\beta = b - 2d$, close enough to $\beta + B$.

$$\Omega^1(R/I) = \Omega^1 R / I(\Omega^1 R) + (\Omega^1 R) I + dI$$

$$\Omega^1(R/I)_\natural = \Omega^1 R / [R, \Omega^1 R] + I\,dR + dI$$

$(I\Omega^1 R = I\,dR)$.

$$X(R/I^{n+1}) = \left\{ R/I^{n+1} \qquad \Omega^1 R \big/ [R, \Omega^1 R] + I^{n+1} dR + dI^{n+1} \right\}$$

— a quotient of $X(R)$. $\varprojlim$ is interesting.

Better:

Def $X^{2n+1}(R,I) = \{ R/I^{n+1} \leftrightarrows \Omega^1 R/[R,\Omega^1 R] + I^{n+1} dR + I^n dI \}$

$X^{2n}(R,I) = \{ R/I^{n+1} + [R,I^n] \leftrightarrows \Omega^1 R/[R,\Omega^1 R] + I^n dR \}$

These turn out to be quotient cxs, &

$$X(R/I^{n+1}) \twoheadrightarrow X^{2n+1}(R,I) \twoheadrightarrow X^{2n}(R,I) \twoheadrightarrow X(R/I^n)$$

so they have a common $\varprojlim$ which we write

$$\hat{X}(R,I).$$

Thm. Under $X(RA) \cong \Omega A$,

$$X^n(RA, IA) = \Omega A / F^n \Omega A$$

Sept 11

Check X's are quotient complexes.

$dI^{n+1} \subset \sum I^j \, dI \, I^{n-j} \quad \subset [R, \Omega^1 R] + I^n dI$

$b(I^{n+1} dR) \subset [I^{n+1}, R]$

$b(I^n dI) \subseteq [I^n, I]$

$d[I^n, R] \subseteq [\Omega^1 R, R] + [R, \Omega^1 R]$

Still reviewing,

Next ~~express~~ take $R = RA$, $I = \ker(RA \to A)$.
and use the model $R = \Omega^+ A$,
then $I^n = \bigoplus_{k \geq n} \Omega^{2k} A$

$$\begin{array}{ccc} R & \underset{\delta}{\overset{\beta}{\leftrightarrows}} & \Omega^1 R_\natural \\ \parallel & & \parallel \\ \Omega^+ & \leftrightarrows & \Omega^- \end{array}$$

$\Omega^{\pm}$ has op's b, d, κ, B

$$\beta = d - (1+\kappa) d$$

$|y| = 2n$

$$x\delta y = -\left(\sum_{j=0}^{n-1} \kappa^{2j}\right) b(x \circ y) + \left(\sum_{j=0}^{2n-1} \kappa^j\right) dx\, y + \left(\sum_{j=0}^{2n} \kappa^j\right) x\, dy$$

$$\Rightarrow \quad \delta = \left(-N_{\kappa^2} b + B\right) \qquad \boxed{x\delta a = x\, da}$$

$F^n = b\Omega^{n+1} \oplus \Omega^{n+1} \oplus \cdots \subset \Omega$
is stable under $b, d,$ & so the rest

Lemma. Under our ident. $X(R) = \Omega$, $X^q = \Omega / F^q \Omega$.

pf The subspace $I^n \delta R$ of $\Omega^1 R / [R, \Omega^1 R] = \Omega^-$
is spanned by $x\delta y$ $\quad x, y$ even, $|x| \geq 2n$.
R is gen'd by A so wma $y = a_1 \circ \cdots \circ a_s$.
By the cocycle id.,

$$x\delta(a_1 \circ \cdots \circ a_s) = (a_2 \circ \cdots \circ a_s \circ x)\delta a_1 + \cdots$$

we see that $x\delta y$ is a l.c. of form $x'\delta a$
with $x' \in I^n$, $a \in A$: $\quad I^n \delta R = I^n \delta A$

$$I^n \delta R = I^n \delta A = I^n dA \qquad \text{(by box above)} \Rightarrow$$

② $$I^n \delta R = \bigoplus_{k \geq n} \Omega^{2k+1}$$

① $$I^n = \bigoplus_{k \geq n} \Omega^{2k}$$

$$[I^n, R] + I^{n+1} = \beta(I^n \delta R) + I^{n+1}$$

by 2: $$= (b - (1+\kappa) d) \left(\bigoplus_{k \geq n} \Omega^{2k+1} \right) + \bigoplus_{k \geq n} \Omega^{2k+2}$$

③ $$= b\,\Omega^{2n+1} \oplus \bigoplus_{k \geq n+1} \Omega^{2k}.$$

This is enough to see X^{2n}. Next

$I^n \delta I$: spanned by $x \delta y$, $|x| \geq 2n$, $|y| \geq 2$ ~~$|x| \geq 2n$~~

By cocycle cond: again, with $y = a_0 da_1 \cdots da_{2k}$:

$$x \delta y = \underset{\substack{\cap \\ I^{n+1} \delta R}}{(da_1 \cdots da_{2k}) x \delta a_0} + \underset{\substack{\cap \\ I^n \delta (dA\, dA)}}{(\quad) \delta (da_1 da_2)} + \cdots \quad \cdots$$

Use our formula to compute

$$x \delta (da_1 da_2) = -b(x\, da_1\, da_2) + (1+\kappa)(dx\, da_1\, da_2)$$

$$\Rightarrow \quad I^n \delta I \subseteq I^{n+1} \delta R + b\,\Omega^{2n+2}$$

$$= b\,\Omega^{2n+2} \oplus \Omega^{2n+3} \oplus \Omega^{2n+5} \cdots$$

But in fact / if you look $\pm^n \delta I + I^{n+1}\delta R = b\Omega^{2n+2} \oplus \Omega^{2n+3} \oplus \cdots$

This leads to χ^{2n+1}. Δ.

Next we use the spectral decomp. to ~~see that~~ relate $H(\Omega;\beta,\delta)$ to $H(\Omega;b,d)$:

On Ω^n $(\kappa^n - 1)(\kappa^{n+1}-1) = 0$.

Because of κ^2 occurring in δ, we keep track of eigenvalue -1 as well as $+1$.

$P = P_1$ = spectral proj. at 1

P_{-1} = ——— -1.

$P_{\pm 1} = P_1 + P_{-1}$

$P_\pm^\perp = 1 - P_\pm$, at other roots of 1.

Lemma. Let $c_q : \Omega^q \to \Omega^q$ by $c_{2n} = c_{2n+1} = (-1)^n n!$.

Then on $P_{\pm 1}\Omega$, : $c(\beta+\delta)c^{-1} = b + B$.

pf. $\beta = b - (1+\kappa)d$, $\delta = -N_{\kappa^2} b + B$.

$\kappa^q = 0$ on $b\Omega^q$ & on $d\Omega^{q-1}$.
so the gen. eigenspaces equal eigenspaces there.

so κ is diagonalizable on $P_\pm b\Omega$ & on $dP_\pm\Omega$ with eigenvalues ± 1; so there $\underline{\kappa^2 = 1}$.

So on $P_\pm \Omega$,

$$\beta = b - (1+\kappa) d = b - \frac{1}{n+1} B \qquad \text{on } \Omega^{2n+1}.$$

$$\delta = -N_{\kappa^2} b + B = -nb + B \qquad \text{on } \Omega^{2n}$$

& this is what the scaling accomplishes. $\triangle$.

Lemma. On $P_\pm^\perp \Omega$, β is invertible (so $\delta = 0$).

pf. $(b - (1+\kappa)d)^2 = -(1+\kappa)(bd + db)$

$$= -(1+\kappa)(1-\kappa) = \kappa^2 - 1$$

which is invertible on $P_\pm^\perp \Omega$.

So $b - (1+\kappa) d : P_\pm^\perp \Omega^{-} \longrightarrow P_\pm^\perp \Omega^{+}$

is invertible with inverse $(\kappa^2-1)^{-1}\big(b - (1+\kappa)d\big)$.

17 Oct.

Thm. $cP_\pm : \big(X(RA), \beta + \delta\big) \longrightarrow (\Omega A, b + B)$

is a map of complexes, preserving cofiltrations, &

$$H(X^q(RA, IA)) \xrightarrow{\cong} H(\Omega A / F^q \Omega A)$$

Thus $H(\hat{X}(RA, IA)) \xrightarrow{\cong} H(\hat{\Omega} A)$.

So we can read off

$$H_i(X^q(RA, IA)) = \begin{cases} HC_q\, A & q = i + 2\mathbb{Z} \\ HD_{q-1}\, A & q = i - 1 + 2\mathbb{Z} \end{cases}$$

$$H_i(\hat{X}(RA, JA)) = HP_i\, A .$$

pf. We saw $c : P_\pm X(RA) \xrightarrow{\cong} P_\pm \Omega A$.

This also gives iso on quotient complexes.

On $P_\pm^\perp \Omega$, β is invertible, $\delta = 0$.

$F^q \Omega A$ is stable under all operators, so same holds on quotient. etc $\square$.

$$\boxed{\text{So: we can get at the cyclic homology via the } X\text{-construction of the universal extension}}$$

Commutative case.

Then we have the ordinary diff forms

$$\Omega_A^\bullet : A \xrightarrow{d} \Omega_A^1 \longrightarrow \Omega_A^2 \longrightarrow \cdots ,$$

the univ. commutative DGA gen'd by A.

Ω_A^1 = A-mod of Kahler differentials; d is the universal derivation. (to an A-module, vs. A-bimodule)

Also, $\Omega^1_A = I/I^2$, $I = \ker(A\otimes A \xrightarrow{m} A)$.

m here is an alg. hom, so I is an ideal.

Notice that in fact $I = \Omega^1 A$, the non-com. diff'ls.
In fact

$$\Omega^1_A = (\Omega^1 A)_{G}.$$

as is clear from the univ. property. So then

$$X(A): \quad A \underset{b=0}{\overset{d}{\rightleftarrows}} \Omega^1_A$$

$$\Omega^q_A = E^q_A \Omega^1_A$$

Hochschild, Kostant, Rosenberg:
Suppose A is the alg. of functions on a nonsingular variety (eg $A = \mathbb{C}[x_1, \dots x_n]$, $A = \mathbb{C}[t^{\pm 1}], \dots$)
Then

$$HH_q(A) \cong \Omega^q_A \qquad \text{naturally.}$$

Sketch proof.. Get to local question. There, one has coordinates $x^1, \dots x^n$ st. the assoc. Koszul cx

$$0 \leftarrow A \leftarrow A\otimes A \leftarrow A\otimes \mathbb{C}^n \otimes A \leftarrow A\otimes E^2\mathbb{C}^n \otimes A \leftarrow \cdots$$
$$a_0(x^i\otimes 1 - 1\otimes x^i)a_1 \leftarrow\!\shortmid a_0 \otimes e_i \otimes a_1$$

is exact. Using this bimodule resol. to compute HH.:
In $(\ )_G$, $d = 0$, so you get $A\otimes E^q\mathbb{C}^n \cong \Omega^q_A$.

Ad cyclic homology, in this case: say A smooth.

For any com. A $\quad \mu : \Omega A \longrightarrow \Omega_A$, surj. of DGA's.

$\mu d = d\mu$; $\quad \mu b = 0$; $\quad \mu \kappa = \mu$; $\mu P = \mu$.

But $\mu B = \mu \sum \kappa^i d = N d$,

where $N \big| \Omega^q A = \times q$.

So to get a map of mixed complexes we must rescale:

$c'' : \Omega_A \circlearrowleft$ by $c''_n = \dfrac{1}{n!}$.

Then $c''\mu : \Omega A \longrightarrow \Omega_A$ is a map of mixed complexes

When A is smooth, HKR $\Rightarrow c''\mu$ is a quasi-iso, so the cyclic theories agree: this leads to

$$HC_n(A) = \Omega^n_A / d\Omega^{n-1}_A \oplus H^{n-2}_{dR}(A) \oplus H^{n-4}_{dR}(A) \oplus \cdots$$

$$HP_n(A) = \bigoplus_{i \in n} H^i_{dR}(A).$$

Relate this to X-complex: Elts of structure in X:-

Def For a com. alg A, define the Fedosov alg. R_A to be Ω^+_A with product $x \circ y = xy - dx\, dy$

Then of course $RA \longrightarrow R_A$ as an alg. surjection.

② 1-cocycle: $x\,\delta y = -\left(\qquad \cdots \qquad \right) b(x \circ y)$

$|y|=2n.$ $\quad + \left(\sum_{j=0}^{2n-1} \kappa^j\right) dx\, y + \left(\sum_{j=0}^{2n} \kappa^j\right) x\, dy.$

Apply μ: $\quad x\,\delta y = 2n\, dx \cdot y + (2n+1)\, x\, dy.$

Define pairing $\quad \Omega_A^+ \otimes \Omega_A^+ \longrightarrow \Omega_A^- \qquad dy$

$$x \otimes y \longmapsto x\,\delta y \underset{df}{=} |y|\, d(xy) + x\, dy.$$

This is then a 1-cocycle wrt Fedosov product.
(This construction works for any DG comm. alg.

8 Oct.

If A is fgen as alg, then Ω_A^+ quits, so R_A is a nilpotent extension of A.

$x\,\delta y$ is a 1-cocycle on R_A with values in Ω_A^-.
so we get a canonical surjection

$$(\Omega^1 R_A)_q \longrightarrow \Omega_A^-$$

This fits into a map of $\mathbb{Z}/2$-graded complexes:

$$\begin{tikzcd} R_A \arrow[r, shift right, "\delta"'] \arrow[d, "\cong"'] & (\Omega^1 R_A)_q \arrow[l, shift right, "\beta"'] \arrow[d] \\ \Omega_A^+ \arrow[r, shift right, "Nd"'] & \Omega_A^- \arrow[l, shift right, "-2d"'] \end{tikzcd}$$

$-2d$: $\beta(x\,dy) = [x, y]_\circ$

$$= (xy - dx\,dy) - (yx - dy\,dx) = -2\,dx\,dy.$$

$=$

$$\begin{tikzcd} X(RA) \arrow[r] \arrow[d, equal] & X(R_A) \arrow[d] \\ \Omega A \arrow[r, "\not\mu"] & \Omega_A \end{tikzcd}$$

To get scaling right,

$$c'_{2n} = \frac{(-1)^n}{2^n(2n-1)!!} \qquad c'_{2n+1} = \frac{(-1)^n}{2^n(2n+1)!!}$$

where $(2n-1)!! = 1.3.\cdots.(2n-1)$.

Then $c'\mu : X(RA) \longrightarrow (\Omega_A, d)$

is a map of complexes.

From the first thm of Wednesday, we get cP; later, $c''\mu$:

$$\begin{tikzcd} X(RA) \arrow[rr, "cP"] \arrow[dr, "c'\mu"'] & & (\Omega A, b+B) \arrow[dl, "c''\mu"] \\ & (\Omega_A, d) & \end{tikzcd}$$

commutes.

Thm. Assume A smooth. Then

$$c'\mu : \hat{X}(RA, IA) \longrightarrow (\Omega_A, d)$$

is a quasi-iso of $\mathbb{Z}/2$-graded complexes.

Cor: When A is smooth, every periodic cyclic cohomology class of A is realized by a trace or cyclic 1-cocycle on the nilpotent extension RA of A.

$$- \qquad R^* \underset{d^T}{\overset{b^T}{\rightleftarrows}} (\Omega^1 R)_{\natural}$$

A trace is $\ker b^T$

A cyclic 1-cocycle is a Hochschild 1-cocycle ~~etc~~ in $\ker d^T$:

$$0 = (bf)(x,y,z) = f(xy,z) - f(x,yz) + f(zx,y)$$

$$\Rightarrow 0 = (bf)(1,y,z) = f(y,z) - f(1,yz) + f(z,y)$$

$\hookrightarrow$ if $= 0$, then f is skew-sym.

A trace on a nilpotent extension of A determines an elt. of $HP^0 A$

A cyclic 1-cocycle on a nilp. ext. of A — $HP^1 A$

(Actually we've only proven this for the particular nilpotent extension RA / IA^n. "Homotopy Invariance" will do the general case :)

The Homotopy Property.

The Cartan homotopy property for X :-

$u_t : A \longrightarrow \mathbb{C}[t] \otimes R$: poly. family of hom's.

$u_0 : A \to R$, $\quad u_1 : A \to R$.

Q: is $(u_0)_* \simeq (u_1)_* : X(A) \longrightarrow X(R)$. ?

Ans: No, in general.

For: $\quad X(A) \longrightarrow X(\mathbb{C}[t] \otimes R)$.

X is 1st approx to the cyclic thy.
If R is free then OK, but $\mathbb{C}[t] \otimes R$ isn't free. So

$$X(\mathbb{C}[t]) \otimes X(R) \longrightarrow X(\mathbb{C}[t] \otimes R)$$

isn't q'iso. Rem: $X(\mathbb{C}[t]) = \Omega_{\mathbb{C}[t]}$.

If R is quasi-free you can factor:

$$\begin{tikzcd}
X(A) \arrow[r] \arrow[dr, dashed] & X(\mathbb{C}[t] \otimes R) \\
 & X(\mathbb{C}[t]) \otimes X(R) \arrow[u]
\end{tikzcd}$$

But $X(\mathbb{C}[t]) = \mathbb{C}$, so there's only one map, & this is htpy invariance.

Technique for constructing the lift: <u>connections</u>.

Connections.

Right modules: E a rt A-module then

$$E \otimes_A \Omega A$$: "E-valued forms."
- rt ΩA-module

& $\xi \otimes \omega = (\xi \otimes 1)\omega$ which we'll write $\xi\omega$.

Def (Connes) A connection on E is an operator

$$\nabla : E \longrightarrow E \otimes_A \Omega^1 A$$

st $\nabla(\xi a) = (\nabla\xi)a + \xi da$ $\quad \xi \in E$, $a \in A$.

∇ extends to a deg. 1 op. ∇ on $E \otimes_A \Omega A$ st

$$\nabla(\eta\omega) = (\nabla\eta)\omega + (-1)^{|\eta|}\eta\, d\omega$$ $\quad \eta \in E \otimes \Omega A$, $\omega \in \Omega A$

Ex: ① $E = V \otimes A$. Then $E \otimes_A \Omega A = V \otimes \Omega A$,
and $\nabla = 1 \otimes d$ is a connection.

② Grassmannian connection. E a dir. summand of $V \otimes A$: $E \underset{i}{\overset{p}{\rightleftarrows}} V \otimes A$. Then def. ∇ by

$$\begin{tikzcd} E \otimes_A \Omega A \arrow[r, "i"] \arrow[d, "\nabla"'] & V \otimes \Omega A \arrow[d, "1 \otimes d"] \\ E \otimes_A \Omega A & V \otimes \Omega A \arrow[l, "p"'] \end{tikzcd}$$

A general E is a quotient of $E \otimes A$: $E \otimes A \xrightarrow{m} E$
If $s : E \longrightarrow E \otimes A$ is a section as A-mod's, we get the Grassmannian connection.

Prop. By associating to s the Gr. con. $\nabla = m(1 \otimes d)s$, we get a 1-1 corresp. between A mod sections and connections on E.

Cor. E has a connection $\Longleftrightarrow$ E is projective.

Cor. (Narasimhan-Ramanan) Any connection is a Grassmannian connection.

Pf.
$$0 \longrightarrow \Omega^1 A \xrightarrow{j} A \otimes A \xrightarrow{m} A \longrightarrow 0$$
$$a_0 da_1 \longmapsto a_0 a_1 \otimes 1 - a_0 \otimes a_1$$
split $\Rightarrow$
$$0 \longrightarrow E \otimes_A \Omega^1 A \xrightarrow{j} E \otimes A \xrightarrow{m} E \longrightarrow 0 \quad \text{ex.}$$

As vector spaces this splits: $\xi \otimes 1 \longleftarrow\!\shortmid\, \xi$.
The section of j is: ~~$\ldots$~~

$$-j(\xi da) = \xi \otimes a - \xi a \otimes 1 \longleftarrow\!\shortmid\, \xi \otimes a$$

$$\Rightarrow \quad -j(\xi da) \longleftarrow\!\shortmid\, \xi \otimes a \quad \text{ie} \quad -m(1 \otimes d).$$

Then there is a 1-1 corresp between $\mathbb{C}$-lin. sections s and $\mathbb{C}$-linear maps $\nabla : E \longrightarrow E \otimes_A \Omega^1 A$, given by

$$s\xi = \xi \otimes 1 - j(\nabla \xi)$$

$$\nabla = m(1 \otimes d)s.$$

Now the Leibniz property translates ~~to~~ A-linearity:

$$s(\xi a) - s(\xi)a = \xi a \otimes 1 - \xi \otimes a - (\nabla(\xi a) - (\nabla\xi)a)$$

$$= j(\xi da) - (\nabla(\xi a) + (\nabla\xi)a).$$

21 Sept

Def. A left connection $\nabla_\ell : E \to \Omega^1 A \otimes_A E$ etc.

Def. A connection on a bimodule E is a pair (∇_r, ∇_ℓ) (with no relation)

NB: If E has a connection then it's a projective bimodule:

$$\begin{tikzcd} A \otimes E \otimes A \arrow[r] & E \otimes A \arrow[d] \\ A \otimes E \arrow[r] & E \end{tikzcd}$$

split as bimodules. $\Rightarrow E$ is rt. proj.

split as bimodules, and $A \otimes E$ is projective bimodules.

Case of $E = \Omega^1 A$:

Prop. It's equivalent to give:

1. $\nabla_r : \Omega^1 A \to \Omega^1 A \otimes_A \Omega^1 A = \Omega^2 A$

1'. Bimodule splitting of $0 \to \Omega^2 A \xrightarrow{j} \Omega^1 A \otimes A \to \Omega^1 A \to 0$

2. $\phi : \bar{A} \to \Omega^2 A$ st $-(\delta\phi)(a_1, a_2)\ (= -a_1 \phi a_2 + \phi(a_1 a_2) - (\phi a_1) a_2)$ $= da_1 da_2$

ie $-\delta\phi = d \cup d.$

2' A lifting hom. $A \longrightarrow RA/IA^2$, expressing RA/IA^2 as a split extension of A by $I/I^2 = \Omega^1 A$

1⇒2: $\nabla_r : A \otimes \bar{A} \longrightarrow \Omega^2 A$ rt con. Set $\phi a = \nabla_r(da)$
so $\nabla_r(a_0 da_1) = a_0 \phi a_1$. Then a left A-mod map $\Omega^1 A \to \Omega^2 A$ ⇒ lin. $\phi : \bar{A} \longrightarrow \Omega^2 A$.
Leibnitz rule ⇒:

$$\nabla_r(da_1 a_2) = \nabla_r(d(a_1 a_2) - a_1 da_2) = \phi(a_1 a_2) - a_1 \phi a_2$$
$$\parallel$$
$$\nabla_r(da_1)a_2 + da_1 da_2 = \phi(a_1)a_2 + da_1 da_2 . \quad \Delta$$

2⇒1 This arg. reverses.

2': $RA/IA^2 = A \oplus \Omega^1 A$ with Fedosov product

A lifting hom $u : A \longrightarrow RA/IA^2$ is of form $ua = a - \phi a$
where $\phi : \bar{A} \longrightarrow \Omega^2 A$ satisfies:

$$u(a_1)u(a_2) = (a_1 - \phi a_1) \circ (a_2 - \phi a_2)$$

Now work mod IA^2:

$$= (a_1 a_2 - da_1 da_2) - a_1 \phi a_2 - \phi a_1 a_2$$

vs

$$u(a_1 a_2) = a_1 a_2 - \phi(a_1 a_2)$$

ie $-\delta\phi = d \cup d$. Δ.

Prop. Such exist iff $\Omega^1 A$ is a projective bimodule; ie iff every square-0 alg. extension of A is a semidirect product.

For: 1) $\Omega^1 A \otimes A \otimes = A \otimes \bar{A} \otimes A$ is a free A-bimodule
&
2) RA/IA^2 is the universal square-0 extension of A.

=

Other approaches to the Hochschild theory of algebra extensions:

$\{$Iso classes of square-0 extensions of A by the bimodule $M\}$

$$= H^2(A; M).$$

$$= \mathrm{Ext}^2_{A \otimes A^{op}}(A, M)$$

$$= \mathrm{Ext}^1_{A \otimes A^{op}}(\Omega^1 A, M).$$

using $0 \to \Omega^1 A \to A \otimes A \to A \to 0$

=

Def. A is <u>quasi-free</u> iff it satisfies these equiv. condns: $\Omega^1 A$ is projective; every sq. 0 extension splits.

=

Cartan Homotopy Formula.

R, R' alg's, $u: R \to R'$ alg homs,
$\dot{u}: R \to R'$ a derivation rel. u:

$$\dot{u}(xy) = \dot{u}x \; uy + ux \; \dot{u}y.$$

On Ω: $u_*: \Omega R \longrightarrow \Omega R'$