H Miller
Nov 90

Oxford Seminar on Jones-Witten Theory

Michaelmas Term 1988

Preface

This seminar attempted to present Witten's new approach to the Jones polynomial invariants of knots. For various reasons, including inadequate time, it was not possible to give anything like a complete presentation. What we aimed to do was to present the material in as mathematical a manner as possible at the present time, and this dictated that we should approach the theory from the Hamiltonian point of view. Fortunately this aspect has been well studied for many years by algebraic geometers in relation to the moduli spaces of vector bundles over algebraic curves. This point of view has been emphasized in several of the seminars. However in the two seminars by Graeme Segal use has also been made of the theory of loop groups and ideas from conformal field theory. These are clearly alternative ways of developing the theory and these seminars represent a somewhat uneasy mixture of different versions.

The notes of the seminars are just minor modifications of what was presented. No serious attempt has been made to integrate them or produce a coherent polished account. These notes therefore have limited value and are mainly designed for the audience who attended.

Witten's paper is such a masterly exposition of the whole theory, from the point of view of quantum field theory, that there is little point in attempting an alternative along similar lines. Our aim has been to emphasize the algebraic geometric side. The only really new idea embodied here is described in Nigel Hitchin's seminar. This looks a very promising and important approach which may eventually lead to a much clearer understanding of the Jones-Witten theory and its relation to branched coverings.

The main problem with Witten's theory, from a mathematical point of view, is how to justify the path-integral definition. As briefly mentioned in the sixth seminar a combinatorial approach seems to be called for. This looks a promising avenue for exploration.

The first seminar in which I formulate a set of axioms for topological quantum fields is derived from similar axioms of Graeme Segal for conformal field theory. It is now clear (from more recent work of Witten) that topological quantum field theories are quite numerous, so that the axiomatic framework may be a useful guide.

The initial notes for all the seminars were taken by Ruth Lawrence who did a sterling job in transcribing difficult material.

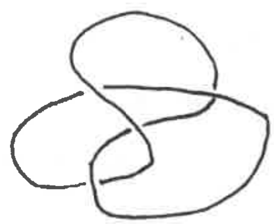
Michael Atiyah
December 1988.

Seminar 1

Professor Atiyah: An introduction to Jones-Witten theory

These seminars will be devoted to the Jones invariants of knots and links and their relation to quantum field theory as developed by Witten.

Here, a knot means a classical knot; that is, a closed non-singular curve in $\mathbb{R}^3$. Such a knot can be pictured by projecting it onto a plane:



A curve with more than one component is called a link. In the theories we shall discuss, we shall deal with links along with knots. Sometimes we need to orient links.

Although knots may be presented in the form of a plane diagram this presentation is highly non-unique. In classifying knots therefore it is extremely useful to have invariants which can help to distinguish inequivalent knots.

One of the most useful invariants, discovered around 1930, was the Alexander polynomial. However, it by no means distinguishes all knots. In fact, the Alexander polynomials of two knots which are mirror images of each other, are equal; whereas there are many knots which differ from their mirror image.

For a link L in three dimensional space, one can consider it as embedded in S^3. The Alexander polynomial is given by looking at the homology of cyclic coverings of $S^3\backslash L$. This shows that the Alexander polynomial essentially extends to

(i) other three dimensional manifolds (not just S^3);

(ii) other dimensions (not just 3).

Around 1984, Vaughan Jones discovered another polynomial invariant of links. The Jones polynomial can distinguish knots from their mirror images, and is thus more powerful than the Alexander polynomial. The interesting question is whether one can now generalise this polynomial as in (i), (ii) above.

In attempts to understand the Jones polynomial, it has come to be related to two-dimensional physics in two different ways:

(a) 2-D statistical mechanics

(b) 2-D conformal field theory.

In this connection, solvable systems and, in particular, the Yang-Baxter equations play a fundamental role in the constructions. From the study of these related areas, a whole series of new polynomials have been found, of which the Jones polynomial is just the first. These series depend on Lie groups; the Jones polynomial corresponds to the Lie group $G = SU(2)$.

A purely combinatorial approach to compute these invariants was also found (c.f. Kauffman). All this leaves many more questions than it answers: why are all these areas connected, and does an intrinsic three-dimensional description exist?

Some good references in this area are:

V.F.R. Jones, Ann. Math. <u>126</u> (1987) 335-388.

W.B.R. Lickorish, Bull. LMS. <u>20</u> (1988) 558-588.

Witten's theory

The breakthrough in understanding some of these connections came in Swansea on July 26th, 1988 at 8 p.m. over dinner!

There are still major problems unsolved in order to provide a suitable mathematical backing to Witten's theory; however, the whole theory fits together with existing ones so nicely that it is impossible to believe that these problems will not be resolved soon.

Witten's theory is an intrinsic 3-dimensional theory and does not require a knot projection for its foundation. Moreover it extends to knots in general 3-manifolds not just S^3. On the other hand, unlike the Alexander polynomial, it does not extend to higher-dimensions. It is specific to 3 dimensions.

The theory uses Lie groups in a natural way: the way physicists use non-abelian groups in gauge theory and quantum field theory. The quantum field theory Witten uses has natural physical antecedents but it still lacks a rigorous mathematical formulation.

Jones' theory produces a polynomial (or finite Laurent series)

$$f(t) = \Sigma a_n t^n$$

for a link. In Witten's version it is the value $f(\exp \frac{2\pi i}{k})$ that is computed for integer values of k.

Topological quantum field theories

Witten's construction produces a whole theory - just as homology theory is more than just the computation of Betti numbers. Witten's theory is really the theory of topological quantum field theories.

We will now try to define what a topological quantum field theory in dimension $d + 1$ means. Here d refers to space dimensions; and 1 refers to time dimension. The main element is a <u>functor</u> Z, called the <u>partition function</u> of the QFT. In the Witten-Jones theory $d = 2$.

Let Σ^d be a closed oriented d-dimensional manifold. The theory assigns to Σ, a complex vector space of finite dimension, $Z(\Sigma)$. This satisfies various axioms:

(0) <u>NATURALITY</u>. This means that an isomorphism $f : \Sigma \to \Sigma'$ induces an isomorphism $Z(f) : Z(\Sigma) \to Z(\Sigma')$ and these compose in the obvious way, i.e. $Z(f_1 f_2) = Z(f_1)Z(f_2)$.

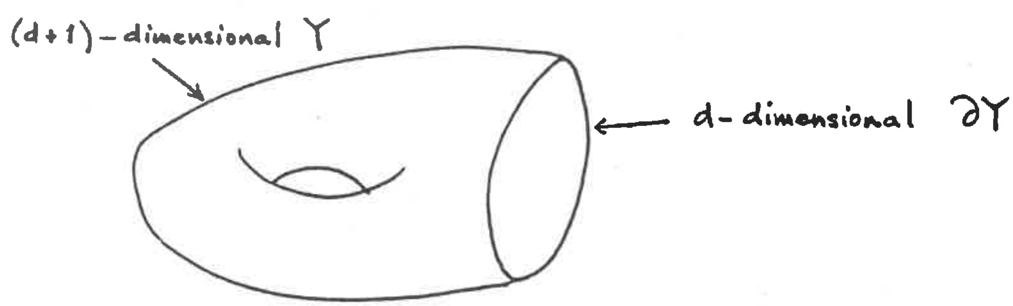
(1) <u>DUALITY</u>. If Σ^* denotes Σ with the opposite orientation then $Z(\Sigma^*) = Z(\Sigma)^*$ the dual space.

(2) <u>MULTIPLICATIVITY</u>. If $\Sigma_1 \cup \Sigma_2$ is the disjoint union of Σ_1 and Σ_2 then $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.

Taking $\Sigma_1 = \Sigma_2 = \phi$ in (2) it follows that $Z(\phi)$ is idempotent so that $Z(\phi) = 0$ or $Z(\phi) = \mathbb{C}$. The first case would imply by (2) that $Z(\Sigma) = 0$ for all Σ. Discarding this trivial case we then have (3) $Z(\phi) = \mathbb{C}$.

Note that property (2) involves a tensor product rather than a direct sum. The significance of this will be discussed later.

The theory also assigns to a $(d+1)$-dimensional compact oriented manifold Y, a vector $Z(Y) \in Z(\partial Y)$; this distinguished element is called the <u>vacuum state</u> corresponding to Y.

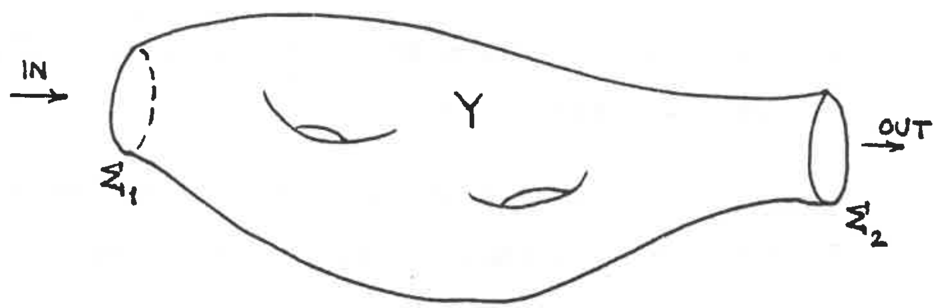


We require $Z(Y)$ to satisfy a naturality axiom as in Axiom (O) .

Notes (i) If Y is a closed $(d+1)$-dimensional manifold so that $\partial Y = \phi$, then $Z(Y) \in Z(\phi) = \mathbb{C}$, i.e. $Z(Y)$ is a complex number. Thus, to each closed $(d+1)$-dimensional oriented manifold, Z associates a numerical invariant.

To start with, we will consider the Witten theory without links: i.e. as a theory on $(d+1)$-dimensional manifolds. Obviously, looking at S^3 without links isn't very interesting! However, the theory of three dimensional manifolds, without links, is roughly like the theory of links in S^3 .

(ii) If ∂Y has 2 components as indicated (so that Y is a cobordism between Σ_1 and Σ_2)



Then:

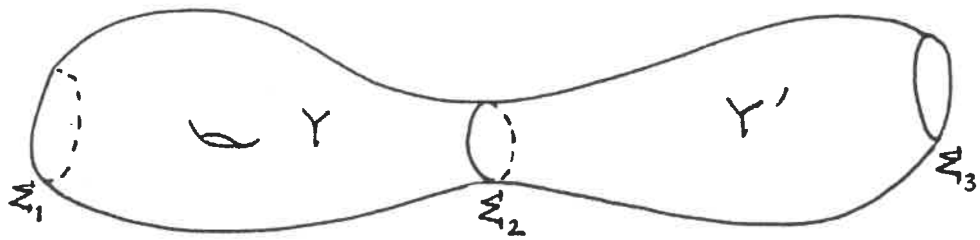
$$Z(\partial Y) = Z(\Sigma_2) \otimes Z(\Sigma_1)^*$$

$$= \mathrm{Hom}(Z(\Sigma_1), Z(\Sigma_2)) .$$

Thus Y specifies a special vector in $Z(\partial Y)$, and thus a linear map from the vector space of Σ_1 to the vector space of Σ_2 .

This suggests another axiom

(4) ASSOCIATIVITY



In the above situation, we require that the composition of the maps:

$$Z(Y) \ : Z(\Sigma_1) \to Z(\Sigma_2)$$

$$Z(Y') \ : Z(\Sigma_2) \to Z(\Sigma_3)$$

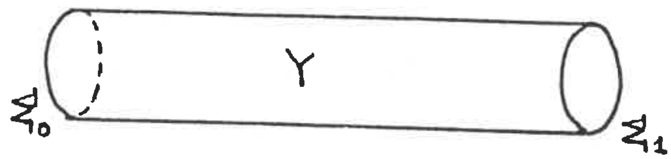
is the distinguished element of $\mathrm{Hom}(Z(\Sigma_1), Z(\Sigma_3))$ corresponding to the cobordism $Y \cup Y'$.

Notes (iii) This axiom can also be formulated as follows. Consider the category C whose objects are closed oriented d-dimensional manifolds and whose morphisms are oriented cobordisms. Then Axiom (4) asserts that Z is a functor from C to the category of finite-dimensional complex vector spaces. However, this point of view is not totally adequate because a $(d+1)$-dimensional manifold Y can be viewed as a morphism in several ways, depending on a decomposition of ∂Y into 2 parts ("incoming" and "outgoing"). The element $Z(\partial Y)$ is the same for all these morphisms. For example, a cylinder $\Sigma \times I$ can be viewed as having IN/OUT boundaries:

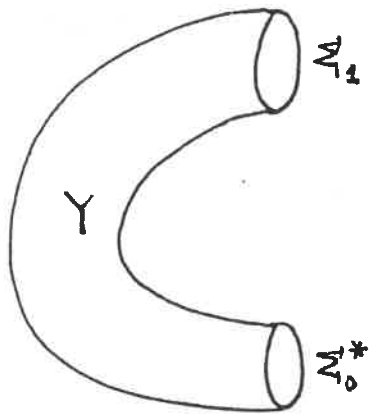
$$\Sigma_0^* \quad \text{and} \quad \Sigma_1$$

or ϕ and $\Sigma_0^* \cup \Sigma_1$

as in the the following diagram.



ϕ



$\Sigma_0^* \cup \Sigma_1$

The element $Z(Y)$ then appears in the two equivalent guises:

$$Z(\Sigma_0) \to Z(\Sigma_1)$$

$$\mathbb{C} = Z(\phi) \to Z(\Sigma_0)^* \otimes Z(\Sigma_1) \; .$$

(iv) The naturality axiom for $Z(Y)$, when applied to the cylinder $Y = \Sigma \times I$ implies a homotopy axiom. More precisely the naturality of $Z(\Sigma)$ implies that the group $\mathrm{Diff}_+(\Sigma)$ of orientation preserving diffeomorphisms of Σ acts on the vector space $Z(\Sigma)$. The naturality of $Z(\Sigma\times I)$ implies that the identity component of $\mathrm{Diff}_+(\Sigma)$ acts trivially on $Z(\Sigma)$. Thus the discrete group $\Gamma(\Sigma)$ of components of $\mathrm{Diff}_+(\Sigma)$ acts on $Z(\Sigma)$.

(v) The associativity axiom implies that the endomorphism of $Z(\Sigma)$ defined by the cylinder $\Sigma \times I$ is an <u>idempotent</u> and more generally that it acts as the identity on the subspace $Z_0(\Sigma) \subset Z(\Sigma)$ spanned by the "vacuum states" $Z(Y)$ as Y runs over all manifolds with $\partial Y = \Sigma$. Only these subspaces contribute to the invariants of closed manifolds so it is reasonable to consider a further axiom:

(5) <u>COMPLETENESS</u>

The vacuum states span $Z(\Sigma)$. As we have just noted this implies that $Z(\Sigma \times I)$ is the <u>identity</u>.

(vi) If Y^* is Y with orientation reversed then we have two vacuum states

$$Z(Y) \in Z(\partial Y)$$

$$Z(Y^*) \in Z(\partial Y)^* .$$

For a general QFT there may be no general relation between these two vectors. However, there is an important class of theories, including the Witten-Jones theory in which there is such a relation. For this we need to assume that the vector spaces $Z(\Sigma)$ have natural identifications with their duals. One possibility is to assume that $Z(\Sigma)$ is a complex (finite-dimensional) Hilbert space so that there is a natural anti-linear isomorphism

$$Z(\Sigma) \to Z(\Sigma)^* .$$

If $\Sigma = \Sigma_1^* \cup \Sigma_2$ this isomorphism just associates to a linear transformation

$$T : Z(\Sigma_1) \to Z(\Sigma_2)$$

its adjoint T^*. We shall therefore use this notation in general. Note that, for $\Sigma = \phi$, T^* is then the complex conjugate of T. We can now impose, as a further axiom,

(6) CONJUGATION

For any oriented (d+1)-dimensional manifold

$$Z(Y^*) = Z(Y)^* .$$

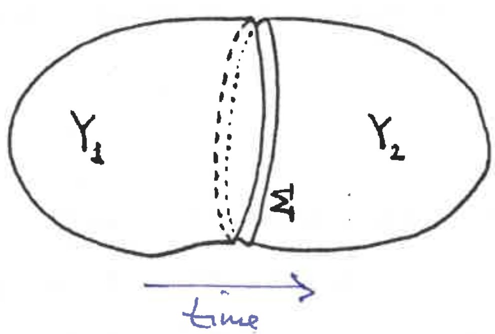
Suppose Y is a closed (d+1)-dimensional manifold. We can cut Y into two parts by a d-dimensional slice Σ:

$$Y = Y_1 \cup Y_2$$

with $\partial Y_1 = \partial Y_2 = \Sigma = Y_1 \cap Y_2$. The slice Σ gives distinguished vectors:

$$v_1 = Z(Y_1) \in Z(\Sigma)$$

$$v_2 = Z(Y_2) \in Z(\Sigma^*)$$



The natural inner product between elements $Z(\Sigma)$ and $Z(\Sigma^*) = Z(\Sigma)^*$ gives:

$$\langle v_1, v_2 \rangle \underset{\boxed{def}}{=} Z(Y) \in \mathbb{C}.$$

the invariant of Y. This mechanism gives a method of calculation of the invariants of closed 3-manifolds, by suitably cutting up the manifold.

Physicists refer to $\langle v_1 | v_2 \rangle$ as a vacuum-to-vacuum expectation value: v_1, v_2 are the vacuum states on Σ corresponding to Y_1, Y_2.

We can modify this story in many different ways. For example, we can put extra conditions on the cobordisms, e.g. by considering framed cobordisms. We can also change the ground ring from $\mathbb{C}$, (the ring over which physicists usually work) to an arbitrary commutative ring. For example, we may consider the theory with ground ring $\mathbb{Z}$. Also we could consider "super-symmetric" theories in which $Z(\Sigma)$ has a mod 2 grading. In all these cases the axioms have to be modified appropriately.

Some examples

How do we know that theories satisfying the above axioms actually exist? The examples below all have physical backgrounds.

(i) $d = 3$

Here lives the Donaldson/Floer theory. This was recently explained by Witten in terms of topological QFT. In the Floer theory, closed three manifolds give rise to Abelian groups, rather than vector spaces as in the above discussion. Donaldson theory assigns integers to four manifolds, rather than complex numbers. Thus this theory is essentially a theory with base ring $\mathbb{Z}$, rather than $\mathbb{C}$. The Donaldson invariants of closed 4-manifolds M and M^* are unrelated so there is no axiom (6).

(ii) d = 2

This is the dimension in which the Witten/Jones theory resides. Recall that Witten produces the values of the Jones polynomial at values $e^{2\pi i/k}$. This is a theory over some algebraic number field very much smaller than $\mathbb{C}$.

(iii) d = 1

There are two theories in this dimension

(a) Conformal field theory. Topological QFTs are so called because the whole structure is set up without reference to a metric, depending only on the topological structures involved. In a similar way, one may consider quantum field theories on Riemann surfaces which depend on the metric only to the extent that they depend on a complex structure on the Riemann surface (a finite number of parameters only is required).

Such quantum field theories are called conformal field theories (CFT). However, the formalism used for such theories is very similar to that used in the $d = 2$ Jones/Witten theory, and hence it is included in this list. The vector spaces are however not finite dimensional. In a CFT we have actions of $\mathrm{Diff}_+ S^1$. (See the seminars by G. Segal).

(b) Gromov/Floer theory. This is similar to Donaldson/Floer theory except that we are working in one dimension. Whereas the base objects in Donaldson/Floer theory are instantons and ASD connections, the basic objects in Gromov/Floer theories are holomorphic maps of curves into complex manifolds. This is connected with the physical theory of σ-models.

(iv) d = 0

Here we are just considering discrete points. To a single point, we associate a vector space V. To a collection of n points, we associate $V^{\otimes n}$. We then have the classical action of the symmetric group S_n. Moreover V itself is usually viewed as a representation of some Lie group e.g. $U(n)$. Thus $d = 0$ encompasses much of classical representation theory.

One can also combine two of these cases. For example, combining $d = 3,1$ is related to four dimensional manifolds Y containing a two dimensional surface X. The boundaries of these spaces give a link in a three dimensional manifold.

Combining $d = 2,0$ leads to the general Witten-Jones theory of links L in a three dimensional manifold Y. Their boundaries give a set of points in $\partial Y = \Sigma$. In the case where Σ is a 2-sphere with a number of points upon it, we obtain an action of the discrete group of components of the diffeomorphism group. This group is the braid group B_n, and hence representations of the braid group are very much in evidence in this theory.

(c) Witten-Jones theory

This is defined on a three manifold Y containing a link L. Put:

$$\partial Y = \Sigma$$

$$\partial L = \{P_1, \ldots, P_n\} .$$

Assume everything here is oriented and framed. Then we have associated with (Y,L) the vector:

$$Z(Y,L) \in Z(\partial Y, \partial L)$$

in the vector space associated with the boundaries $(\partial Y, \partial L)$. In particular, if $\partial Y = \phi$, then $Z(Y,L) \in \mathbb{C}$ is an invariant of the theory, associated to a link in Y.

In this theory, we must fix a compact Lie group G, and an integer k, the level. When we have no links, and are only considering closed manifolds, this is all the data that we require. Given a link, L, with components $L_1, \ldots, L_n$, we must assign to each component an irreducible representaiton λ_i of G: this specifies the $d = 0$ theory that we are "coupling" to the $d = 2$ theory. If we reverse the orientation of a point the representation of G is replaced by its dual.

If we are not considering closed manifolds, then, to each point in ∂L, we must also assign a representation of G. For example, a knot in a closed manifold needs just one representation and Lie group in order to obtain invariants. The Jones polynomial corresponds to the Lie group $SU(2)$ and the standard representation.

We will close by looking at how QFT ideas really enter and why the multiplicative axiom has a tensor product in it, rather than a direct sum.

Start with a closed surface Σ of genus g. Then $H^1(\Sigma, \mathbb{R})$ has dimension $2g$. There is a natural skew form on this, given by:

$$(a,b) \rightarrow \int_\Sigma a \wedge b \, .$$

This makes $H^1(\Sigma,\mathbb{R})$ into a symplectic manifold. Let $H^1(\Sigma,\mathbb{R})$ have a standard symplectic basis:

$$(q_1,\ldots,q_n,p_1,\ldots p_n) .$$

This can be quantised, producing a Hilbert space of the quantised theory:

$$H = L^2(q)$$

$$q \leftrightarrow \text{multiplication}$$

$$p \leftrightarrow \text{differentiation} .$$

Thus taking a Lagrangian subspace of $H^1(\Sigma,\mathbb{R})$, say $\langle q_1,\ldots q_n\rangle$, we consider the q_i to be position coordinates; and $p_1,\ldots p_n$ to be momentum coordinates. Thus we have

Two-dimension Σ	$\rightarrow$	Symplectic vector space $H^1(\Sigma,\mathbb{R})$	$\rightarrow$	Hilbert space H
disjoint union	$\rightarrow$	direct sum $\oplus$	$\rightarrow$	tensor product $\otimes$.

The multiplicative property comes from moving from the classical space $H^1(\Sigma,\mathbb{R})$ to the quantum space H. One can think of elements of $H^1(\Sigma,\mathbb{R})$ as <u>classical fields</u>; and elements of H as <u>quantum fields</u>.

If we changed the Lagrangian subspace of $H^1(\Sigma,\mathbb{R})$ used, from $\langle q_1,\ldots,q_n\rangle$ to $\langle p_1,\ldots,p_n\rangle$, then the Hilbert spaces obtained are isomorphic:

$$L^2(p) \cong L^2(q)$$

the isomorphism being given by Fourier transforms. If we use variables $p_i + iq_i$ instead, then we get another Hilbert space quantisation. However, it is a basic fact of quantum theory that the Hilbert spaces H are canonically isomorphic up to a projective multiplier. That is, the projective Hilbert space is well defined, depending only on the underlying topology of the space Σ.

At this level, we have infinite dimensional Hilbert spaces. However, we can also introduce $H^1(\Sigma,\mathbb{Z})$ which is a lattice inside $H^1(\Sigma,\mathbb{R})$.

Quantizing the torus

$$H^1(\Sigma,U(1)) = H^1(\Sigma,\mathbb{R})/H^1(\Sigma,\mathbb{Z})$$

is formally equivalent to looking at invariant vectors in H. In fact we need to enlarge H to find any fixed vectors. Then we find a one dimensional space given by the classical Θ-function. Alternatively, for each k, one can consider the sub-lattice of $H^1(X,\mathbb{Z})$ of level k. The subspace of invariant vectors under this smaller lattice gives a finite dimensional vector space. Actually these statements need to be refined and the situation will be dealt with more carefully in the next seminar.

For general non-abelian groups, replace $\mathbb{R}/\mathbb{Z} = U(1)$ by any compact Lie group G. Then $H^1(\Sigma,\mathbb{R}/\mathbb{Z})$ is generalized to $H^1(\Sigma,G)$ the space parametrising the classes of representations:

$$\pi_1(\Sigma) \to G\ .$$

This space is still a symplectic manifold (with singularities). When this is quantised we obtain the finite dimensional Hilbert spaces of the theory. There are now definite advantages in this quantisation procedure as we have replaced a non-linear theory by a linear theory:

manifold		vector spaces
(family of reps)	→	(functions on space of reps)
non-linear		linear

If we view the symplectic manifold $H^1(\Sigma,G)$ as a non-linear "homology space", then its quantization may be viewed as a "quantum homology group". The quantum character is exemplified by its multiplicative rather than additive properties.

Seminar 2

Graeme Segal : The Abelian Theory

In this lecture, we shall discuss how the functional Z, discussed in the last seminar, which assigns to each surface Σ a finite dimensional vector space, can be realized concretely. We shall follow the construction in still more detail in the case where the Lie group, required as part of the data for the model, is Abelian. This requires the use of theta functions. In fact a spin structure on the surface Σ is required in order to assign a unique space $Z(\Sigma)$, and the subtleties of this are related to theta characteristics.

We shall also touch on how, for a three-manifold M whose boundary is Σ, one can assign a vector:

$$Z(M) \in Z(\Sigma) .$$

In fact, we shall only describe how to obtain the distinguished ray in $Z(\Sigma)$ given by M, and shall not go into how the normalisation is carried out.

Review of standard theory

Let us begin by reviewing the basic structure of the theory. As given data, we start with a compact Lie group G and an integer $k \in \mathbb{N}$, specifying the level. Given this data, we require

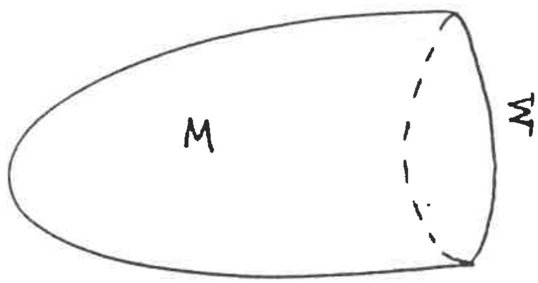
Σ	(2-dimensional surface with spin structure)	$\to$	$Z(\Sigma)$	(finite dimensional vector space)
M	(3-dimensional manifold with $\partial M = \Sigma$)	$\to$	$Z(M)$	(vector in $Z(\Sigma)$)

satisfying certain axioms (see last seminar).

Thus a closed 3-dimensional manifold M has associated with it,

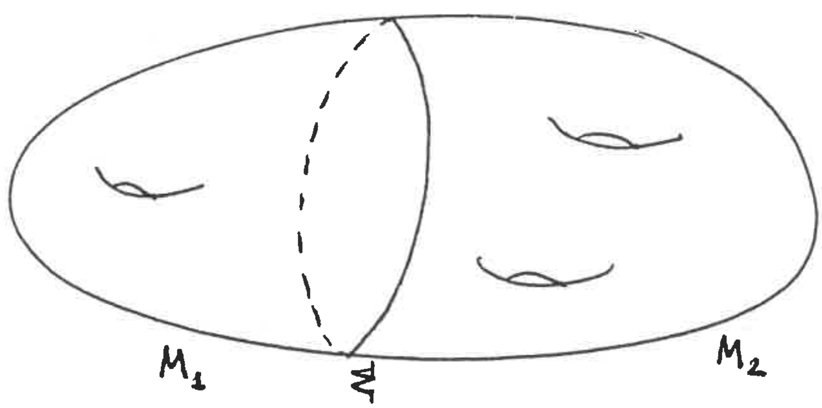
$$Z(M) \in Z(\phi) = \mathbb{C}$$

i.e. a complex number associated to it.



To compute $Z(M)$ for a closed manifold M, one cuts the manifold M up, by a surface Σ, into (say) the open parts M_1, M_2, each with:

$$\partial M_1 = \partial M_2 = \Sigma$$



Then, looking at the left-hand half of the diagram, there is a distinguished vector $Z(M_1)$ in $Z(\Sigma)$ specified by M_1 with $\partial M_1 = \Sigma$.

Similarly, looking at the right-hand half of the diagram, there is a distinguished vector $Z(M_2)$ in:

$$Z(\Sigma) = Z(\Sigma)^*$$

since $\partial M_2 = \Sigma$ with opposite orientation, and one of the axioms for Z states that the vector spaces associated to a surface with its two opposite orientations are dual.

The natural contraction gives:

$$\langle Z(M_1), Z(M_2)\rangle = Z(M) .$$

This enables $Z(M)$ to be evaluated by cutting up the manifold M.

Construction of Z

$Z(M)$ is constructed in Witten's original paper [W] by using path integrals. Although this is not a mathematically rigorous definition, it is nevertheless instructive to investigate this method.

Let $\mathcal{A} = \{G\text{-connections in a trivial } G\text{-bundle on } M\}$. If G is simply connected, the triviality condition is superfluous. Define the gauge group:

$$\begin{aligned} \mathcal{G} &= \{\text{automorphisms of a trivial } G\text{-bundle}\} \\ &= \mathrm{Map}(M,G) . \end{aligned}$$

One can now consider the moduli space $\mathcal{A}/\mathcal{G}$.

On $\mathcal{A}$, one defines the Chern-Simons action as follows. If $A \in \mathcal{A}$, let:

$$S(A) = \frac{1}{4\pi}\int_M \mathrm{tr}(A\,dA + 2/3\,A^3) .$$

In the above expression, A is a 1-form on M, with values in the Lie algebra:

$$A \in \Omega^1(M, g) .$$

If g is a matrix algebra, A is a matrix valued 1-form and so $AdA + 2/3\, A^3$ is a matrix valued 3-form. The trace in the above expression for $S(A)$ is then the natural matrix trace, and $\mathrm{tr}(AdA + 2/3\, A^3)$ is an ordinary 3-form which can be integrated over M.

In general, for a general Lie algebra g, one replaces:

$\mathrm{tr}\, AdA$ by $\langle A, dA\rangle$

$\mathrm{tr}\, A^3$ by $\frac{1}{2}\langle A, [A,A]\rangle$.

We then define

$$\underline{\underline{Z(M) = \int_{A/G} e^{kS(A)} dA}}$$

If $M = \Sigma \times \mathbb{R}$, a connection A on M can be thought of as a path in the space of connections on Σ, for by choosing the appropriate gauge, one can assume that the component of A in the $\mathbb{R}$-direction vanishes

The action Lagrangian has only first order time derivatives and thus the associated Hamiltonian vanishes. This is a situation with which one is unfamiliar in physics.

Let us now define, in a 2-dimensional context,

$$A = \{\text{connections on } \Sigma\} = \Omega^1(\Sigma; g)$$

$$G = \mathrm{Map}(\Sigma, G) .$$

A connection $A \in \mathcal{A}$ is transformed by an element $g \in \mathcal{G}$ according to:

$$(g, A) \to gAg^{-1} - dg \cdot g^{-1} .$$

An infinitesimal gauge transformation is specified by an element:

$$\xi \in \mathrm{Map}(\Sigma; g) = \mathrm{Lie}(\mathcal{G})$$

and its action on $\mathcal{A}$ is given by:

$$(\xi, A) \to [\xi, A] - d\xi = -d_1\xi$$

where d_1 is the covariant derivative with respect to the connection A .

The space $\mathcal{A}/\mathcal{G}$ here is the classical phase space, and not the classical configuration space. The dynamics of the system is completely determined by the symplectic structure on $\mathcal{A}/\mathcal{G}$, since the Hamiltonian vanishes. The Hilbert space we wish to associate to the surface Σ is the quantisation of this classical phase space.

On any symplectic manifold X^{2n} , the symplectic structure provides, for any Lagrangian submanifold $Y^n \subseteq X^{2n}$ with local parameters $q_1, \ldots, q_n$, a conjugate set of functions

$$p_1, \ldots, p_n$$

on a small region of X , so that $q_1, \ldots, q_n, p_1, \ldots, p_n$ provide a local coordinate system for X . Then the 1-form:

$$\sum_i p_i dq_i$$

can be integrated along paths: this corresponds to the Chern-Simons action.

The symplectic structure (for level k) on A is defined by:

$$S(A_1,A_2) = k\int_{\Sigma} tr(A_1 A_2)$$

for $A_1, A_2 \in A$. We could in principle try to quantise this infinite dimensional space A, obtaining a vast Hilbert space H and then select out the subspace (of an enlargement of G) invariant under G.

The alternative is to attempt to quantize the symplectic quotient $A/\!/G$. Let

$$P_\xi : A \to \mathbb{R}$$

be the Hamiltonian function corresponding to the infinitesimal automorphism of A induced by $\xi \in \mathrm{Lie}(G)$. Putting these maps together, we obtain the moment map:

$$P : A \to \mathrm{Lie}(G)^* .$$

In order to be able to include links in this general theory, we will include the possibility of Σ not being closed. This introduces various boundary terms. From the definition of P_ξ it is necessary that:

$$dP_\xi(A,a) = -k\int_{\Sigma} tr(D_A\xi . a) \qquad \text{(i)}$$

for $A \in A$, $\xi \in \mathrm{Lie}(G)$.

However, by integration by parts, this condition can be written as:

$$dP_\xi(A,a) = k\int_{\Sigma} tr(\xi . D_A a) - k\int_{\partial\Sigma} tr(\xi . a) .$$

Hence we can take:

$$P_\xi(A) = k\int_\Sigma \mathrm{tr}(\xi\cdot F_A) - k\int_{\partial\Sigma}\mathrm{tr}(\xi\cdot A) \qquad \text{(ii)}$$

since the curvature of the connection A is given by:

$$F_A = dA + A^2$$

so that $\delta F_A = da + aA + Aa = D_A a$.

Thus (ii) defines the moment map.

However, if $\xi,\eta \in \mathrm{Lie}(G)$, then:

$$\begin{aligned}\{P_\xi,P_\eta\}(A) &= dP_\eta(A,\delta_\xi A)\\ &= -k\int_\Sigma \mathrm{tr}(D_A\eta\cdot D_A\xi) \qquad \text{from (i)}\\ &= -k\int_{\partial\Sigma}\mathrm{tr}(\eta\cdot D_A\xi) + k\int_\Sigma \mathrm{tr}(\eta[F_A,\xi])\end{aligned}$$

and $P_{[\xi,\eta]}(A) = k\int_\Sigma \mathrm{tr}([\xi,\eta]\cdot F_A) - k\int_{\partial\Sigma}\mathrm{tr}([\xi,\eta]\cdot A)$, from (ii) .

Thus

$$\begin{aligned}(P_{[\xi,\eta]} - \{P_\xi,P_\eta\})(A) &= -k\int_{\partial\Sigma}\mathrm{tr}\{[\xi,\eta]A - \eta\cdot D_A\xi\}\\ &= k\int_{\partial\Sigma}\mathrm{tr}(\eta d\xi) \qquad \text{(iii)} .\end{aligned}$$

The right hand side of (*) is a Lie algebra cocycle defining an extension $\tilde{g}_\Sigma$ of the Lie algebra:

$$g_\Sigma = \Omega^0(\partial\Sigma;g)$$

of the gauge group, by $\mathbb{R}$. Thus we have the exact sequence:

$$0 \to \mathbb{R} \to \tilde{g}_\Sigma \to g_\Sigma \to 0$$

and the moment map can be extended to :

$$\tilde{P} : A \to (\tilde{g}_\Sigma)^*$$

$$A \to \{(\lambda,\xi) \to \lambda + P_\xi(A) \quad \text{for } \lambda \in \mathbb{R}\} .$$

The image of A is thus entirely in the hyperplane:

$$1 \times g_\Sigma^* \subseteq \tilde{g}_\Sigma^* .$$

We can now identify:

$$\Omega^2(\Sigma,g) \oplus \Omega^1(\partial\Sigma,g)$$

with a subspace of g^*, and then :

$$\tilde{P}(A) = (1,F_A,A|_{\partial\Sigma}) .$$

The gauge group G acts in a natural way on $F_A,A|_{\partial\Sigma}$, on the right, where we regard $A|_{\partial\Sigma}$ as a connection. The orbits of G on:

$$1 \times \Omega^2(\Sigma,g) \times \Omega^1(\partial\Sigma,g)$$

are in one-to-one correspondence with the conjugacy classes of G :

$$(1,F_A,A|\partial\Sigma) \to (\text{monodromy of } A|\partial\Sigma) .$$

Each G-orbit contains a representative with $A|_{\partial\Sigma}$ constant on each component, when the associated monodromy is $\exp(2\pi A)$. For $A|_{\partial\Sigma}$ constant, the isotropy group consists of $g \in G$ for which $g|_{\partial\Sigma}$ is constant on each component, and:

$$gAg^{-1} = A$$

on these components.

The quotient symplectic manifold A/G obtained from the action of G on A is now given by:

$$\tilde{P}^{-1}(\text{constant})/(\text{isotropy group}) .$$

Then $P^{-1}(1,0,A)$ consists of all those connections which are flat, where A is constant.

If Σ has genus g, $\pi_1(\Sigma)$ has $2g$ generators with one relation. Thus,

$$\mathrm{Hom}(\pi_1\Sigma;G)$$

is the subset of the product of $2g$ copies of G, obtained by imposing the relation. The classical phase space is obtained by dividing out by the action of G, by conjugation, and is thus finite dimensional. In general :

$$\begin{aligned}&\{\text{isomorphism classes of flat bundles}\}\\ &\qquad = \mathrm{Hom}(\pi_1\Sigma;G)/(\text{conjugation})\\ &\qquad = (\text{classical phase space}).\end{aligned}$$

In general we get singularities. However, when G is abelian, there are no singularities.

<u>The case of G abelian</u>

When $G = T$ is abelian, its action on $\mathrm{Hom}(\pi_1\Sigma;G)$ given by conjugation is trivial; and so the phase space is simply:

$$V = H^1(\Sigma;T) .$$

If g denotes the genus of Σ, then:

$$V \cong T^{2g} .$$

One can write this as :

$$H^1(\Sigma;T) = H^1(\Sigma,t)/\Lambda$$

where $t = \text{Lie } T$, $\Lambda = H^1(\Sigma, \pi_1(T))$. So Λ is a lattice in $H^1(\Sigma, t)$. This is because one can identify $t/\pi_1(T)$ with T , using the exponential map. Hence $H^1(\Sigma; T)$ is compact.

The symplectic structure on $H^1(\Sigma, t)$ is given by:

$$H^1(\Sigma, t) \times H^1(\Sigma, t) \to H^2(\Sigma, \mathbb{R}) \to \mathbb{R} .$$

The first part is given by the cup product followed by the inner product on t ; and the second part is:

$$k \times \int_\Sigma$$

where k is the 'level'. This defines a non-degenrate skew form on the vector space $H^1(\Sigma, t)$.

We must now quantise this space $H^1(\Sigma, t)$. In the semi-classical approximation the dimension of the Hilbert space H of the physical system associated to the symplectic structure is:

$$\int_{\substack{\text{classical phase}\\ \text{space}}} \text{(Liouville measure)}$$

where this Liouville measure is $\omega^n/n!$ and ω is the 2-form associated with the symplectic structure on $H^1(\Sigma, t)$.

Since the phase space here is compact, the above integral is finite. The classical phase space has dimension $2g \cdot \dim T$. So $\dim H$ is proportional to :

$$\int_{\text{phase space}} \text{(volume form)} \propto k^{g \dim T}$$

since the symplectic form is proportional to k .

Normalisation of the inner product

A compact Lie group has a 3-form:

$$\langle A_1, [A_2, A_3] \rangle .$$

If G is a simple Lie group (non-abelian), then there is a least normalisation for the inner product which makes the above 3-form into an integral cohomology class; and for SU_n it turns out that this normalisation makes the inner product just the trace. If G is abelian, one requires that $\langle , \rangle$ is integral on $\pi_1 T \subseteq Z$. This corresponds to the case $k = 1$; and other k's correspond to inner products given by k times that inner product corresponding to $k = 1$.

θ-functions

Suppose V is a real symplectic vector space with skew form:

$$S : V \times V \to \mathbb{R} .$$

The Heisenberg group $\tilde{V}$ is defined to be:

$$\mathbb{T} \times V$$

where the group law is defined by:

$$(u_1, v_1)\ (u_2, v_2) = (u_1 u_2 e^{\pi i S(v_1, v_2)}, v_1 + v_2) \qquad \text{(iv)}$$

for $u_1, u_2 \in \mathbb{T}$, $v_1, v_2 \in V$.

Suppose that Λ is a lattice in V on which S is integral. Then $\tilde{\Lambda} = \{\pm 1\} \times \Lambda$ is an abelian subgroup of $\tilde{V}$.

Definition. A splitting of $\tilde{\Lambda} \to \Lambda$ is a map $\varepsilon : \Lambda \to \{\pm 1\}$ specified by a map:

$$\sigma : \Lambda \to \{0,1\}$$

<u>with</u> $\varepsilon = (-1)^{\sigma}$, <u>such that</u>

$$\sigma(v_1+v_2) = \sigma(v_1) + \sigma(v_2) + S(v_1,v_2) \pmod 2 .$$

For any choice of σ, let $\Lambda_\sigma \subseteq \tilde{V}$ be the corresponding subgroup. On the torus:

$$T = V/\Lambda$$

we have the $\mathbb{T}$-bundle, $P_\delta = \tilde{V}/\Lambda_\sigma$. A section of the corresponding complex line bundle:

$$L_\sigma = P_\sigma \times_{\mathbb{T}} \mathbb{C}$$

is a map $F : \tilde{V} \to \mathbb{C}$ such that:

$$\left.\begin{array}{ll} F(u_1u_2,v) = u_1F(u_2,v) & \forall u_1,u_2 \in \mathbb{T} \\ F((u\,,\,v)\cdot(\varepsilon_\lambda,\lambda)) = F(u,v) & \forall \lambda \in \Lambda \end{array}\right\} \text{(v)}$$

where $\varepsilon_\lambda = (-1)^{\sigma(\lambda)}$

Clearly F is defined by $F(1,v)$, if it satisfies (v), since

$$F(u,v) = uF(1,v) .$$

Thus, if $f(v) = F(1,v)$, then a section F of L_σ is equivalent to a map $f : V \to \mathbb{C}$ such that :

$$F(v+\lambda) = \varepsilon_\lambda e^{\pi i S(\lambda,v)} f(v), \quad \forall \lambda \in \Lambda.$$

We now want to put a complex structure on V, and hence T. Write $V_{\mathbb{C}} = W \oplus \overline{W}$ where W is a positive isotropic subspace. Then:

$$T = V/\Lambda \xrightarrow{\cong} \overline{W}\backslash V_{\mathbb{C}}/\Lambda .$$

The $\mathbb{T}$-bundle P_σ is contained in the $\mathbb{C}^\times$-bundle:

$$P_\sigma^{\mathbb{C}} = \overline{W} \backslash \tilde{V}_{\mathbb{C}} / \Lambda_\sigma$$

where

$$\tilde{V}_{\mathbb{C}} = \mathbb{C}^\times \times V_{\mathbb{C}}$$

with group law (i), and $\overline{W}$ denotes the subgroup $1 \times \overline{W} \subseteq \tilde{V}_{\mathbb{C}}$.

Then $L_\sigma = P_\sigma^{\mathbb{C}} \times_{\mathbb{C}^\times} \mathbb{C}$, and a holomorphic section of L is a holomorphic map:

$$F : \tilde{V}_\sigma \to \mathbb{C}$$

satisfying (v), and:

$$F((1,\overline{w})\cdot(u,v)) = F(u,v) \qquad \forall \overline{w} \in \overline{W} .$$

That is, a holomorphic section of L_σ is given by a holomorphic map $F : W \to \mathbb{C}$ such that :

$$f(w+\mu) = \varepsilon_{\mu+\overline{\mu}}\, e^{\pi i S(\overline{\mu},2w+\mu)} f(w) \qquad \text{(vi)}$$

for $\mu \in \Lambda_W = \{\mu \in W ; \mu + \overline{\mu} \in \Lambda\}$. Such functions f are <u>θ-functions</u>; and the choice of splitting ε is called a <u>θ-characteristic</u>.

<u>Example</u>

Take $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$, and define a skew form on V by

$$S(v_1,v_2) = \det(v_1\ v_2)$$

i.e. $S\left(\begin{pmatrix}x_1\\y_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\end{pmatrix}\right) = x_1y_2 - x_2y_1$. A complex structure on V is described by a point λ in the upper half plane in such a way that the corresponding decomposition:

$$V = W \oplus \overline{W}$$

has W spanned by $\xi \in \mathbb{C}^2$, where:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \xi + \overline{\xi}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \tau\xi + \overline{\tau\xi}$$

i.e. $(\xi\overline{\xi})\begin{pmatrix} 1 & \tau \\ 1 & \overline{\tau} \end{pmatrix} = \underline{I}$. Here τ is defined so that Λ is generated by $1, \tau$ over $\mathbb{Z}$.

Define $\tilde{\theta}(z) = f(z\xi)$ for $z \in \mathbb{C}$. Then the condition (vi) , for f to correspond to a holomorphic section of L_σ, is:

$$\left.\begin{aligned} \tilde{\theta}(z+1) &= \varepsilon_1 e^{-\pi i A.(2z+1)}\ \tilde{\theta}(z) \\ \tilde{\theta}(z+\tau) &= \varepsilon_2 e^{-\pi i A.\overline{\tau}(2z+\tau)}\ \tilde{\theta}(z) \end{aligned}\right\}$$

where $A = S(\xi,\overline{\xi}) = (\overline{\tau}-\tau)^{+}$

and ε_1, ε_2 denote:

$$\varepsilon_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}\ ,\ \varepsilon_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

respectively.

It is usual to define:

$$\theta(z) = e^{\pi i A z^2}\tilde{\theta}(z) \ .$$

Then θ satisfies the conditions:

$$\left.\begin{aligned} \theta(z+1) &= \varepsilon_1\theta(z) \\ \theta(z+\tau) &= \varepsilon_2 e^{-2\pi i z-\pi i\tau}\ \theta(z) \end{aligned}\right\} \qquad \text{(vii)}$$

There are four choices for ε_1 and ε_2, and for each choice there is a unique function θ. For example, if $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon$, then (vii) implies that

$$\theta(z) = \sum_{n\in\mathbb{Z}} \varepsilon^n q^{\frac{1}{2}n^2} e^{2\pi i n z}$$

up to a constant multiple, where $q = e^{2\pi i\tau}$.

To obtain a quantisation, we still need to pick a polarisation of V (i.e. a complex structure) so that :

$$V = W \oplus \overline{W} .$$

We think of V as the classical phase space, which is classically given by parameters $p_1,\ldots,p_n$, $q_1,\ldots,q_n$. The skew form on V is given by:

$$S((\underline{p}_1,\underline{q}_1),(\underline{p}_2,\underline{q}_2)) = \underline{p}_1\cdot\underline{q}_2 - \underline{p}_2\cdot\underline{q}_1 \qquad \text{(viii)}$$

where $\underline{p}_1$, $\underline{p}_2$, $\underline{q}_1$, $\underline{q}_2$ are n-component vectors. A quantisation then requires a choice of 'half' the phase space: for example, either the space parametrised by the $p_i (q_i=0)$, or the space parametrised by the $q_i (p_i=0)$. The quantisations obtained from such a choice are all isomorphic, as was mentioned in the last seminar. Thus, for the example of the two subspaces above, the quantisations are:

(a) L^2 functions of $q_1,\ldots,q_n$

(b) L^2 functions of $p_1,\ldots,p_n$

and the isomorphism between them is given by a Fourier transform.

Another way of formulating this discussion is to consider those holomorphic functions $F : \tilde{V}_{\mathbb{C}} \to \mathbb{C}$ such that :

$$F((u_1,\tilde{w})(u_2,v)) = u_1 F(u_2,v)$$

i.e. which are left-$(\mathbb{C}^{\times}\times W)$-equivariant. These form an irreducible representation of $\tilde{V}$, and $\tilde{V}_{\mathbb{C}}$, which acts on:

$$(\mathbb{C}^{\times}\times W)\backslash\tilde{V}_{\mathbb{C}}$$

on the right. This is the Heisenberg representation H; it is the unique irreducible representation of the Heisenberg group $\tilde{V}$ with dimension >1. Then θ-functions form a vector space which from this point of view is the subspace:

$$H^{\Lambda_\sigma} \subseteq H$$

pointwise fixed under the action of $\Lambda_\sigma \subseteq \tilde{V}$.

From any description of the representation H, one obtains a description of H^{Λ_σ}. The classical description of H is as $L^2(Q)$ where:

$$V = Q \oplus Q^*$$

with Q, Q^* isotropic, and skew form defined on V by (viii). Here Q, Q^* denote the subgroups $1 \times Q$, $1 \times Q^*$ of $\tilde{V}$ acting on $L^2(Q)$ by:

$$(1,k) : f(q) \to f(q+k) \qquad \text{for } k \in Q$$

$$(1,\alpha) : f(q) \to e^{2\pi i\alpha q} f(q) \qquad \text{for } \alpha \in Q^*$$

and q denotes the contraction between Q, Q^* given by S.

<u>Example</u>

Consider the case of $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$ again. The action of Λ on $L^2(\mathbb{R})$ is given by:

$$(n,0) : f(q) \to f(q+n)$$

$$(0,n) : f(q) \to e^{2\pi i n q} f(q) .$$

Then $H_\sigma \subset \{\text{functions } \mathbb{R} \to \mathbb{C}\}$ consists of those functions f such that :

$$\left.\begin{aligned} f(t+1) &= \varepsilon_1 f(t) \\ f(t)e^{2\pi i t} &= \varepsilon_2 f(t) \end{aligned}\right\} .$$

When $\varepsilon_2 = 1$, this implies:

$$f(t) = \sum_{n\in\mathbb{Z}} \varepsilon_1^n \delta(t-n)$$

and when $\varepsilon_2 = -1$, it implies :

$$f(t) = \sum_{n\in\mathbb{Z}} \varepsilon_1^n \delta(t-n-\tfrac{1}{2}) .$$

<u>Construction of $Z(\Sigma)$</u>

Suppose Σ is a Riemann surface. Then

$$H^1(\Sigma,\mathbb{C}) \cong \mathbb{C}^{2g}$$

canonically splits into two parts $H^{1,0} \oplus H^{0,1}$, namely holomorphic and anti-holomorphic differentials. These are complex conjugate subspaces. A complex structure L on Σ gives a natural isomorphism:

$$H^1(\Sigma,\mathbb{R}) \cong H^{1,0} .$$

So V/Λ is a complex torus, and we have:

$$\dim_{\mathbb{C}} \tilde{V}_{\mathbb{C}} = g \dim T + 1 .$$

The quantisation of $V = H^1(\Sigma,\mathbb{R})$ is $H_V = L^2(Q)$ where V is split into $Q \oplus Q^*$ by I .

Traditionally, one picks a basis for $H^1(\Sigma,t)$:

$$\alpha_1,\ldots,\alpha_g,\ \beta_1,\ldots,\beta_g$$

such that the intersection form evaluated on α_i,β_j is δ_{ij}. Then one thinks of:

$$Q = \langle\alpha_1,\ldots,\alpha_g\rangle$$

$$Q^* = \langle\beta_1,\ldots,\beta_g\rangle$$

and the θ-functions are given by the Λ-invariant part of H, the quantisation resulting in:

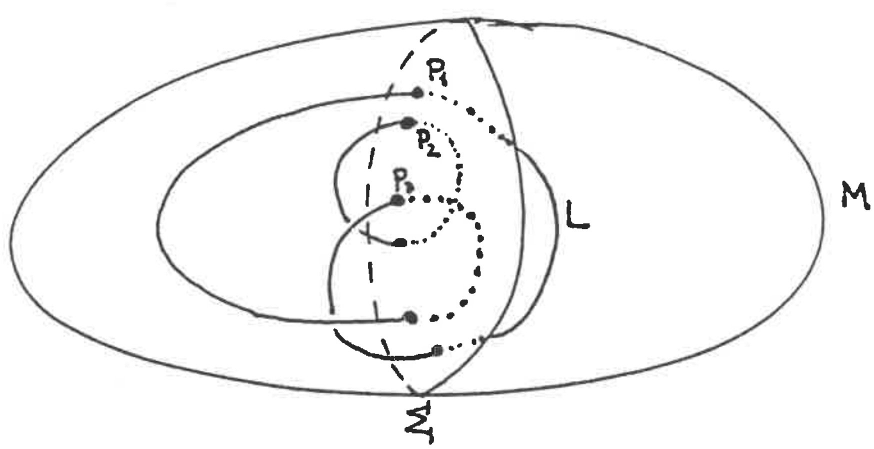
Q acting as translation

P acting as multiplication by $e^{\pi i\beta j}$

on functions on Q .

Relation to knot theory

Suppose $L \subseteq M$ is a knot in a closed 3-manifold M . Cut M into two parts M_1, M_2 by a surface Σ . Then L cuts Σ in a set of points $\{P_i\}$.



On the surface Σ we thus have a set of marked points, and the functor Z is extended to assign to such data a vector space.

The 'phase space' associated to this data increases by the addition of a flag being chosen on each marked point, giving an extra parameter in G/T . Details will be explained in the next seminar.

If M is a 3-manifold with boundary Σ , then we should associate to M a vector $Z(M)$ in $Z(\Sigma)$. Divide the cycles on Σ into α cycles and β cycles by considering the image of the restriction map:

$$H^1(M,t) \to H^1(\Sigma,t) \ .$$

The image is an isotropic subspace of $H^1(\Sigma,t)$ on which the intersection form vanishes: it is a Lagrangian subspace, using the terminology of Hamiltonian mechanics. This can be used to assign a basis of α-cycles and β-cycles, and hence specify a particular θ-function, up to a normalisation factor, in the space of all θ-functions. This is the basic idea used to specify the distinguished ray in $Z(\Sigma)$ given by M . The actual normalisation of $Z(M)$ is more tricky to specify.

Seminar 3

Michael Atiyah : Moduli spaces of vector bundles

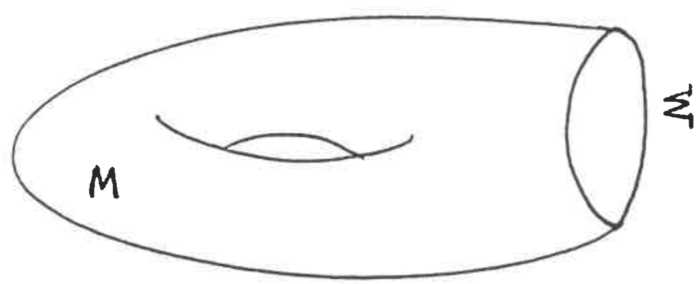
Let us first of all recall what has been covered in the previous two lecturs.

1. We start by fixing a compact Lie group G with an integer k, called the level. We aim to develop a theory (denoted by Z) based on this data. This theory associates:

(a) to each (framed surface Σ, a finite dimensional vector space $Z(\Sigma)$;

(b) to each oriented (framed) 3-manifold with boundary Σ, a vector $Z(M)$ in the vector space $Z(\Sigma)$.

These are required to satisfy certain axioms (see first lecture).

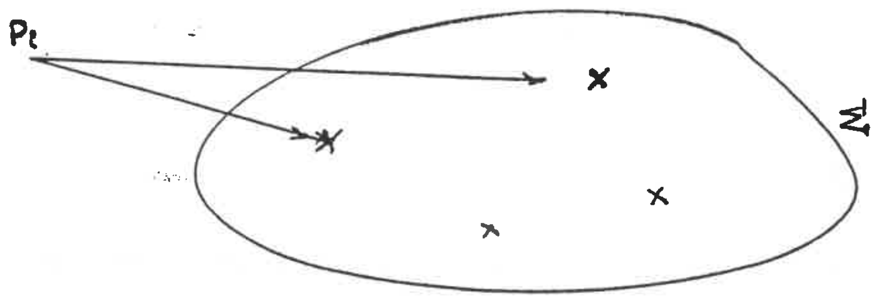


In particular

$$Z(\phi) = \mathbb{C}$$

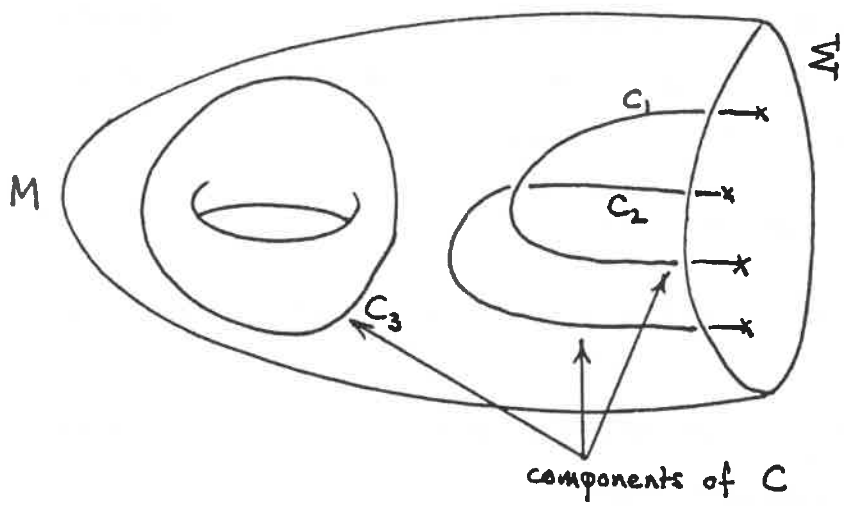
so that for a closed oriented framed 3-manifold M, there is an invariant $Z(M) \in \mathbb{C}$.

This story has a generalization for a surface Σ with a finite set of marked points $P_1, \ldots, P_n$ on Σ.



To each of these points P_i, we associate an irreducible representation, λ_i, of the Lie group G. Having fixed the data $\underline{\lambda}$, $\underline{P}$ (and G, k as before), our functor Z has to produce a vector space $Z(\Sigma,\underline{P},\underline{\lambda})$.

We also have to generalize the assignment of the vector $Z(M) \in Z(\Sigma)$ for every 3-manifold M with $\partial M = \Sigma$, to the case with marked points.



We consider the situation in which M is a 3-manifold containing a curve C. To each component of C we require a representation of G to be assigned giving a collection $\underline{\mu}$. Then, we write:

$$\partial(M,C,\underline{\mu}) = (\Sigma,\underline{P},\underline{\lambda})$$

if

$$\partial M = \Sigma$$

$$\partial \underset{\cdot}{C} = \{P_i\} = C \cap \Sigma$$

while the representations λ_i corresponding to points P_i at the end of a component C_j of the curve C are either μ_j or its dual (depending on the orientation of C_j with respect to the end P_i) .

Under these conditions, we wish to assign a distinguished vector:

$$Z(M,\mathbb{C},\underline{\mu})$$

in the vector space $Z(\Sigma,\underline{P},\underline{\lambda})$, satisfying appropriate properties.

The special case of a closed 3-manifold generalises here to the situation in which $C \subseteq M$ is a knot or link. There are no points P_i , so that:

$$\partial(M,C) = (\phi,\phi)$$

and thus $Z(M,C) \in Z(\phi,\phi) = \mathbb{C}$ giving an invariant of knots and links.

In all the above consistent framings are needed, as we shall indicate later.

2. Now we shall review the last seminar explaining how this construction works in the abelian case : that is, we look at $G = U(1)$. To a surface Σ , we can associate $H^1(\Sigma,U(1))$. If Σ is a surface of genus g we associate:

$$H^1(\Sigma,U(1)) = H^1(\Sigma,\mathbb{R})/H^1(\Sigma,\mathbb{Z})$$

a torus of dimension $2g$. This space can be thought of as the space of abelian representations of the fundamental group:

$$\mathrm{Hom}(H_1(\Sigma), U(1)) .$$

So, to each surface we have associated a torus of higher dimension.

Recall that a symplectic structure on a space of dimension $2n$ is a closed 2-form ω with $\omega^n \neq 0$. On a torus, we can define a simple symplectic structure which is invariant under translation. Such a structure is defined by giving a skew form on the tangent space to the torus at one point, using translation to define the structure at other points. Since the tangent space looks like:

$$H^1(\Sigma, \mathbb{R}) ,$$

one can define the symplectic structure by using the skew form:

$$H^1(\Sigma, \mathbb{R}) \otimes H^1(\Sigma, \mathbb{R}) \to H^2(\Sigma, \mathbb{R}) = \mathbb{R}$$

by taking the cup product. Since this procedure is compatible with $\mathbb{Z}$, thus ω is an integral class. Hence there exists a line bundle L over the torus with a connection, whose curvature is ω, up to factors of 2π.

A closed integral 2-form on a simply connected space defines a line bundle uniquely up to isomorphism. If the space is not simply connected then the line bundle is arbitrary up to tensoring by flat line bundles. Of course the torus is far from simply connected, and thus the line bundle is not unique : we shall make some more comments on this below.

In the case of a general level k, we replace the line bundle L by $L^{\otimes k}$ to obtain the more general vector spaces $Z(\Sigma)$.

Quantisation

We have shown how to associate a higher dimensional torus with a line bundle over it, to a surface Σ. Next we must show how to quantise this to obtain a vector space.

In order to carry out the quantisation, we need some extra data, namely a complex structure σ on Σ. This complex structure induces a complex structure on $H^1(\Sigma, U(1))$. When endowed with a complex structure, $H^1(\Sigma, U(1))$ is called the Jacobian $\mathrm{Jac}(\Sigma_\sigma)$ of Σ_σ.

One can directly define:

$$\mathrm{Jac}(\Sigma_\sigma) = (\text{isomorphism classes of holomorphic line bundles of degree zero on } \Sigma_\sigma)$$

as a complex torus, of complex dimension g.

This abelian case is basic to the general case. So we will look at this in more detail first.

Sheaf theory

We have two exact sequences:

$$\begin{tikzcd}
 & H^2(\Sigma_\sigma, \mathbb{Z}) \\
H^1(\Sigma, U(1)) \arrow[r] & H^1(\Sigma_\sigma, \mathcal{O}^*) \arrow[u] \\
H^1(\Sigma, \mathbb{R}) \arrow[u] \arrow[r] & H^1(\Sigma_\sigma, \mathcal{O}) \arrow[u] \\
H^1(\Sigma, \mathbb{Z}) \arrow[u] \arrow[r] & H^1(\Sigma_\sigma, \mathbb{Z}) \arrow[u]
\end{tikzcd} \qquad (*)$$

with a natural map between them. Here, the right hand side has:

$\mathcal{O}$ = sheaf of holomorphic functions on Σ_σ

$\mathcal{O}^*$ = sheaf of non-zero holomorphic functions on Σ_σ

and the map:

$$H^1(\Sigma_\sigma,\mathcal{O}) \to H^1(\Sigma_\sigma,\mathcal{O}^*)$$

is given by $f \to e^{2\pi i t}$.

From these sequences it easily follows that there is a natural isomorphism

$$H^1(\Sigma,U(1)) \to \mathrm{Jac}(\Sigma_\sigma) .$$

We would now like to define

$$Z(\Sigma_\sigma) = H^0(\mathrm{Jac}(\Sigma_\sigma),L^k)$$

the space of holomorphic sections of the holomorphic line-bundle L^k on the Jacobian. As we mentioned earlier, although the curvature of L is fixed by the symplectic structure, this does not determine L uniquely because the Jacobian is not simply connected. The solution to this problem is to shift from the Jacobian to the copy of J_{g-1} which parametrizes line-bundles of degree $g - 1$. This shift can be obtained by tensoring line-bundles of degree zero on Σ by a fixed square root $K^{\frac{1}{2}}$ of the canonical bundle. This is equivalent to a spin structure. On J_{g-1} there is a natural choice for L corresponding to the distinguished Θ-divisor on J_{g-1} (representing effective divisors of degree $g - 1$). Finally therefore we can define $Z(\Sigma_\sigma)$.

All this was explained in greater detail in the last seminar where it was shown that for $k = 1$ the space $Z(\Sigma_\sigma)$ has dimension 1 and is generated by a classical Θ-function. Also it was shown that for $k > 1$ the space $Z(\Sigma_\sigma)$ has dimension k^g and is spanned by Θ-functions of level k.

The key proposition in this theory relates the vector spaces $Z(\Sigma_\sigma)$ when we vary the complex structure σ on Σ. Recall that a choice of complex structure was necessary to facilitate the quantisation procedure; but we would like the result $Z(\Sigma_\sigma)$ to depend only on Σ, and not the particular choice of σ.

Fix a basis of cycles for Σ and then integrate a basis of holomorphic forms around these cycles. This produces a $2g \times g$ dimensional matrix, where g is the genus of Σ, called the period matrix. So we can think of σ as defining an element:

$$\sigma \in H$$

of the Siegel upper half plane, defined analogously to the complex upper half plane by:

$$H = \{\text{complex symmetric matrices } A + iB \text{ where } B \text{ is positive definite}\}.$$

Just as the complex upper half plane is acted upon by the symplectic group, so similarly H is acted upon by the symplectic group $Sp(2g,\mathbb{R})$ with isotropy group $U(g)$. Hence:

$$H = Sp(2g,\mathbb{R})/U(g)$$

is a homogeneous subgroup of $Sp(2g,\mathbb{R})$.

The period matrix determines the complex structure σ of the Riemann surface and by varying σ we get the Teichmuller subspace T of H. A general point of H arises from an

Abelian variety which is not the Jacobian of a curve. However the definition of $Z(\Sigma_\sigma)$ extends to all Abelian varieties and so can be defined for all points of H. The key fact about this construction can be stated as

<u>Proposition</u>. <u>There is a holomorphic vector bundle</u>:

$$\begin{array}{c} Z \\ \downarrow \\ H \end{array}$$

<u>the fibre over</u> $\sigma \in H$ <u>being the space</u> $Z(\Sigma_\sigma)$. <u>There is a natural connection on this bundle whose curvature is a scalar matrix</u>.

The connection enables one to parallel transport around a path in the base H to nearby fibres. If this connection had zero curvature, then one could identify all the fibres, and we would have obtained an intrinsic quantisation $Z(\Sigma)$ independent of σ. However, one actually finds that the curvature is not zero, but a scalar, so that only the projective spaces can be naturally identified.

The curvature is in fact a scalar multiple of the distinguished 2-form on H given by the Kähler metric. The value of the scalar multiple can be computed and this is needed for the detailed development.

We shall now go on to discuss the <u>non-abelian case</u>. We think of k as large. This will not be a great restriction, since the end product is supposed to be an invariant polynomial evaluated at $e^{2\pi i/k}$; and thus the polynomial can be determined from knowledge for large k only.

So now we start with a compact Lie group G, and consider, instead of the torus of the abelian theory, the symplectic manifold (with singularities) $H^1(\Sigma,G)$. Essentially, the singularities correspond to representations which are reducible. Of course, when $G = U(1)$, all representations are one-dimensional and thus irreducible, so there are no singularities.

When $SU(2) = G$, reducible representations reduce to $U(1)$, so that the singular points correspond to points on the space corresponding to $U(1)$.

We must now show how to define a symplectic structure on this manifold by giving a skew-symmetric form on the tangent spaces to $H^1(\Sigma,G)$. However, if:

$$\alpha \in \mathrm{Rep}(\pi_1(\Sigma),G) = H^1(\Sigma,G)$$

is a point on the manifold, then the tangent space at α is given by

$$H^1(\Sigma,\mathrm{Ad}_\alpha(G)) .$$

This is the first homology with respect to a flat bundle describing a twisted coefficient system on Σ: the coefficients being the Lie algebra.

The skew form is defined as before:

$$H^1(\Sigma,\mathrm{Ad}_\alpha(G)) \times H^1(\Sigma,\mathrm{Ad}_\alpha(G)) \to H^2(\Sigma,\mathbb{R}) \to \mathbb{R}$$

where the first map is given by the cup product, and we contract on the Lie algebra by using a suitable fixed inner product on the Lie group .

Thus we have a symplectic manifold associated to Σ, and since the above form is integral, as before, it defines a line bundle on the space.

<u>Notes:</u> (1) Here we have ignored the singularities, but in fact that can be treated properly and the line-bundle extends over them (2) For a semi-simple group (with no abelian factor) $H^1(\Sigma,G)$ is simply-connected so we do not have the spin-structure trouble of the abelian case.

Just as in the abelian case, once we fix a complex structure on Σ, we can identify $H^1(\Sigma,G)$ with the moduli space of stable holomorphic bundles over Σ_σ with group G^c (the complexification of G). This is the theorem of Narasimhan and Seshadri which has been reproved by Donaldson more in the spirit of our presentation.

Once again, we proceed as we did in the abelian case, putting:

$$\underline{\underline{Z_\sigma = H^0(M_\sigma, L^k)}} \quad .$$

We now get a counterpart of the key proposition:

<u>Proposition.</u> <u>There is a holomorphic vector bundle</u>

$$\begin{array}{c} Z \\ \downarrow \\ \mathcal{T} \end{array}$$

<u>where</u> $\mathcal{T}$ <u>is Teichmuller space, the fibre over</u> $\sigma \in \mathcal{T}$ <u>being</u> $Z(\Sigma_\sigma)$. <u>There is a natural connection on this bundle whose curvature is a scalar multiple of the natural Kähler metric.</u>

The rest of the discussion that held in the abelian case holds now, with H replaced by τ.

There are alternative approaches to this proposition:

A. Use the theory of Riemann surfaces with boundary. This leads to the theory of representations of loop groups, conformal field theory.

B. Reduction to the abelian case (see Seminar 5 by Nigel Hitchin).

We have now described (roughly) how $Z(\Sigma)$ is defined. The other part of the functor Z associates to a 3-manifold M with boundary Σ, a distinguished vector:

$$Z(M) \in Z(\Sigma) .$$

In our approach this might be done as follows. There is a natural restriction map:

$$H^1(M,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z})$$

whose image is a Lagrangian subspace of $H^1(\Sigma,\mathbb{Z})$. This should be used to define a ray in $Z(\Sigma)$: the actual multiple is tied up with the structure of the projective multipliers and is more subtle.

Generalisation with marked points

When we try to generalise to the case of a Riemann surface Σ with marked points P_i and associated representations λ_i, the two halves of the theory generalise as follows:

1. Unitary theory. We must look at representations of $\pi_1(\Sigma)$ taking account of the points P_i. The correct generalisation is to consider representations:

$$\pi_1(\Sigma\setminus\{P_1,\ldots,P_n\}) \to G$$

whose monodromy around the P_i's lie in given conjugacy classes of elements of order k in G. (The monodromy around

P_i is the image of a small loop around P_i under the above map.) The conjugacy classes in G are labelled by the representations: this is how the data of λ_i at the P_i enters this half of the story (see later for details of this link).

2. Holomorphic theory. We must fix a complex structure σ on the Riemann surface Σ, together with the marked points P_i and associate representations λ_i. One considers holomorphic bundles on Σ, with given data at the points P_i, dependent on λ_i/k. There is then an appropriate moduli space M_σ.

A recent theorem of Seshadri states that M_σ can be identified with the space of representations obtained from 1.

The theory obtained is similar to the basic case of Σ without marked points.

To understand the labelling of representations of compact Lie groups and conjugacy classes of the group, recall the first lecture, in which the different theories were neatly fitted into the different dimensions 1, 2, 3, 4. Recall that the theory said to correspond to dimension zero was that of group representations. Let us review the basic facts.

The Borel-Weil theorem

Let G be a compact Lie group and T a maximal torus. Then G/T is called the flag manifold. It can also be expressed as a homogeneous space of the complexification G^C:

$$G/T = G^C/B$$

where B is a Borel subgroup. When $G = U(n)$, $G^C = GL(n,C)$ and B consists of upper triangular matrices. The representation theory of G (or G^c) can be understood in terms of line-bundles on the flag manifold.

A unitary character $\lambda : T \to U(1)$ extends to a holomorphic character $B \to \mathbb{C}^*$ and so defines a homogeneous holomorphic line-bundle L_λ on G^c/B. Then G (or G^c) acts on the space V_λ of holomorphic sections of L_λ. For suitably "positive" λ this space is non-zero, and the Borel-Weil theorem that these V_λ describe all the irreducible representations of G.

The line-bundle L_λ has a natural G-invariant connection where curvature is (for generic λ) a symplectic form on G/T. The moment map

$$\mu_\lambda : G/T \to g^*$$

then identifies G/T as a co-adjoint orbit. The space V_λ is the <u>quantization</u> of G/T with the symplectic structure defined by λ.

Using an invariant quadratic form we can identify the Lie algebra of G with its dual, so that the representation V_λ is associated to the Lie algebra conjugacy class (or G-orbit) which is the image of μ_λ.

<u>Unitary group case</u>. When $G = U(n)$, $G^c = GL(n,\mathbb{C})$ and g consists of skew-Hermitian matrices. A character $U(n) \to U(1)$ is given by a vector $\lambda = (\lambda_1,\dots,\lambda_n)$ of integers. The associated class of skew-Hermitian matrices has (up to a factor 2π) eigenvalues $(i\lambda_1,\dots,i\lambda_n)$.

Let us return now to consider (with $G = U(n)$) a surface Σ with marked points $P_1,\dots,P_r$ and associated integral vectors $\underline{\lambda}_1,\dots,\underline{\lambda}_r$. The relevant moduli space (for level k) now consists of classes of representations

$$\pi_1(\Sigma - \{P_1,\ldots,P_r\}) \to U(n)$$

with the property that the monodromy round each point P has eigenvalues

$$\exp \frac{2\pi i\lambda_1}{k}, \ldots \exp \frac{2\pi i\lambda_n}{k} \quad .$$

(We have for simplicity omitted here the index i of P_i and the corresponding additional index for the λ) . Note in particular that such monodromy matrices are of order k . This means that, passing to a k-fold branched cover of Σ , we get a flat bundle over the unpunctured surface.

The algebraic geometric approach to these moduli spaces is to consider holomorphic vector bundles over Σ_σ with a distinguished flag at each marked point P_i . The moduli space is then constructed by factoring out by flag-preserving isomorphisms. Roughly speaking this means that our new moduli space M'_σ (using marked points) is fibered over the ordinary moduli space M_σ with fibre a product of flag manifolds (one copy for each marked point). The large moduli space M'_σ will then have a line-bundle (and associated symplectic structure) which partly comes from the original line-bundle on M_σ and partly from the line-bundles L_λ on the flag manifolds.

The details are of course more intricate than this rough outline indicates. In particular care has to be taken with the notion of stability.

The Hilbert space of the quantum theory, for a surface Σ with marked points, is obtained by quantizing the base moduli space M'_σ . Again the key proposition will concern the variation of this with σ having only a scalar curvature.

Much of this theory has been developed differently in work on conformal field theory, using the techniques of representations of loop groups and the Virasoro algebra. This approach, indicated in Graeme Segal's seminars (2) and (4), involves cutting out small holes around the marked points and considering the resulting surface with boundary. The approach sketched above, in the framework of algebraic geometry, has been developed (for quite independent reasons), by Seshadri, who uses the term "vector bundle with parabolic structure" to indicate the data at the marked points. The most natural approach (with a view to Witten's theory) is probably to follow the Atiyah-Bott-Donaldson path to moduli spaces. This links the differential geometry naturally to the algebraic geometry.

In algebraic geometry think of the surface Σ as an algebraic curve, over which we have a vector bundle, fibre $\mathbb{C}^n$: the flag manifold is the space of flags i.e. sets of spaces (V_i) with $V_1 \subseteq V_2 \subseteq \dots \subseteq V_n = \mathbb{C}^n$ and:

$$\dim(V_j) = j$$

for $j = 1,2,\dots,n$. At each point P_i, we fix a flag.

The moduli space M_σ , of vector bundles on Σ is defined by dividing out by the action of isomorphisms between bundles. If we change the notion of isomorphism by requiring that not only the vector bundles correspond, but also the flags at each P_i , then we obtain a larger moduli space M'_σ , since we are dividing it by a much smaller group than for M_σ .

Seminar 4

Graeme Segal : Fusion rules and the Verlinde algebra

Start with a compact Lie group G and an integer k, the level. The aim is to produce a functor Z, associating to a Riemann surface Σ, a finite dimensional vector space $Z(\Sigma)$ (see earlier seminars). We also require such an association to exist when we are given marked points

$$x_1, \ldots, x_\ell \in \Sigma$$

on the surface Σ, each of which is labelled with an irreducible representation, V_i, of G.

Once the level k is fixed, only finitely many labels come into play. Representations of G are classified by their highest weights. Thus, if $V_1, \ldots V_\ell$ have highest weights $\lambda_1, \ldots, \lambda_\ell$ then the labels that are relevant are precisely those for which:

$$\|\lambda_i\|^2 \leq 2k \quad \text{for} \quad i = 1, 2, \ldots, \ell .$$

Here $\| \ \|$ comes from the assumed norm on the Lie algebra corresponding to G. If G is simple, $\| \ \|$ is normalised so that:

$$\|\alpha\|^2 = 2$$

when α is a short root.

Thus the functor Z maps:

$$\left\{\begin{matrix} \text{Data of } \Sigma, \text{ points } x_i \\ \text{with representations } V_i \end{matrix}\right\} \to Z\left(\Sigma \begin{matrix} V_1, \ldots, V_\ell \\ x_1, \ldots, x_\ell \end{matrix}\right)$$

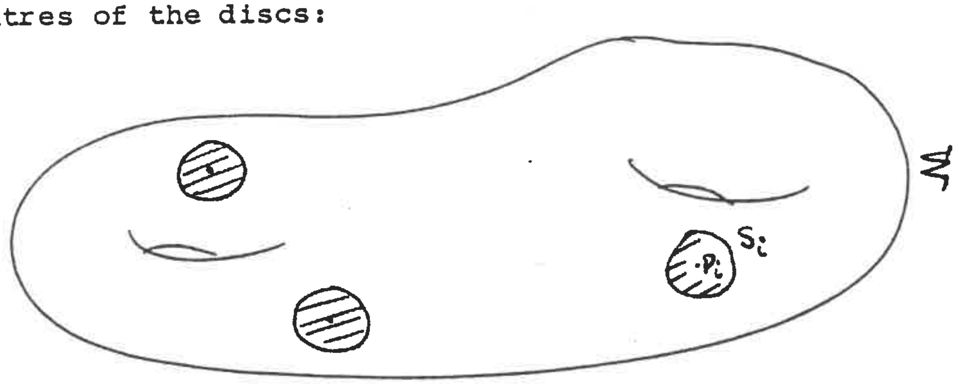
Using only the above data, the vector space $Z(\Sigma)$ is defined only as a complex projective space. To define the vector space absolutely, it is necessary that more structure is given on Σ : a spin structure or framing. In this lecture, we shall only look at this projective space, and suppress the framing.

Instead of considering the Riemann surface Σ with marked points, an alternative point of view is to consider taking out a finite number of discs, giving:

$$\partial\Sigma = S_1 \amalg S_2 \ldots \amalg S_\ell$$

a disjoint union of circles S_i. We shall assume that each component S_i has a given parametrization, oriented corresponding to the orientation of Σ (as a boundary component). The data associated with Σ now reduces to an irreducible representation of G for each component of the boundary $\partial\Sigma$.

Given such a Σ together with the associated data, one can glue standard discs into the holes in Σ, and this gives a Riemann surface with marked points corresponding to the centres of the discs:



Conversely, starting from a Riemann surface with marked points, near to each marked point one chooses a local parameter, and then one defines discs on Σ . Thus these two viewpoints are equivalent.

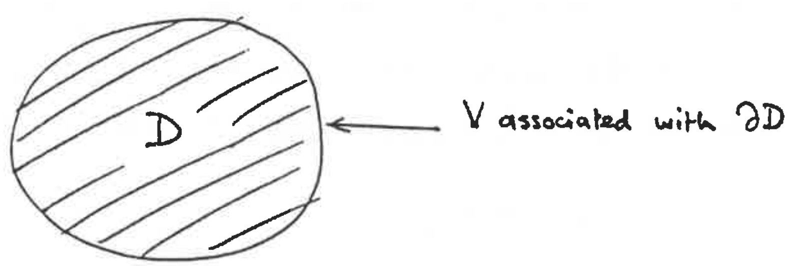
In what follows, we shall use this alternative viewpoint, with $(\Sigma,\partial\Sigma)$ where $\partial\Sigma$ is a disjoint union of parametrised circles. The parametrisation gives a preferred method for gluing boundaries together.

Axioms

The functor Z is required to satisfy the following axioms (see [S1]) .

1. $Z(D,V) = \begin{cases} \mathbb{C} & \text{if } V \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$

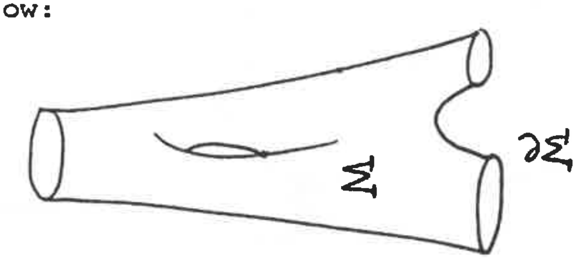
This gives the vector space associate to a disc, where V is the representation given on the boundary circle:



2. $Z(\overline{\Sigma},\overline{V}_i) = Z(\Sigma, V_i)^*$

Here $\overline{\Sigma}$ refers to the surface Σ with the opposite orientation; and this axiom states that the vector space changes to the dual when the orientation of the surface is changed.

One can picture a general surface of the required form as shown below:



The remaining two axioms 3) and 4) specify how Σ behaves under the operations of surgery. Any surgery can be expressed in terms of simple operations, either gluing two surfaces Σ_1, Σ_2 and identifying their boundaries, or identifying two components of the boundary of some surface Σ. To cover all possibilities, one of the axioms gives Z for a disjoint union of two surfaces; and the other axiom specifies the behaviour of Z on gluing two components of the boundary of a surface Σ.

3. $\underline{Z(\Sigma_1 \amalg \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)}$

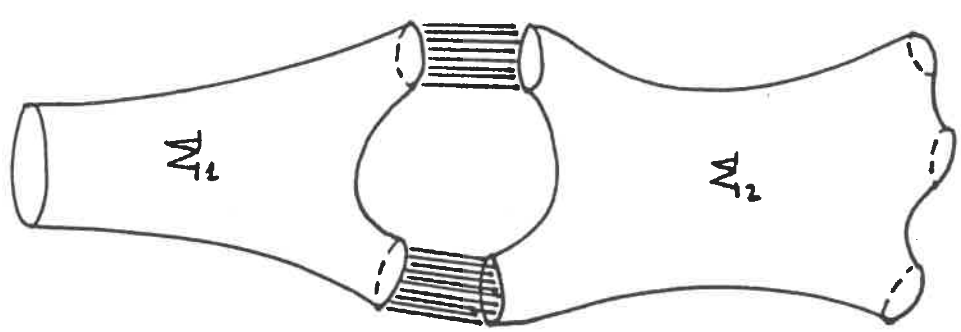
Here $\Sigma_1 \amalg \Sigma_2$ denotes the disjoint union of Σ_1, Σ_2, and the labels used on the boundary of $\Sigma_1 \amalg \Sigma_2$ are those induced from the labels on the boundaries of Σ_1, Σ_2 used in the right had side.

4. If $v : \Sigma \to \Sigma$ denotes the map which sews together pairs of boundary compoments (circles), using the given parametrisations on $\partial\Sigma$ to defining the sewing, then the natural map:

$$\bigoplus_{v} Z(\Sigma_v) \to Z(\check{\Sigma})$$

is an isomorphism. Here Σ_v is obtained from $\check{\Sigma}$ by cutting along a simple closed curve, and giving the label V to the two new edges.

Using 3), 4) we can see how Z behaves when we glue two surfaces Σ_1, Σ_2 joining some of the boundary components.



Boundary components are glued with parameters θ, $-\theta$ on the two components being corresponded. We need a map:

$$Z(\Sigma_1, V_1^{(i)}\text{'s}) \otimes Z(\Sigma_2, V_2^{(j)}\text{'s}) \to Z(\Sigma_1 \cup \Sigma_2, V\text{'s})$$

which is defined when the boundary components which are identified have dual labels. The remaining data on the boundaries for Σ_1, Σ_2 combine to give data on $\Sigma_1 \cup \Sigma_2$.

The map is given by axioms 3) and 4) via the space:

$$Z(\Sigma_1 \amalg \Sigma_2, V_1^{(i)}\text{'s}, V_2^{(j)}\text{'s})$$

i.e. we have:

$$\begin{array}{ccc} Z(\Sigma_1, V_1^{(i)}) \otimes Z(\Sigma_2, V_2^{(j)}) & & \\ & \searrow \text{axiom 3)} & \\ & & Z(\Sigma_1 \amalg \Sigma_2, V_1^{(i)}, V_2^{(j)}) \\ & \swarrow \text{axiom 4)} & \\ Z(\Sigma_1 \cup \Sigma_2, V) & & \end{array}$$

An alternative method for considering these two axioms was suggested by Michael Atiyah. Instead of considering:

$$Z\begin{Bmatrix} \Sigma_{x_1,\ldots,x_\ell}^{V_1,\ldots V_\ell} \end{Bmatrix}$$

one can define

$$Z(\Sigma, x_1, \ldots, x_\ell) = \bigoplus_{V_1,\ldots,V_\ell} Z\begin{Bmatrix} \Sigma_{x_1,\ldots,x_\ell}^{V_1,\ldots,V_\ell} \end{Bmatrix} V_1 \otimes \ldots \otimes V_\ell.$$

This is a representation of G^ℓ, the tensor product of ℓ copies of G. In this notation,

$$\Sigma \to \check{\Sigma}$$

ℓ boundary components $\to$ $(\ell-2)$ boundary components and axiom 4) above states that:

$$Z(\Sigma)^G = Z(\check{\Sigma})$$

where G acts diagonally. This is equivalent to axiom 4), since by Schur's lemma,

$$(V_1 \otimes V_2)^G = \begin{cases} \mathbb{C} & \text{if } V_1 \cong V_2^* \\ 0 & \text{otherwise} \end{cases} .$$

Consequences of the axioms

1) In order to obtain examples of functors, we begin by defining vector spaces $Z(\Sigma)$ associated to Riemann surfaces, i.e. surfaces equipped with a complex (or, equivalently, conformal) structure. We then require $Z(\Sigma)$ to have the following additional property:

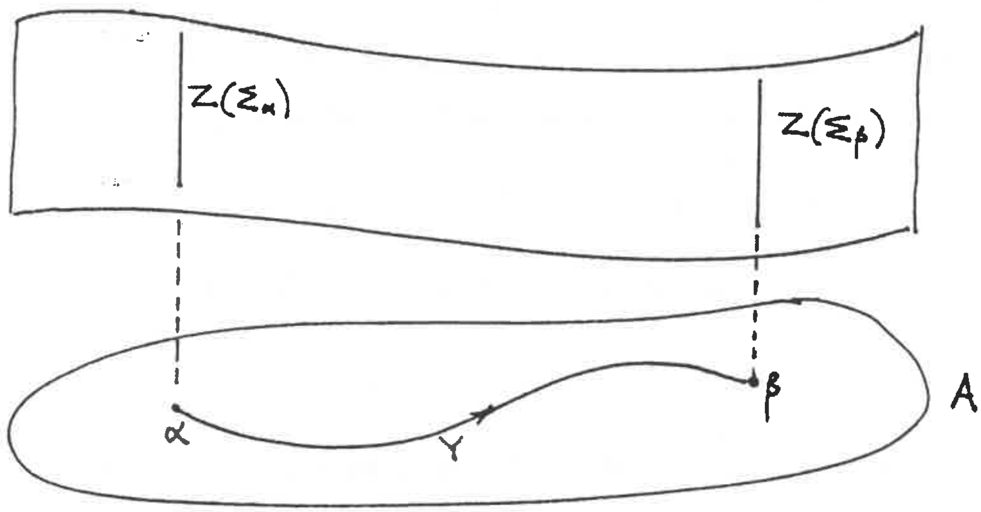
> "If $\{\Sigma_\alpha\}_{\alpha\in A}$ is a holomorphic family of Riemann surfaces parametrised by a complex manifold A, then $\{Z(\Sigma_\alpha)\}_{\alpha\in A}$ is a holomorphic vector bundle over A."

One can then prove the following result:

Theorem. There exists a canonical projective flat connection on $\{Z(\Sigma_\alpha)\}_{\alpha\in A}$ where $\{\Sigma_\alpha\}_{\alpha\in A}$ is a holomorphic family of Riemann surfaces parametrised by a complex manifold A.

That is, to each path Y from α to β in the parameter space, A, there is defined a map:

$$Y_* : Z(\Sigma_\alpha) \to Z(\Sigma_\beta)$$



The connection being projective means that this map is determined by γ up to phase only. The connection being flat means that the map depends only on the homotopy class of the path γ joining α to β.

Thus the given data gives a projective space which is dependent only on the surface Σ, as a smooth manifold, and not on any other data. That is, we have $Z(\Sigma,g)$ defined for each given complex structure g. If $\tilde{g}$ is another complex structure, then an isomorphism

$$Z(\Sigma,g) \xrightarrow{\cong} Z(\Sigma,\tilde{g}) \qquad (*)$$

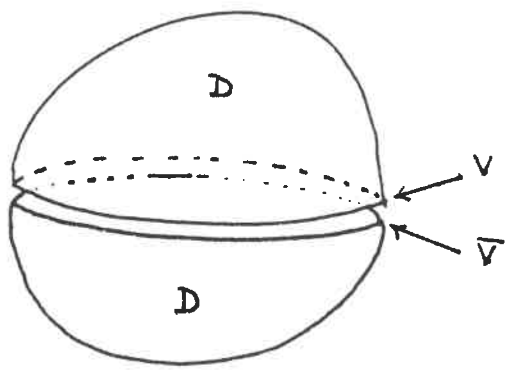
as projective spaces, is obtained by choosing a path from g to $\tilde{g}$ in the space of complex structures on Σ. Since the space of complex structures is contractible, there is no monodromy.

If an isomorphism (*) existed as vector spaces, and not only as projective spaces, then we would have a canonical vector space independent of additional structure. Even though (*) is only an isomorphism between projective spaces, we can still pin the isomorphism down up to an indeterminate root of unity. This is determined by a choice of framing.

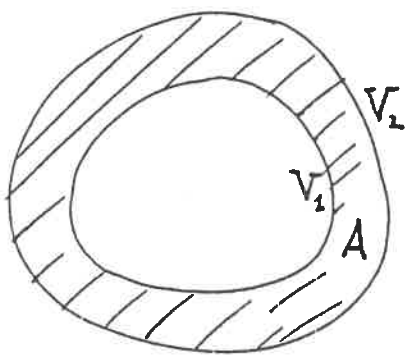
2) The sphere S^2 can be obtained by glueing together two discs, D, along their common boundary. Then:

$$
\begin{aligned}
Z(S^2) &\cong \bigoplus_V Z(D \amalg D, V, \bar{V}) && \text{by axiom 4)} \\
&= \bigoplus_V Z(D,V) \otimes Z(D,\bar{V}) && \text{by axiom 3)} \\
&= \mathbb{C} && \text{by axiom 1)}
\end{aligned}
$$

since the only non-trivial term in the sum comes from the trivial representation V.



3) Consider the annulus:



Then the data required is two representations, one for each component in ∂A. We shall now show that:

$$
Z(A; V_1, V_2) \cong \begin{cases} \mathbb{C} & \text{if } V_1^* \cong V_2 \\ 0 & \text{if } V_1^* \not\cong V_2 \end{cases}
$$

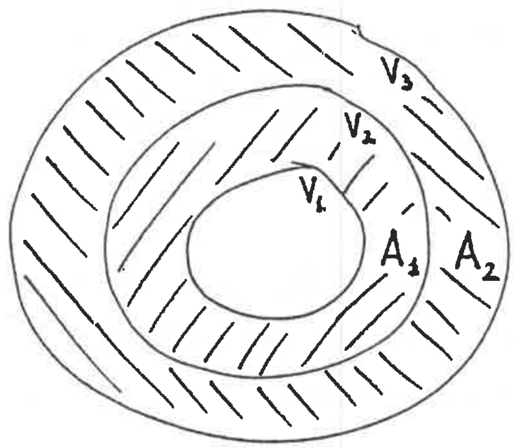
although the isomorphism is not canonical.

Define $n_{V_1,V_2} = \dim Z(A,V_1^*,V_2)$ when V_1, V_2 are two representations of G. If we glue two annuli inside each other as shown below, then by axiom 4),

$$Z(A,V_1^*,V_3) = \bigoplus_{V_2} Z(A \amalg A, V_1^*, V_2, V_2^*, V_3)$$

$$= \bigoplus_{V_2} Z(A,V_1^*,V_2) \otimes Z(A,V_2^*,V_3) \quad \text{by axiom 3)} .$$

Thus $n_{V_1,V_3} = \sum_{V_2} n_{V_1V_2}\, n_{V_2V_3}$.



So the matrix n is a matrix of positive (or, at least, non-negative) integers which is idempotent. Hence n is essentially the identity matrix, with maybe a block of zeros. Any such block of zeros corresponds to labels V such that:

$$Z(\Sigma_x^V) = 0$$

for all Σ_x . Disregarding this uninteresting part, n is the identity, proving the statement.

4) Consider the torus T . This can be obtained from an annulus A , by glueing together the two boundary circles. Thus

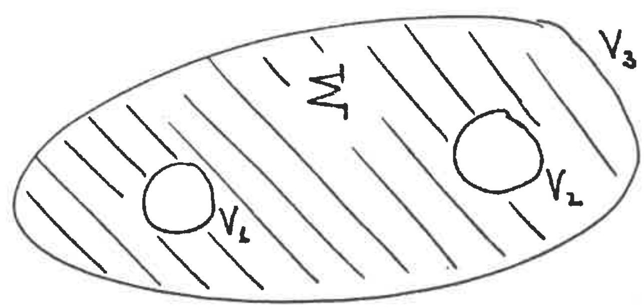
$$Z(T) = \bigoplus_V Z(A, V^*, V) \qquad \text{by axiom 4)}$$

$$= \bigoplus_V \mathbb{C}_{[V]} \qquad \text{by above example}$$

and so dim $Z(\text{torus})$ = number of possible labels.

Verlinde algebra

Consider the surface



Let $n_{V_1V_2V_3} = \dim Z(\Sigma, V_1, V_2, V_3)$ where Σ is the disc with two holes cut out, and the representations associated with the three components of $\partial\Sigma$ are V_1, V_2, V_3^* as shown above.

Define $R = \bigoplus_V \mathbb{Z}\varepsilon_V$ where ε_V is a formal symbol. This gives a free abelian group, R, on the labels V. Make R into a ring by imposing the multiplication law:

$$\varepsilon_{V_1}\varepsilon_{V_2} = \sum_{V_3} n_{V_1V_2V_3}\, \varepsilon_{V_3} \tag{i}$$

This makes R into a ring. R is commutative since we can flip the holes around, up to an isomorphism of surfaces. Associativity of R is proved so long as:

$$(\varepsilon_{V_1}\varepsilon_{V_2})\varepsilon_{V_3} = \varepsilon_{V_1}(\varepsilon_{V_2}\varepsilon_{V_3}) \tag{ii}$$

for all representations V_1, V_2, V_3 of the group G.

Now

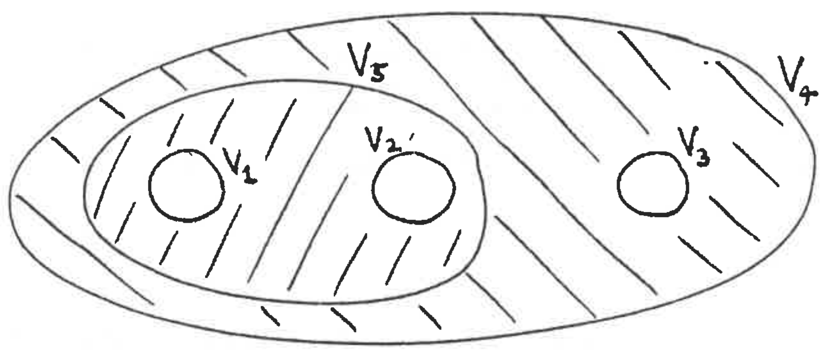
$$(\varepsilon_{V_1}\ \varepsilon_{V_2})\varepsilon_{V_3} = \sum_{V_5} n_{V_1V_2V_5}\varepsilon_{V_5}\varepsilon_{V_3}$$

$$= \sum_{V_5}\sum_{V_4} (n_{V_1V_2V_5}\ n_{V_5V_3V_4})\varepsilon_{V_4}\ .$$

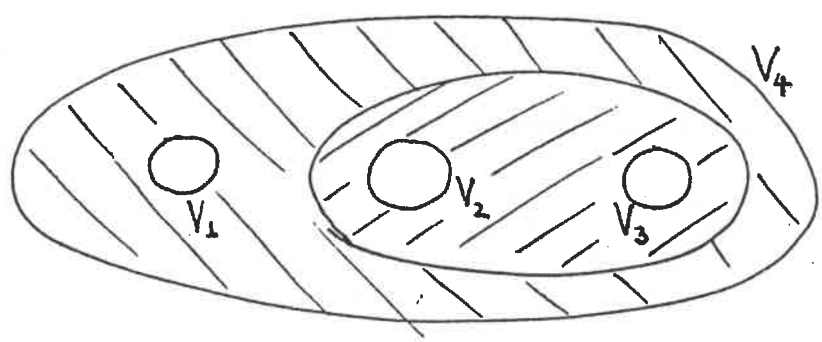
The coefficient of ε_{V_4} in the left hand side of (i) is given by the dimension of the space

Z(disc with three holes, V_1, V_2, V_3, V_4)

due to the cutting:

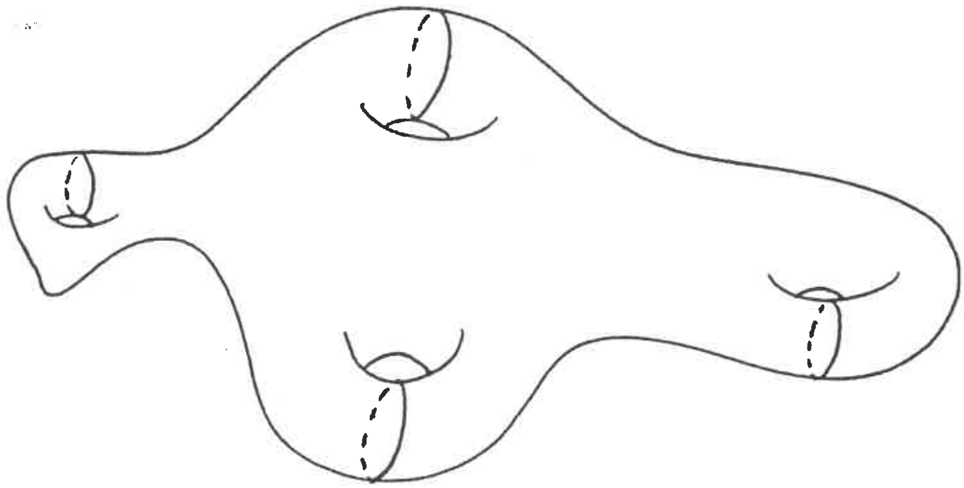


This is the same as is obtained by cutting the disc with three holes in a different way:



This gives rise to the coefficient of ε_{V_4} in the right hand side of (ii), and hence we have associativity.

Suppose now that the the ring R was known. Then one could easily calculate dim $Z(\Sigma)$ for any Riemann surface Σ, with arbitrary holes (marked points), by suitably cutting into elements with three boundary components.



R is known as the Verlinde algebra (see [V]) associated to the compact Lie group G, and level k. As k increases, more and more representations come into play, and in fact:

$$R(k) \text{ '}\to\text{' } R(G)$$

as $k \to \infty$, where $R(G)$ is the free abelian group on the set of representations of G, that is, the representation ring of G. Here '$\to$' means that the coefficients $n_{V_1V_2V_3}$ in (i), considered as functions of k, have :

$$n_{V_1V_2V_3}(k) = \text{some constant } n^{(\infty)}_{V_1V_2V_3}$$

for sufficiently large k, and that $n^{(\infty)}_{V_1V_2V_3}$ is given by:

$$V_1 \otimes V_2 = \sum_{V_3} n^{(\infty)}_{V_1V_2V_3} V_3 .$$

For $G = SU(2)$, the structure of R can be quite precisely determined.

Associated to the group G and closed surface Σ, consider:

$$M_1 = \{\text{homomorphisms } \pi_1(\Sigma) \to G\}/\text{conjugation}$$

where we are dividing out by the action of G given by conjugation. This is equivalent to the space of isomorphism classes of flat principal G-bundles on Σ. The Narasimhan-Seshadri theorem shows that the moduli spaces M_1, M_2 are isomorphic, where:

M_2 = moduli space of holomorphic principal $G_{\mathbb{C}}$-bundles on Σ_σ

in the sense of algebraic geometry.

To define M_2, it is necessary to put a complex structure σ on Σ. M_1 is defined without using a complex structure, and has a natural symplectic structure.

Over the moduli space M of holomorphic vector bundles there is a natural line bundle:

$$L = \det(\bar{\partial}) \quad .$$

We then define $Z(\Sigma)$ as the space of the holomorphic sections of $L^{\otimes k}$. [Note that if G is not $SU(n)$ or $U(n)$, $\det(\bar{\partial})$ is not the basic line bundle we require; L must be defined as a root of $\det(\bar{\partial})$, determined by the Coxeter number of G.]

As pointed out in the previous seminar we should think of the correspondence getween M_1 and M_2 as analogous to the Borel-Weil theorem in the representation theory of compact Lie groups. In the latter situation, we have the identification

$$G/T \cong G_{\mathbb{C}}/B$$

where T is a maximal torus in G; and B is a Borel subgroup in the complexified group $G_{\mathbb{C}}$. Here G/T is a symplectic manifold, with a G-invariant integral symplectic form for each weight λ, i.e. for each character:

$\lambda : T \to \mathbb{T}$

and $G_{\mathbb{C}}/B$ has a Kähler structure.

In the case of $G = U(n)$, the complexified group is:

$G_{\mathbb{C}} = GL(n,\mathbb{C})$

and one can use:

$T = \{\text{diagonal matrices}\}$;

$B = \{\text{upper triangular matrices}\}$.

The symplectic form on G/T is the first Chern class of the line bundle:

$L = G \times_T \mathbb{C}$

where T acts on $\mathbb{C}$, by $\lambda : T \to \mathbb{T}$. The character extends to a holomorphic homomorphism:

$\lambda : B \to \mathbb{C}^*$.

So L can be regarded as a holomorphic bundle $G_{\mathbb{C}} \times_B \mathbb{C}$. Holomorphic sections of L then give representation spaces of G .

Going back to the case of Σ with marked points, that is:

$\partial\Sigma = S_1 \amalg S_2 \ldots \amalg S_\ell$,

any connection on Σ will give rise to a monodromy on going around the boundary circles. The algebraic structure considered, as the analogue of M_1 , consists of pairs:

(flat G-bundle, element of $\mathfrak{g}$ for each pole S_i)

such that the monodromy around S_i is conjugate to:

$\exp(2\pi i\xi_i)$

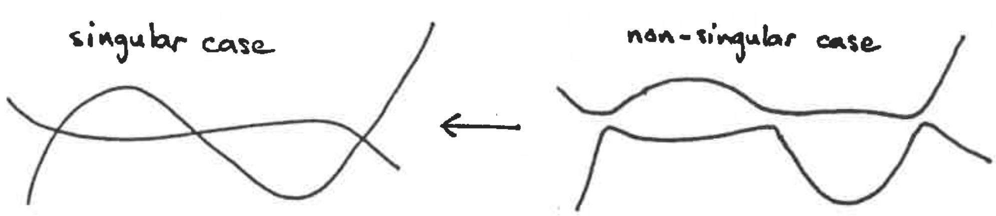
where ξ_i are fixed elements of the Lie algebra. So we fix the conjugacy class of the monodromy of the flat connection, on going around the components of $\partial\Sigma$.

The corresponding holomorphic structure (the analogue of M_2) in the case of marked points, is given by holomorphic G-bundles equipped with a reduction of the structural group from $G_{\mathbb{C}}$ to B at each marked point. For example, if $G_{\mathbb{C}} = GL_n(\mathbb{C})$, an element of M is given by a holomorphic G-bundle together with a choice of a flag in the fibre of the vector bundle at each marked point.

There are two very different approaches here:

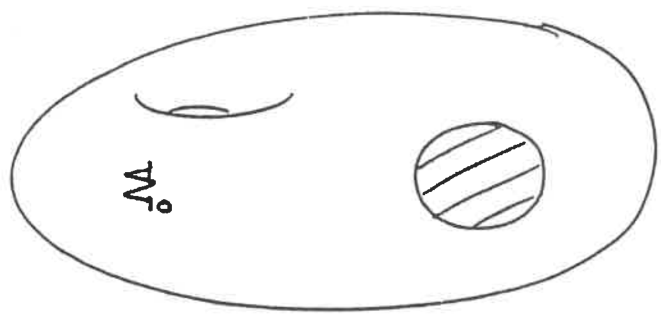
1) 2 - D representation theory, which involves representations of loop groups, and the structure of $\mathrm{Diff}(S^1)$ (G. Segal's approach);

2) theory of D-modules, as developed by Tsuchiya & Yamada (see [Y]) in which one considers a Riemann surface as an algebraic curve with marked points. Glueing two such spaces together along a boundary, is thought of in this interpretation, as considering the algebraic curve given by the union of the two other algebraic curves with the marked points corresponding to the identified boundaries, identified. This algebraic curve has singularities at the identified marked points, and can be deformed into a non-singular curve, so that as the singularity is approached, a cycle is collapsed to a point.



The technique used is to carefully consider what happens during the deformation. The essential point, equivalent to the axioms (3), (4) given here is that over spaces Z_σ extend to the Mumford compactification of the moduli space of curves.

For the rest of this lecture we will restrict ourselves to 1). Consider a curve on Σ which cuts Σ into two parts, Σ_0 and a disc D.



Any holomorphic bundle on Σ is trivial on Σ_0 and on D, since any non-compact Riemann surface is a Stein manifold. Hence, any holomorphic bundle on Σ can be produced by glueing trivial bundles on Σ_0, D by an attaching map along the boundary. This is specified by an element of the loop group $LG_{\mathbb{C}}$.

Let $G_\Sigma = \mathrm{Hol}(\Sigma_0, G_{\mathbb{C}})$. Then clearly multiplication of the attaching map by an element of G_{Σ_0} on one side, or by an element of G_D on the other side will not change the isomorphism class of the holomorphic bundle on Σ.

Thus the set of isomorphism classes of holomorphic bundles on Σ is in one-one correspondence with the points of :

$$G_{\Sigma_0} \backslash LG_{\mathbb{C}} / G_D .$$

In general, moduli spaces arise from the action of a group H on an algebraic variety A, giving,

$$A/H$$

as the moduli space. To define this so as to get a 'good' moduli space, it is necessary to consider refinements involving semi-stable points in A. For more details, see [A]. A good check that the moduli space has been defined well, is that the space of holomorphic functions (or rather sections of the relevant line-bundle) on it is equivalent to, in our case, the space of holomorphic functions on $LG_{\mathbb{C}}$ invariant under the actions of G_D, G_{Σ_0} (and in general, the space of holomorphic functions on A invariant under the action of H).

Loop groups

If G is a compact group with loop group LG then the tangent space to LG at some point is Lg, where g is the Lie algebra of the Lie group G.

There is then an important central extension $\tilde{Lg}$:

$$0 \to \mathbb{R} \to \tilde{Lg} \to Lg \to 0 .$$

In the dual situation, we would have

$$0 \leftarrow \mathbb{R} \leftarrow (\tilde{Lg})^* \leftarrow (Lg)^* \leftarrow 0 .$$

The space $(\tilde{Lg})^*$ can be identified with the set of operators

$$\lambda d/d\theta + \xi$$

on $\Omega^0(S^1;g)$ where $(\lambda,\xi) \in \mathbb{R} \oplus Lg$ and g is identified with g^* by using the inner product.

Similarly, one can construct a central extension of $LG_{\mathbb{C}}$:

$$0 \to \mathbb{C}^\times \to \widetilde{LG}_{\mathbb{C}} \to LG_{\mathbb{C}} \to 0 \ .$$

This extension splits when restricted to G_D and G_{Σ_0}:

$$\begin{tikzcd} & \mathbb{C}^\times \arrow[d] & \\ & \widetilde{LG}_{\mathbb{C}} \arrow[d] & \\ G_{\Sigma_0} \subseteq & LG_{\mathbb{C}} & \supseteq G_D \end{tikzcd}$$

but the lifting maps can be changed by homomorphisms of G_D, G_{Σ_0} into $\mathbb{C}^\times$. Thus the lifts are unique when G has no abelian factors, but correspond to choices of a spin structure on Σ in the abelian case. For more details see [S2].

So we have a principal $\mathbb{C}^\times$-bundle given by:

$$\begin{tikzcd} G_{\Sigma_0} \backslash \widetilde{LG}_{\mathbb{C}} / G_D \arrow[d] \\ G_{\Sigma_0} \backslash LG_{\mathbb{C}} / G_D \end{tikzcd}$$

and the line bundle associated in the natural way, i.e.

$$(G_{\Sigma_0} \backslash \widetilde{LG}_{\mathbb{C}} / G_D) \times_{\mathbb{C}^\times} \mathbb{C} \ .$$

Define $Z(\Sigma) = \mathrm{Hol}_{G_{\Sigma_0} \times G_D \times \mathbb{C}^\times}(\widetilde{LG}_{\mathbb{C}}, \mathbb{C})$. Thus $Z(\Sigma)$ consists of those functions:

$$f : \widetilde{LG}_{\mathbb{C}} \to \mathbb{C}$$

which are holomorphic, and:

$$f(gx) = gf(x) \qquad \forall g \in G_{\Sigma_0} \times G_D \times \mathbb{C}^\times \ .$$

Thus

$$Z(\Sigma) = \mathrm{Hol}_{G_D \times \mathbb{C}^\times}(\widetilde{LG}_{\mathbb{C}}, G_{\mathbb{C}})^{G_{\Sigma_0}}$$

where $(\tilde{LG_{\mathbb{C}}},\mathbb{C})^{G_{\Sigma_0}}$ denotes the part fixed by G_{Σ_0} .

Now the space

$$L_G/G \cong L_{G_{\mathbb{C}}}/G_D \cong \tilde{L}_{G_{\mathbb{C}}}/\mathbb{C}^{\times} \times G_D$$

plays the role in loop group theory of the flag manifold:

$$G/T \cong G_{\mathbb{C}}/B$$

in the theory of finite dimensional affine groups. Namely, irreducible representations of loop groups come from holomorphic sections of the above line bundle.

In particular,

$$\mathrm{Hol}_{G_D \times \mathbb{C}^{\times}}(\tilde{LG_{\mathbb{C}}},\mathbb{C})$$

is called the basic representation of $LG_{\mathbb{C}}$, and thus:

$$Z(\Sigma) = (\text{part of basic representation of } LG_{\mathbb{C}} \text{ invariant under } G_{\Sigma_0}) \ .$$

From this point of view, we include labels as follows. To include a label at a marked point x , we put a disc around x , and then there is a natural map:

$$G_D \to G_{\mathbb{C}}$$

given by evaluation at the centre. This acts on V , and we generalise the basic representation of $LG_{\mathbb{C}}$ to:

$$[\mathrm{Hol}_{G_D \times \mathbb{C}^{\times}}(\tilde{LG_{\mathbb{C}}};V)]$$

a representation induced on $LG_{\mathbb{C}}$ from V . The generalised version of $Z(\Sigma)$, for one label, is then:

$$[\mathrm{Hol}_{G_D \times \mathbb{C}^\times}(\widetilde{LG}_{\mathbb{C}};V)]^{G_{\Sigma 0}} .$$

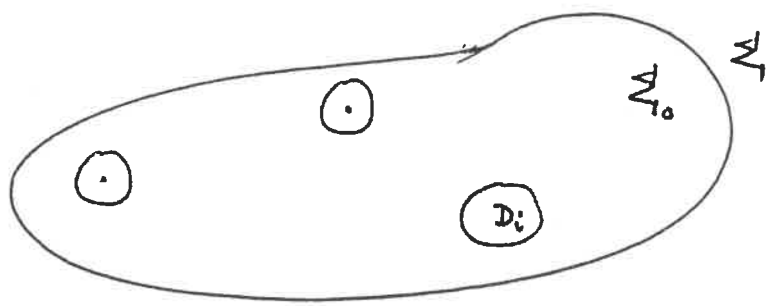
For a space Σ with ℓ marked points, we describe the associated moduli space as:

$$G_{\Sigma_0} \backslash (LG_{\mathbb{C}})^{\ell} / (G_D)^{\ell}$$

where Σ_0 is the part of Σ outside the ℓ discs. So $(LG_{\mathbb{C}})^{\ell}$ is a product of ℓ copies of factors $LG_{\mathbb{C}}$; and G_{Σ_0} corresponds to a trivialisation outside the discs, $(G_D)^{\ell}$ corresponds to a trivalisation inside the discs. So we get a representation:

$$Z(\Sigma,V_1,\ldots,V_\ell) = (E_{V_1} \otimes \ldots \otimes E_{V_\ell})^{G_{\Sigma 0}}$$

where E_{V_i} denotes the space of holomorphic functions on $\widetilde{LG}_{\mathbb{C}}$ with values in V_i , invariant under the action of $\mathbb{C}^\times \times G_D$ for the i^{th} disc.



Remark. In comparing the loop group description of the vector space $Z(\Sigma,V_1,\ldots,V_\ell)$ with the 2-dimensional gauge theory point of view the following remark may be helpful. The space A of G-connections on Σ can, once we fix a complex structure σ on Σ, be identified with the space C of complex structures on a fixed G^c-bundle over Σ . The gauge group $G = \mathrm{Map}(\Sigma,G)$

acts on A and its complexification $\mathcal{G}^c = \mathrm{Map}(\Sigma, G^c)$ acts holomorphically on C. The natural line-bundle L on $A = C$ is holomorphic and acted on by $\mathcal{G}^c$. The vector space $Z(\Sigma)$, at level k, is the space of $\mathcal{G}^c$-invariant (or $\mathcal{G}$-invariant) holomorphic sections of L^k. Given any finite-dimensional representation of $\mathcal{G}$ we can equally well look for the space of sections of L^k which transform accordingly. In particular we get representations of $\mathcal{G}$ from marked points P_i by the evaluation $\mathcal{G} \to G$ and then using the representation V_i of G. Thus we can define

$$Z(\Sigma, V_1, \ldots, V_\ell) = \mathrm{Hom}_{\mathcal{G}}(V_1 \otimes \ldots \otimes V_\ell, \Gamma(L^k))$$

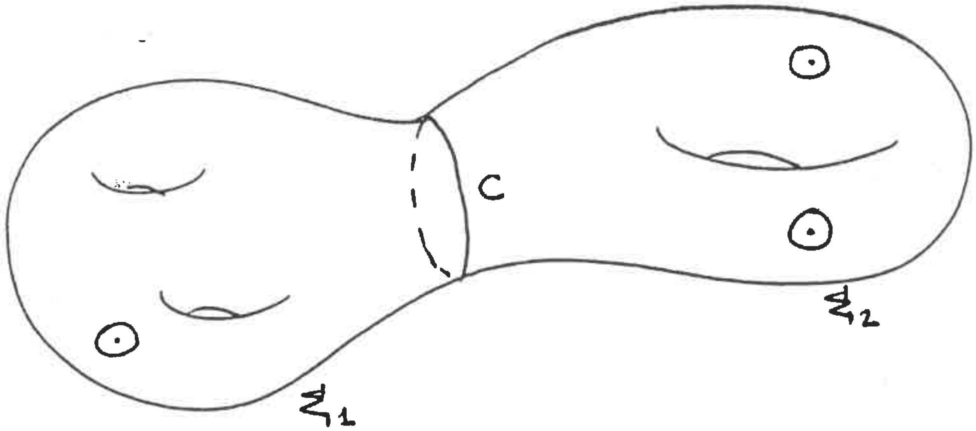
where $\Gamma(L^k)$ is the space of all holomorphic sections of L^k over C and $\mathrm{Hom}_{\mathcal{G}}$ stands for the space of $\mathcal{G}$-equivariant linear maps. The approach adopted in this section amounts essentially to cutting out discs around the marked points and taking boundary values of everything. Thus the 2-dimensional problem is reduced to a 1-dimensional one, which has some technical advantages.

Verlinde property

To end, we shall explain how to obtain the Verlinde property (see [V]) from this point of view. Consider a surface Σ cut by a curve C into two halves:

$$\Sigma = \Sigma_1 \cup \Sigma_2 .$$

We assume Σ has ℓ (>0) marked points, and C is chosen so that it does not pass through any of these points. Let ℓ_1, ℓ_2 be numbers of marked points on ℓ_1, ℓ_2.



Any function on the boundary curve $C = -\partial\Sigma_1 = \partial\Sigma_2$ can be written as a product of holomorphic functions defined on the two sides Σ_1', Σ_2' so long as holes exists on at least one side. Here, Σ_0 is that part of Σ outside the discs, and:

$$\Sigma_0 = \Sigma_1' \cup \Sigma_2'$$

where

$$\Sigma_i' = \Sigma_0 \cap \Sigma_i \quad \text{for} \quad i = 1,2 \; .$$

$$\left.\begin{aligned} \text{Thus} \quad LG_{\mathbb{C}} &= G_{\Sigma_1'} \cdot G_{\Sigma_2'} \\ \text{and} \quad G_{\Sigma_0} &= G_{\Sigma_1'} \cap G_{\Sigma_2'} \end{aligned}\right\} \qquad \text{(iii)}$$

Denote by E^1, E^2 the spaces $E_1 \otimes \dots \otimes E_{\ell'}$, E , $E_{\ell_1+1} \otimes \dots \otimes E_{\ell_1+k}$ respectively.

We have:

$$\begin{aligned} Z(\Sigma) &= \{(E_1 \otimes \dots \otimes E_{\ell_1}) \otimes (E_{\ell_1+1} \otimes \dots \otimes E_{\ell_1+k})\}^G \\ &= \mathrm{Map}(\tilde{G}_\sigma; E^1 \otimes E^2)^{G_{\Sigma_1} \times G_{\Sigma_2} \times \mathbb{C}^\times} \end{aligned}$$

where 'Map' denotes the space of holomorphic maps fixed under the action of $G_{\Sigma_1} \times G_{\Sigma_2} \times \mathbb{C}^\times$. This action is induced from the action of $G_{\Sigma_1} \times G_{\Sigma_2}$ on $G_{\mathbb{C}}$ by :

$$(g_1, g_2) \cdot g_3 = g_1 g_3 g_2^{-1}$$

which is restricted (by [U]), and has isotropy group G_Σ. So:

$$Z(\Sigma) = \{\mathrm{Hol}_k(\widetilde{LG}_{\mathbb{C}}) \otimes E^1 \otimes E^2\}^{G_{\Sigma_1} \times G_{\Sigma_2}}$$

where Hol_k denotes the space of holomorphic functions:

$$f : \widetilde{LG}_{\mathbb{C}} \to \mathbb{C}$$

such that

$$f(uy) = u^k f(y) \quad \text{for} \quad u \in \mathbb{C}^\times \subseteq \widetilde{LG}_{\mathbb{C}} .$$

The Peter-Weyl theorem for loop groups (see [S2]) states that there is an inclusion:

$$\mathrm{Hol}_k(\widetilde{LG}_{\mathbb{C}}) \hookrightarrow \bigoplus_E E^* \otimes E \tag{iv}$$

where E runs through the positive energy irreducible representations of level k. The map is given by:

$$E^* \otimes E \to \mathrm{Hol}(\widetilde{LG}_{\mathbb{C}})$$

$$(\eta \otimes \xi) \to (g \to \langle \eta, g \cdot \xi \rangle) .$$

This defines an inclusion (iv) with dense image, which induces an isomorphism between the parts of the spaces on either side of (iv) fixed under the action of:

$$G_{\Sigma_1} \times G_{\Sigma_2}$$

for any surfaces Σ_1, Σ_2 such that:

$$\partial\Sigma_2 = -\partial\Sigma_1 = S^1 \ .$$

Thus

$$Z(\Sigma) = \{\bigoplus_E E^* \otimes E \otimes E^1 \otimes E^2\}^{G_{\Sigma 1} \times G_{\Sigma 2}}$$

$$= \bigoplus_E \{E^* \otimes E^1\}^{G_{\Sigma 1}} \otimes \{E \otimes E^2\}^{G_{\Sigma 2}}$$

since G_{Σ_1}, G_{Σ_2} act independently on the two halves of $E^* \otimes E \otimes E^1 \otimes E^2$

$$= \bigoplus_E Z(\Sigma_1, V^*) \otimes Z(\Sigma_2, V)$$

since when $E = E_V$, $Z(\Sigma_1, V) = (E^1 \otimes E_V)^{G_{\Sigma 1}}$.

Hence we have derived the Verlinde property.

Seminar 5

Nigel Hitchin: Reduction to the Abelian case

The aim of our programme is to associate to a compact surface Σ, compact Lie group G and integer k (level), a finite dimensional vector space $Z(\Sigma)$ which satisfies certain axioms, as described in previous seminars.

We recall that $Z(\Sigma)$ is, strictly speaking, only well-defined as a projective space.

Choose a complex structure on Σ. This makes Σ into a Riemann surface. One then considers the moduli space:

$$M = \mathrm{Hom}(\pi_1(\Sigma), G)/G \qquad (*)$$

where G acts on $\mathrm{Hom}(\pi_1(\Sigma), G)$ by conjugation. A complex structure σ on Σ induces a complex structure M_σ on M, where the suffix σ denotes the dependence on the complex structure. By Narasimhan-Seshadri-Ramanathan theory, M_σ can be be interpreted as the moduli space of semi-stable G^c-bundles.

Thus M has a topological definition, given by (*), but it acquires a complex structure when one is put on Σ.

We can think of $\mathrm{Hom}(\pi_1(\Sigma), G)$ as the space of flat G-connections, and thus M as the space of equivalence classes of flat G-connections. To a flat G-connection, one can associate a covariant derivative. Taking:

$$G = U(n)$$

the unitary group, for simplicity, we see that a flat G-connection is associated to a rank n vector bundle V over Σ. Given a complex structure σ, we obtain a $\bar{\partial}$ operator

$$\bar{\partial} : \Omega^0(\Sigma;V) \to \Omega^{0,1}(\Sigma;V) \ .$$

We can also consider the determinant line bundle:

$$L = \det(\bar{\partial})$$

over the moduli space M_σ. Then define

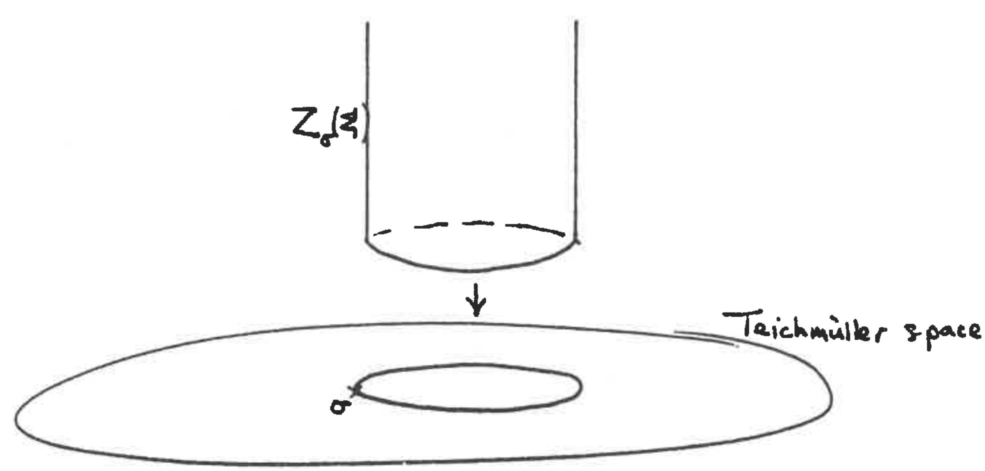
$$Z_\sigma(\Sigma) = H^0(M_\sigma;L^k)$$

the space of holomorphic sections of the line bundle L^k over M. In fact, if $\deg V$ is appropriate, so that index $\bar{\partial} = 0$ then there is a natural section of L vanishing on the subspace of M_σ for which there is a non-trivial solution, s, to the equation

$$\bar{\partial}s = 0$$

i.e. for which V has a non-trivial holomorphic section.

This defines a vector space $Z_\sigma(\Sigma)$ associated to Σ_σ, k and G. To eradicate the dependence of this vector space on the complex structure σ, our main aim is to produce a flat connection on the space of $Z_\sigma(\Sigma)$'s over Teichmüller space (the space of complex structures on Σ modulo the action of identity component of $\mathrm{Diff}_+\Sigma$).



Remark. The complex structure on the moduli space induced by a complex structure σ on Σ depends in fact only on the isomorphism class of σ. For example for $G = U(1)$ the complex structure of the Jacobian is determined directly by the period matrix of σ. The isomorphism $\Sigma_\sigma \to \Sigma_{\sigma'}$ to be used here should be restricted to those homotopic to the identity so that the underlying space M of (classes of) representations $\pi_1(\Sigma) \to G$ is fixed. For general G recall that for a representation $\alpha : \pi_1(\Sigma) \to G$ giving a point of M the tangent space T_α to M at α is given by

$$T_\alpha = H^1(\Sigma, Ad_\alpha(G))$$

and the complex structure α on Σ then induces a complex structure on T_α by the identification

$$H^1(\Sigma, Ad_\alpha(G)) \to H^{0,1}(\Sigma, Ad_\alpha(G)) .$$

This shows that isomorphic σ give the same complex structure T_α. Thus we have a natural family of complex structures on M parametrised by Teichmüller space and the associated family of vector spaces Z_σ forms a holomorphic vector bundle $Z(\Sigma)$ over Teichmüller space.

Since Teichmüller space is simply connected, once a flat connection is defined on the above vector bundle $Z(\Sigma)$, we can identify, by parallel translation, any two vector spaces corresponding to different complex structures.

In this lecture, we shall give a definition of such a flat connection. It arises by associating to Σ, G, a family of abelian varieties (algebraic tori). For $G = U(n)$, these are Jacobians of curves. We shall show how one can define a connection in the non-abelian case, when one knows how to define a flat connection for the case of $G = U(1)$ (abelian case). Thus this has reduced the non-abelian case to an abelian case. However the reduced system is no longer over Σ, but over an n-fold covering of Σ.

Thus we have translated a non-abelian problem on Σ to an abelian problem on a (higher genus) covering space of Σ: there is a trade-off between complexity of the base, and the abelian nature of G.

Note that the discrete group $\Gamma(\Sigma)$ of components of $\mathrm{Diff}_+(\Sigma)$ acts on Teichmüller space and on the bundle $Z(\Sigma)$. The flat connection we shall define on $Z(\Sigma)$ will not be invariant under $\Gamma(\Sigma)$, although the induced projective connection will be invariant.

Method of construction

The construction of the flat connection on $Z(\Sigma)$ over Teichmüller space uses 'Higgs bundles' (see [N2]). These are points (V,Φ) where:

V = holomorphic vector bundle ;

Φ = Higgs field .

For $G = U(n)$, this means that $\Phi \in H^0(\Sigma;\mathrm{End}\, V \otimes K)$ where K is the canonical bundle.

Such a pair (V,Φ) is said to be stable according to a condition which is the analogue of the stability condition for vector bundles.

We can consider the moduli space M_σ of stable vector bundles V. Consider a smooth point in M_σ. The tangent space at such a point in M_σ is naturally identified with:

$$H^1(\Sigma;\mathrm{End}\ V) .$$

By Serre duality,

$$H^1(\Sigma;\mathrm{End}\ V)^* \cong H^0(\Sigma;\mathrm{End}\ V \otimes K)$$

and so we think of Φ as a cotangent vector to M_σ. Then we can consider T^*M_σ as a subset of $\mathcal{M}$, where $\mathcal{M}$ is the moduli space of stable Higgs fields (V,Φ); and M_σ is the moduli space of stable bundles V. The complement of:

$$T^*M_\sigma \subseteq \mathcal{M}$$

has codimension at least g. So T^*M_σ is a large open subset of $\mathcal{M}$. Think of $\mathcal{M}$ as like the cotangent bundle of M_σ, but slightly enlarged.

The large moduli space $\mathcal{M}$ is a holomorphic symplectic manifold. That is, there exists a natural holomorphic skew form which is non-degenerate and restricts on T^*M_σ to the canonical skew form. In fact Hartog's theorem can be used (for $g \geq 2$) to extend the canonical 2-form from T^*M_σ to $\mathcal{M}$.

Consider $\det(\lambda-\Phi) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$. Since

$$\Phi \in H^0(\Sigma;\mathrm{End}\ V \otimes K)$$

we have $a_i \in H^0(\Sigma,K^i)$. Note in particular, $a_2 \in H^0(\Sigma,K^2)$, that is, a_2 is a quadratic differential. The space $H^0(\Sigma,K^2)$ is the cotangent space of Teichmüller space.

So, given any (V,Φ) the characteristic polynomial of Φ gives a collection of differentials of different degrees, given by the coefficients a_i :

$$\chi : M \to \bigoplus_{i=1}^{n} H^0(\Sigma;K^i) = W .$$

This map has some important properties:

1) χ is proper [M is not compact, although M_σ is compact (if one adjoints singular points); but M is compact in fibre directions of the above maps].

2) The generic fibre is an abelian variety.

Here, an abelian variety refers to a complex compact torus which is also an algebraic variety. M is a holomorphic symplectic manifold and it can be checked that the number of functions in χ is precisely $\frac{1}{2}\dim(M)$.

These functions are Poisson commuting, and there is a general theorem, on integrable Hamiltonian systems, which states that in this situation, the generic fibre is a torus.

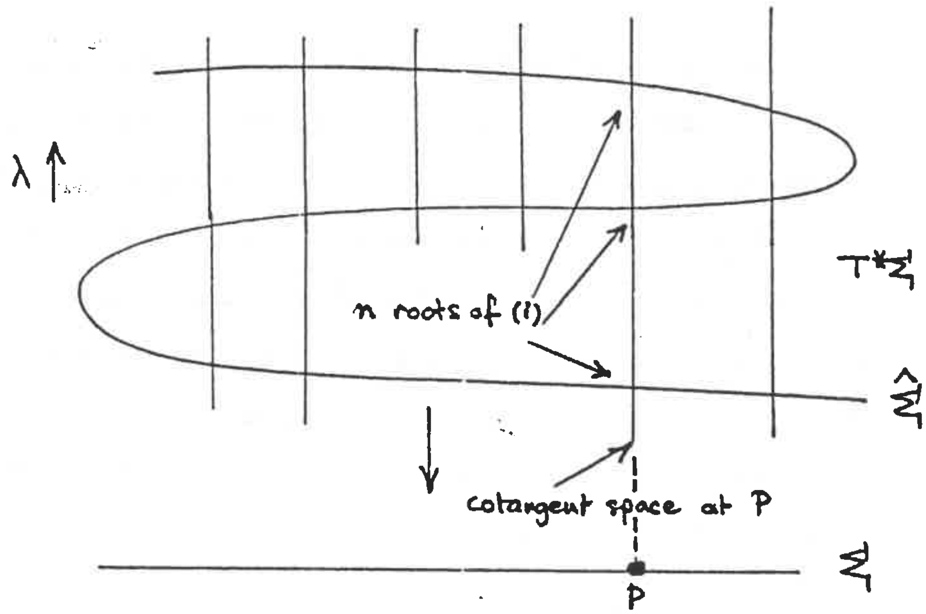
Now $\det(\lambda-\Phi) = 0$ defines an algebraic curve:

$$\hat{\Sigma} \subseteq T^*\Sigma$$

since Φ is a matrix valued 1-form, and λ is a holomorphic form. If we think of the cotangent space $T^*\Sigma$ as fibred over Σ , then on a generic fibre,

$$\det(\lambda-\Phi) = 0 \qquad \text{(i)}$$

has n roots : we think of λ as a parameter in the fibre direction:

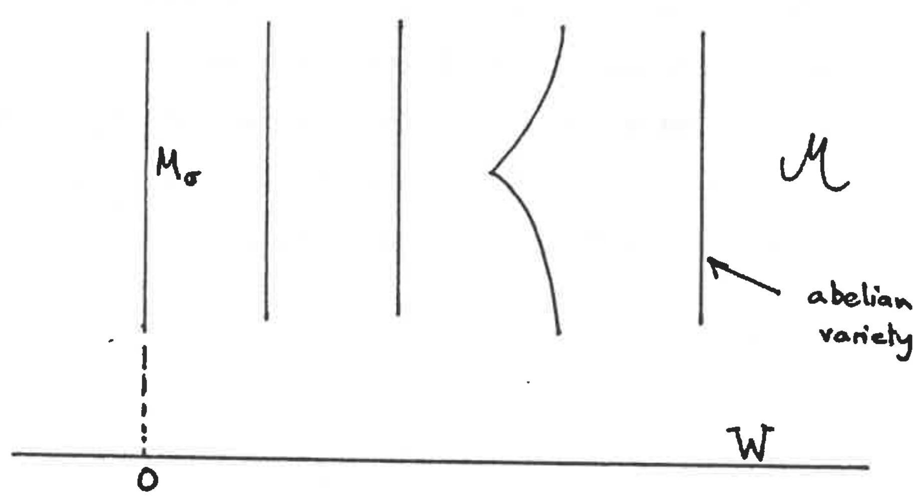


Hence $\hat{\Sigma}$ is an n-fold branched covering of Σ .

Now $\ker(\lambda - \Phi)$ is generically of dimension one, and defines a line bundle U over $\hat{\Sigma}$. It is not hard to see that sheaf theoretically,

$$\pi_* U^* \cong V^*$$

where $\pi_* U^*$ is the direct image sheaf. This means the fibre of V^* at a generic point p of Σ , is the direct sum of the fibres of U^* over the n points in $\pi^{-1}(p)$. The equation of $\hat{\Sigma}$ is given by the coefficients of the characteristic polynomial. The only other data involved in an element of M is the line bundle U .



The generic fibre is $\mathrm{Jac}(\hat{\Sigma})$, a non-singular abelian variety. Some fibres may be singular: they occur when $\hat{\Sigma}$ acquires singularities, or even becomes reducible.

If $\deg U = 0$, then the trivial bundle gives a cross-section of M over W. One should think of the fibres of M as groups, and not merely as inert abelian varieties. In the case $\deg U \neq 0$, we do not get a unique canonical cross-section, but we still have a finite choice of cross-sections.

Since $\pi_* U^* \cong V^*$, we have

$$H^0(\Sigma, V^*) = H^0(\hat{\Sigma}, U^*) \ .$$

So if V^* has a holomorphic section, so does U^* on $\hat{\Sigma}$. Thus the line bundle on the moduli space of bundles V^*, i.e.

$$L = \det(\bar{\partial})$$

has sections vanishing where:

$$H^0(\Sigma, V^*) \neq 0 \ .$$

This corresponds to a line bundle U^* for which :

$$H^0(\hat{\Sigma}, U^*) \neq 0 \ .$$

The locus of such line-bundles in the Jacobian of line bundles of degree $\hat{g} - 1$ is called the θ-divisor. [Narasimhan has made a detailed study of this situation].

The determinant line bundle L on M_σ, pulls back to M, and is such that its restriction to $\mathrm{Jac}(\hat{\Sigma})$ corresponds to the natural θ-divisor on $\mathrm{Jac}(\hat{\Sigma})$.

Abelian theory

Consider an abelian variety A of dimension g which is principally polarised. That is, there exists an integral cohomology class of ω of type (1,1) which is positive and such that

$$\int_A \frac{\omega^g}{g!} = 1 \ .$$

For example, a Jacobian satisfies these requirements.

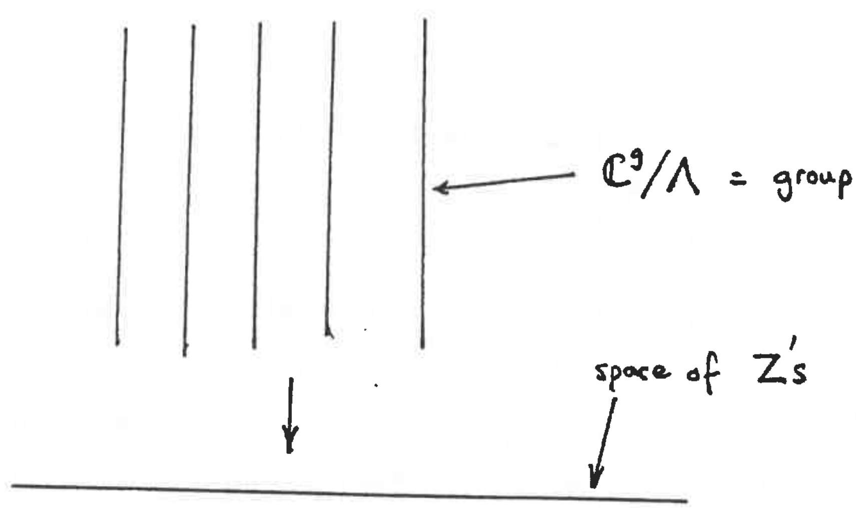
Suppose Z_{ij} is a $g \times g$ complex symmetric matrix, with $\operatorname{Im} Z_{ij} > 0$. Then the $g \times 2g$ matrix:

$$(I \mid Z)$$

defines a lattice Λ, since its columns are $2g$ vectors in $\mathbb{C}^g$ which are linearly independent over $\mathbb{R}$. Consider the abelian variety:

$$\mathbb{C}^g/\Lambda \ .$$

Then for each Z satisfying the above conditions, we have shown how to construct an abelian variety.



We have a space of abelian varieties of dimension g, fibred over the space of complex symmetric matrices with positive imaginary parts (this is called the Siegel upper half space). Every principally polarised abelian variety corresponds to such a Z: that is, we have a fibration of the space of all principally polarised abelian varieties over the Siegel upper half plane.

Now define, for each k and $u \in \mathbb{C}^g$:

$$\theta_m(u,Z) = \sum_{\substack{\ell \in \mathbb{Z}^g \\ \ell \equiv m \bmod k}} e^{\pi i/k \langle \ell, Z\ell\rangle} e^{2\pi i \langle \ell, u\rangle}$$

where $m \in (\mathbb{Z}/k)^g$. This is the classical theta function.

Properties of theta functions

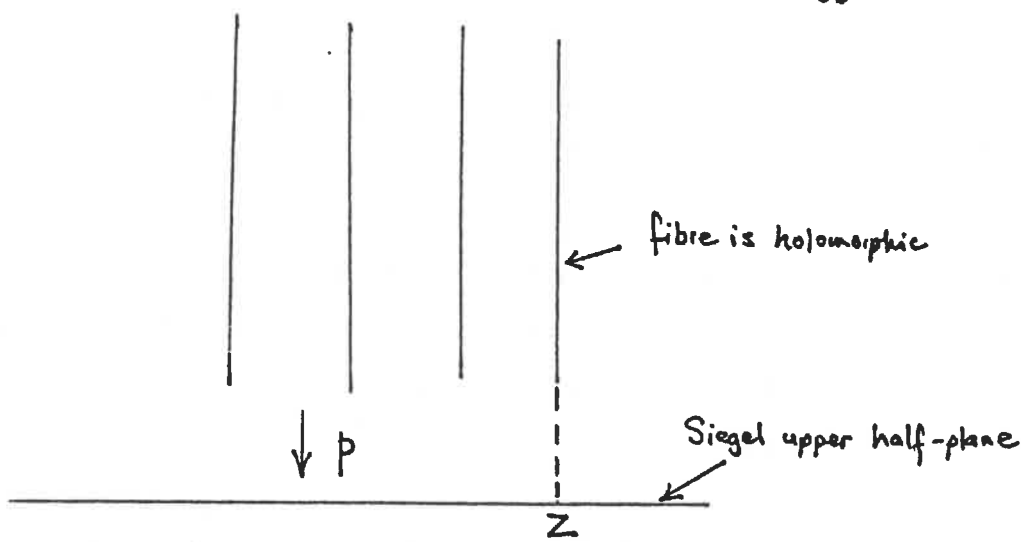
1. Fix Z. Then each θ_m can be interpreted as a globally defined holomorphic section of Θ^k, where Θ is the line bundle generalising the classical θ-divisor of the Jacobian.

2. As m varies, we obtain k^g different θ-functions. By the Riemann-Roch theorem Θ^g has k^g holomorphic sections. The k^g θ-functions obtained as m varies are a basis for the sections of Θ^g. Thus, for fixed Z, all sections are linear combinations of θ_m's.

3. θ_m satisfies a differential equation:

$$\frac{1}{k}\frac{\partial^2\theta_m}{\partial u_i \partial u_j} = 2\pi i(1+\delta_{ij})\frac{\partial\theta m}{\partial Z_{ij}} \qquad \text{(ii)}$$

Equation (ii) is similar to a heat equation in one dimension. It specifies the variation of θ in the Z-direction in terms of that in the u-direction, and hence effectively defines a flat connection on the bundle over Z whose fibre is the space of holomorphic sections of Θ^k.



Consider this vector bundle over the Siegel upper half space. By the above, this is a rank k^g vector bundle over the base.

The flat connection is defined so as to make the θ-functions $\theta_m(u,Z)$ into covariant constant sections over the Siegel upper half plane. There are enough constant sections to give a completely flat connection.

The left hand side of (ii) is a differential operator. If it had been a matrix transformation, then one would immediately recognise (ii) as defining a connection. The fact that the θ's satisfy (ii) implies that they are covariant constant.

Alternative approach

This definition of the connection does not appear to be independent of coordinates. To see that it is, we shall give another way of construction the connection. We have an exact sequence:

$$0 \to L \otimes S^2T_F^* \to J_F^2(L) \to J_F^1(L) \to 0$$

where $J_F^2(L)$ consists of holomorphic 2-jets of sections of L along fibres (i.e. given by the first three terms of a

Taylor series expansion). The natural map:

$$J^2_F(L) \to J^1_F(L)$$

given by "losing one of the terms of the Taylor series expansion" has kernel $L \otimes S^2T^*_F$, where $S^2T^*_F$ denotes the symmetric square of the cotangent bundle. There is also an exact sequence given by:

$$0 \to L \otimes T^*_B \to J^1(L) \to J^1_F(L) \to 0$$

where the map:

$$J^1(L) \to J^1_F(L)$$

is given by restriction to the fibre, and has kernel consisting of those jets vanishing in the fibre direction. T^*_B , T^*_F denote the cotangent bundles to the base and fibre respectively.

However, here the base is the space of symmetric matrices, and the tangent bundle of each fibre is trivial. Thus:

$$T^*_B \cong S^2T^*_F$$

(in fact this is the Kodaira-Spencer map). Hence we have:

$$\begin{array}{ccccccccc} 0 & \to & L\otimes S^2T^*_F & \to & J^2_F(L) & \to & J^1_F(L) & \to & 0 \\ & \text{natural} & \| & & & & \| & & \\ 0 & \to & L\otimes T^*_B & \to & J^1(L) & \to & J^1_F(L) & \to & 0 \, . \end{array}$$

One can show, by carefully considering the two extensions, that $J^2_F(L) \cong J^1(L)$. The existence of a differential equation satisfied by the θ's is equivalent to the statement that an isomorphism:

$$J_F^2(L) \cong J^1(L)$$

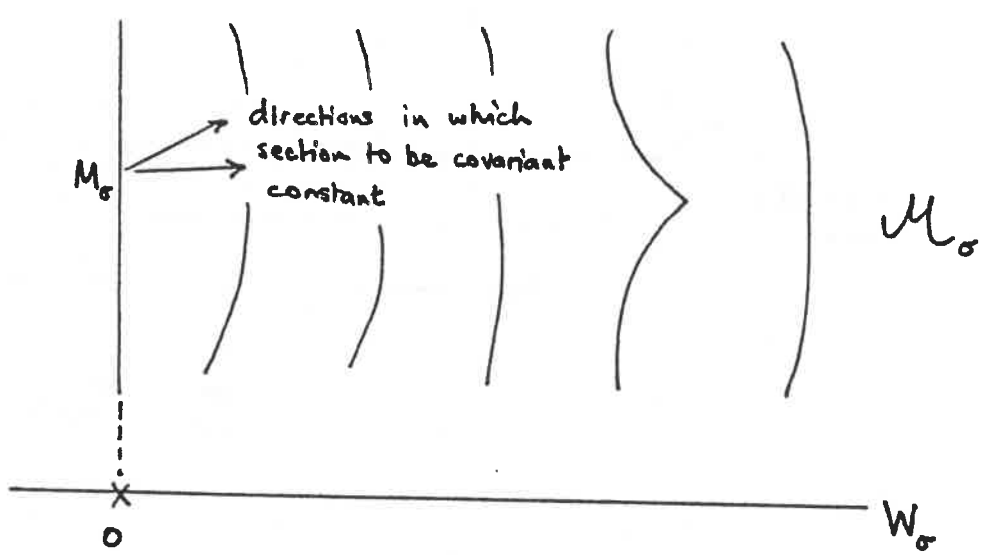
exists. This isomorphism is not unique; in fact any two such isomorphisms differ by a differential form coming from the base. This is equivalent to the statement that the global connection does not exist canonically on the vector bundle, but only on the projective space.

Whichever way one looks at it, ultimately the θ-function differential equation (ii) needs to be used. So we have a family of abelian varieties (Jacobians in $G = U(n)$ case) and a flat connection over this space. Restricting to $\mathcal{M}$ over W, we obtain a flat connection on the direct image of corresponding line bundles on W.

Proposition. A key fact which must be proved is: If a section on M_σ is pulled back to the cotangent bundle, and then extended by Hartog's theorem to a section on the whole of:

$$\mathcal{M} \supseteq T^*M_\sigma$$

then it is covariant constant in the horizontal direction:



M_σ is a family of abelian varieties, parametrised by the vector space W_σ. The suffix σ has been introduced to indicate dependence on the complex structure σ. We have a flat connection on the vector bundle over W_σ, given by the direct image sheaf.

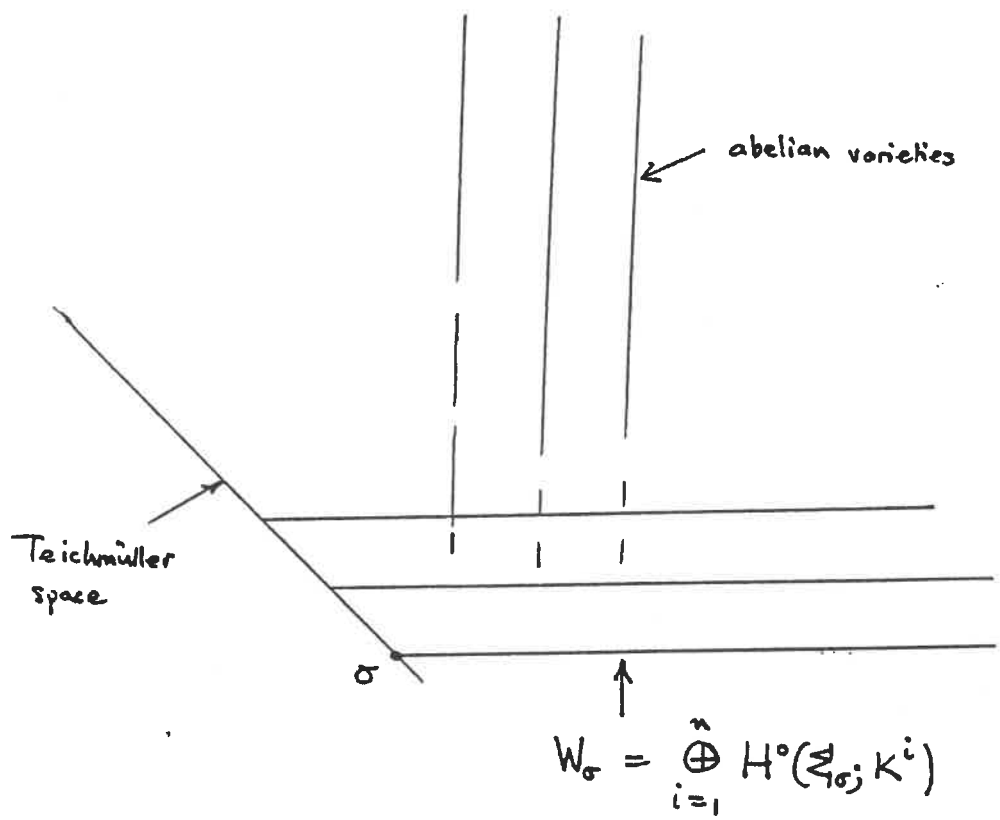
As σ is varied, that is, the modulus of the Riemann surface is varied, the abelian varieties will vary in a bigger subspace of all abelian varieties.

Thus the abelian variety will now be parametrised by:

(i) an element of Teichmüller space, giving σ ;

(ii) an element of:

$$W_\sigma = \bigoplus_{i=1}^{n} H^0(\Sigma_\sigma, K^i) .$$



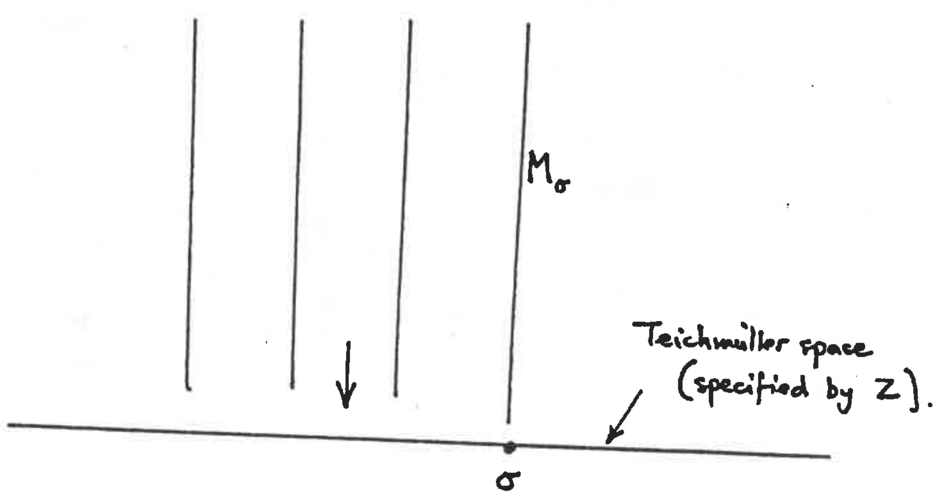
Each fibre is an abelian variety, and we can put a flat connection on this big bundle, since Teichmüller space is simply connected and thus so is the base. The restriction to W_σ is the same as the connection imposed before, and thus we have a flat connection in σ-directions also:

Theorem. A section form M (pulled back from M_σ to T^*M_σ and then extended by Hartog's theorem to M) is covariant constant in the W_σ-directions.

Thus a section of the bundle $H^0(M_\sigma, L^k)$ on Teichmüller space has a covariant derivative which vanishes in the W_σ-direction and hence is well-defined on Teichmüller space, giving the required connection.

The argument that proves the theorem can be applied locally: we can see from the differential equation (ii) that we only need the θ-function and its first two-derivatives in the u direction, to determine its first derivative in the parameter direction, and thus the whole θ function for different σ.

So consider M_σ, the moduli space of stable bundles over a point in Teichmüller space, as a fibre of a bundle over Teichmüller space:



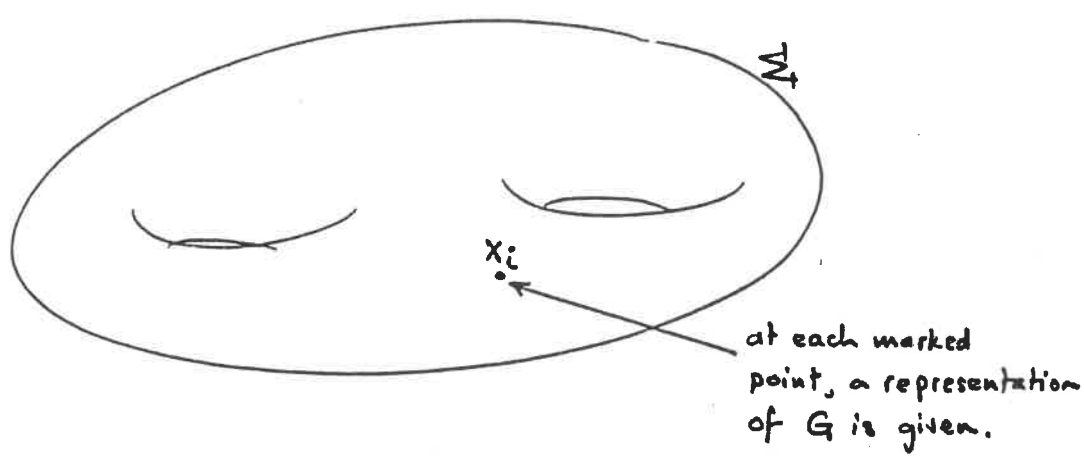
We have seen that there exists a differential equation relating first order derivatives with respect to Z with second order derivatives in the moduli space direction, of θ-functions. Such a differential operator gives a symbol:

$$L \otimes S^2T^*M \to H^0(\Sigma;K^2) .$$

Here $H^0(\Sigma;K^2)$ is the cotangent bundle of Teichmüller space; and the above symbol is a holomorphic quadratic form on the cotangent bundle of moduli space: in fact it is precisely the coefficient a_2 in the characteristic polynomial $\det(\lambda-\Phi)$, for $G = SU(n)$.

The differential operator defines a connection, which is flat because of the existence of sufficiently many θ-functions. Generally, when one has a situation of the moduli space of vector bundles, it is rather difficult to actually find the symbol.

In the case of a Riemann surface with marked points $x_1,\ldots,x_\ell$ and representations of G attached to each point, we must replace M_σ, the moduli space of bundles by the moduli space of bundles with "parabolic structure".

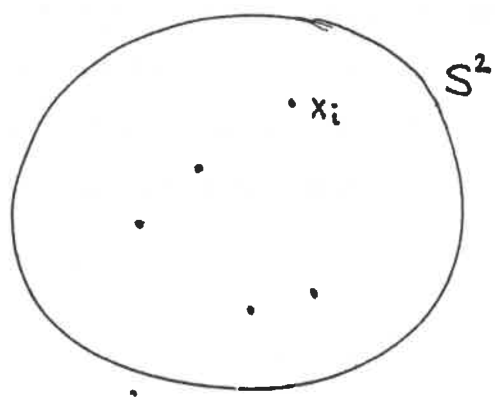


Essentially we are looking at holomorphic bundles over Σ which have a reduction to a Borel subgroup at each marked point. Stability is defined in terms of a degree, defined as in the case without marked points, except that a contribution is required from each of the marked points. (See Mehta & Seshadri).

Generalise Higgs bundles so that

> Φ = meromorphic section of $K \otimes \mathrm{End}\, V$ having simple poles at $x_1, \ldots, x_\ell$ with residues which are nilpotent (in nilpotent part of Borel subalgebra) .

We can then imitate the whole procedure above. Applying this to a sphere we can actually derive from these methods, a formula for the flat connection, which is in fact already known: the Kohno connection (see [K])



A bundle on S^2 is generically trivial, and so we can write:

$$\Phi = \sum_{j=1}^{\ell} N_i dz/(z-x_i)$$

where N_i are nilpotent, and:

dual to a 2-form is a 1-form i.e. $*F_A$ is a 2-form. However A is an affine space, and so its tangent space at any point consists of Lie algebra valued 1-forms.

Thus F is a 1-form on A. Its value on a tangent vector to A is given by multiplying by F_A and integrating over M, contracting on the Lie algebra variables. Define:

$$\begin{aligned} \mathcal{G} &= \text{group of gauge transformations} \\ &= \operatorname{Map}(M,G)\ . \end{aligned}$$

Clearly $\mathcal{G}$ acts on A; and F is $\mathcal{G}$-invariant. Moreover, in the fibration

$$\begin{array}{c} A \\ \downarrow \mathcal{G} \\ A/\mathcal{G} \end{array}$$

F vanishes in the vertical (fibre) direction, and thus comes from the base. So F is a well-defined 1-form on $A/\mathcal{G}$.

Also $dF = 0$ i.e. F is a closed 1-form. Thus one would expect F can be expressed in the form:

$$F = df$$

for some function f, where f is a $\mathcal{G}$-invariant scalar valued function on A determined up to a constant. One can fix this constant by requiring that:

$$f = 0$$

on the trivial connection.

This works if A is simply connected. Otherwise, one can only expect f to be locally defined, and globally, it will be multi-valued. In fact, $\mathcal{A}$ is not simply connected, and f is well-defined only up to integral multiples of some constant. This f is the Chern-Simons functional. It is well-defined modulo integers, and is G-invariant.

Explicit formula

Define:

$$L(A) = \frac{1}{4\pi}\int_M \mathrm{Tr}(A \wedge dA + 2/3\, A \wedge A \wedge A)$$

where $A \in \mathcal{A}$. Here Tr stands for a suitable normalization of the Killing form. For $SU(n)$ it is the standard trace. Here L is a multiple of f: the notation L has been used so as to be consistent with Witten's paper.

One now verifies that L is invariant under the subgroup:

$$G_0 \subseteq G$$

given by the connected component of G containing the identity. Here G, G_0 differ by a copy of $\mathbb{Z}$; and under a generator of G/G_0, L is not invariant: it pick up a multiple of 2π.

Thus $e^{ikL(A)}$ is a well-defined function of A, for $k \in \mathbb{Z}$. Witten's invariant of 3-manifolds is now defined by:

$$Z(M) = \int_{\mathcal{A}} \exp(ikL(A))\,\mathcal{D}A$$

This is a very elegant definition provided one believes that the integral makes sense! More generally, we consider a closed oriented curve:

$$C \subseteq M$$

and fix an irreducible representation, λ, of G, in addition to the data required previously: G, k.

A connection A on M then defines a parallel transport along any curve in M. In particular, around C, one obtains a monodromy element $\mathrm{Mon}_C(A)$. Then:

$$\mathrm{Tr}_\lambda \mathrm{Mon}_C(A) \equiv W_C(A)$$

evaluated by taking the trace in the representation λ. Here $W_C(A)$ is known as a Wilson line. Define:

$$Z(M,C) = \int_{\mathcal{A}} \exp(ik\mathcal{L}(A)) \cdot W_C(A) \mathcal{D}A \quad .$$

This is a generalization of $Z(M)$. In physicist's language,

$$Z(M) = \langle 1 \rangle$$

$$Z(M,C) = \langle W_C(A) \rangle$$

where $\langle \ \rangle$ denotes the expectation value.

Of course, one can similarly deal with several components $C_1,\ldots,C_r$, to each one associating a different irreducible representation of G. Then

$$Z(M,C_1,\ldots,C_r) = \langle W_{C_1}(A) W_{C_2}(A) \ \ldots \ W_{C_r}(A) \rangle .$$

It is important to notice that the above definitions involve no metrics or volumes. This is an indication that we have defined topological invariants.

Stationary phase approximation

To see if the above definitions make any sense, we first of all consider the stationary phase approximation $k \to \infty$. One should think of the parameter k as something like $1/h$ where h is Planck's constant. The classical limit comes from $h \to 0$.

For the rest of this talk we shall only be concerned with $Z(M)$: the generalisations $Z(M,C_1,\ldots,C_r)$ are similar, and only slightly more complicated. There is enough complexity in $Z(M)$!

In the stationary phase approximation, the dominant part comes from the stationary points of the exponent. That is, at points where:

$$dL = 0$$

$$\Rightarrow \quad F_A = 0 \text{ , by definition of } f$$

i.e. A is a flat connection and thus corresponds to a representation of $\pi_1(M)$:

$$\alpha : \pi_1(M) \to G \text{ .}$$

Then, the stationary phase approximation to $Z(M)$ gives a sum of contributions, one from each of the representations α . Thus we need only look at the integral for $Z(M)$ locally. Suppose α is a flat connection, and:

$$A = \alpha + \beta$$

where β is "small" .

Then $L(A) = L(\alpha) + \frac{1}{4}\pi \int_M \mathrm{Tr}(\beta \wedge d_\alpha \beta) + \text{cubic terms.}$

There are no linear terms, since $dL = 0$ at α. Here, $d_\alpha\beta$ is the covariant derivative of β with respect to the connection α .

Define $Q(\beta) = \frac{1}{4}\pi \int_M \mathrm{Tr}(\beta \wedge d_\alpha\beta)$. This is the quadratic term in the expansion of $L(A)$ above. One can think of Q as a quadratic form in an infinite number of variables. Here:

$$Q(\beta) = \tfrac{1}{4}\pi\langle\beta, {*d_\alpha\beta}\rangle$$

where $\langle , \rangle$ is the inner product on 1-forms:

$$\langle\alpha,\beta\rangle = \mathrm{Tr} \int \alpha \wedge {*\beta} \text{ .}$$

Thus Q is given by a self-adjoint operator, $*d_\alpha$. This Q is related to the de Rham complex with respect to the coefficient system given by α. Let g_x be the flat G-bundle on M given by the connection α, with fibre the Lie algebra g. Then we have a de Rham complex:

$$\Omega^0_\alpha \xrightarrow{d_\alpha} \Omega^1_\alpha \xrightarrow{d_\alpha} \Omega^2_\alpha \xrightarrow{d_\alpha} \Omega^3_\alpha \ .$$

Since α is flat, $d_\alpha^2 = 0$.

We shall assume for simplicity that α is a non-degenerate representation; i.e. that the above complex has no cohomology:

$$H^*(M,g_\alpha) = 0 \ .$$

Since H^2, H^3 are dual to H^0, H^1, these conditions essentially reduce to:

$$H^0 = 0 \ , \quad H^1 = 0 \ .$$

Here $H^0(M,g_\alpha) = 0$ corresponds to α being an irreducible representation and $H^1(M,g_\alpha) = 0$ corresponds to this representation being isolated (since $\dim H^1$ is essentially the number of deformation parameters of the representation).

In this case, $Q(\beta)$ is degenerate on $d\Omega^0_\alpha$, since $*d_\alpha$ vanishes on the image of

$$d_\alpha : \Omega^0_\alpha \to \Omega^1_\alpha \ .$$

This corresponds to the fact that f is invariant under G; $d\Omega^0_\alpha$ corresponds to infinitesimal gauge transformations. Factoring out $d\Omega^0_\alpha$, we find that Q is non-degenerate on $\Omega^1/d\Omega^0_\alpha$.

Classical Gaussian Integrals

We start with the one-dimensional integral:

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}}\, e^{-\mu x^2} = \frac{1}{\sqrt{\pi}} .$$

Analytically continuing, we get putting $\mu = i\lambda$:

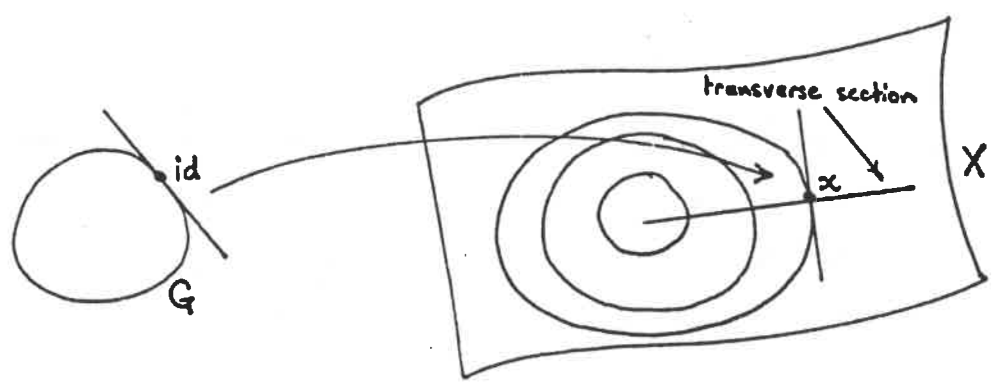
$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}}\, e^{+i\lambda x^2} = |\lambda|^{-\frac{1}{2}} \exp\left(\frac{\pi i}{4} \operatorname{sgn} \lambda\right) .$$

The n-dimensional form of this is as follows. Suppose Q is a non-degerate quadratic form on $x_1,\ldots,x_n$. Then:

$$\int \exp(i\lambda Q(x)) \frac{dx}{\pi^{n/2}} = |\det Q|^{-\frac{1}{2}} \exp\left|\frac{\pi i}{4} \operatorname{sgn} Q\right| .$$

This holds for non-degenerate quadratic forms only (no zero eigenvalues).

Suppose we have the action of a compact group G (e.g. S^1) on a Euclidean space, X , and $Q(x)$ is a G-invariant quadratic form. Take a transversal slice of the space for the G action. We must take into account some form of Jacobian: in fact, the appropriate quantity is the volume of an orbit.



Thus G acts on X . At a point $x \in X$ we have a map:

$G \to X$

$g \to g(x)$.

This gives a map, B , from the tangent space of G at the identity to the tangent space of the orbit at x . The Jacobian of the map corresponds to the volume of the orbit. Thus:

$$(\det B^*B)^{\frac{1}{2}} = \frac{\text{vol. of orbit}}{\text{vol of } G}$$

is the appropriate scaling factor.

Hence we obtain the modulus

$$\frac{(\det B^*B)^{\frac{1}{2}}}{|(\det Q)|^{\frac{1}{2}}} \qquad \text{(i)}$$

and phase factor $\exp\left|\frac{\pi i}{4} \operatorname{sgn} Q\right|$. (ii)

<u>Application to our situation</u>

In our case,

B = (infinitesimal map from Lie algebra to tangent space of manifold)

$= (d_\alpha : \Omega^0_\alpha \to \Omega^1_\alpha)$

$\Rightarrow \quad B^*B = d^*_\alpha d_\alpha = \Delta^0_\alpha$, the Laplace operator on Ω^0_α .

Now Q is given by $*d_\alpha$ on $\Omega^1/d\Omega^0$. Consider the operator:

$$L = d_\alpha{}^* + {}^*d_\alpha$$

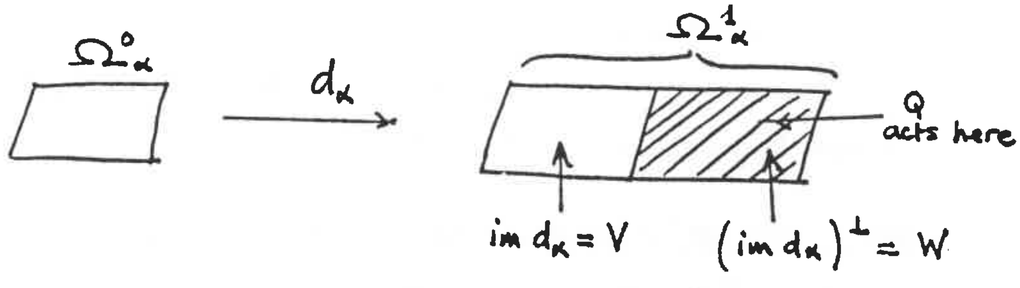
acting on odd forms Ω^{odd}_α . This is $\Omega^1 \oplus \Omega^3$; by duality, one can replace Ω^3 by Ω^0 , and thus L can be thought of as acting on $\Omega^0 \oplus \Omega^1$. L is closely related to Q .

We can think of $\Omega^1_\alpha = V \oplus W$ where:

$$V = \operatorname{Im}(d_\alpha : \Omega^0_\alpha \to \Omega^1_\alpha)$$

and $W = V^\perp$ in Ω^1_α . Then Q acts on W ; and L acts on $\Omega^0_\alpha \oplus V \oplus W$ by :

$$\begin{pmatrix} 0 & B & 0 \\ B^* & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}$$



A quadratic form of the type:

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$$

always has zero signature. Of course in this case, we have not assigned any meanings to det Q or sgn Q; but in any "sensible" definition one would hope that:

$$\operatorname{sgn} L = \operatorname{sgn} Q$$

$$|\det L| = (\det B^*B)\,|\det Q| \tag{iii}$$

So, if we can make sense of det L, det B*B , then we can write down the local contribution to the stationary phase approximation.

Here we have left out the level k ; see later. In finite dimensions, such a factor changes the resultant integral by an appropriate power of k , which we will see in our case is zero .

Regularisation of determinants and signatures

Let Δ be a Laplace operator, with positive eigenvalues λ. Then we can define the zeta function:

$$\operatorname{Tr} \Delta^{-s} = \sum_\lambda \lambda^{-s} = \zeta(s) .$$

The function $\zeta(s)$ is a meromorphic function, defined in the first instance for $\mathcal{R}(s)$ sufficiently large. It can be analytically continued to the whole complex plane, leaving isolated poles. Here $s = 0$ is not a pole, and $\zeta(0), \zeta'(0)$ are well-defined.

Formally, $\zeta(0)$ is the dimension of the Hilbert space. In odd dimensions, $\zeta(0) = 0$.

<u>Ray-Singer definition $\det \Delta = \exp(-\zeta'(0))$</u>

Formally, one sees that:

$$\zeta'(0) = \sum_\lambda d/ds(\lambda^{-s})\big|_{s=0}$$

$$= \sum_\lambda (-\log \lambda)$$

and thus $\exp(-\zeta'(0)) = \prod_\lambda(\lambda) = \det \Delta$.

The above definition makes sense as a real number, and is used by physicists to make sense of Gaussians occurring in QFT. We wish to do this for the Laplacian with twisted coefficients, Δ^0_α. This makes $\det(\Delta^0_\alpha)$ well-defined, and thus gives the B*B term.

Similarly, $L^2 = \Delta^0 \oplus \Delta^1$, the direct sum of the Laplace operators on Ω^0, Ω^1. Thus

$$(\det L)^2 = (\det \Delta^0_\alpha) \cdot (\det \Delta^1_\alpha)$$

and hence $|\det L|$ is well-defined, giving $|\det Q|$, from (iii).

Thus one can evaluate (i), obtaining:

$$\underline{\underline{\frac{(\det \Delta^0_\alpha)^{\frac{3}{4}}}{(\det \Delta^1_\alpha)^{\frac{1}{4}}}}}$$

Ray-Singer proved that:

$$T_\alpha = \frac{(\det \Delta_\alpha^0)^{3/2}}{(\det \Delta_\alpha^1)^{\frac{1}{2}}}$$

the square of the above expression, is independent of Riemannian metric. The Riemannian metric is used to obtain a *-operator, which is necessary to make sense of the divergent quantities.

To prove independence of metric, one differentiates T_α with respect to the metric as parameter, and shows that this vanishes. Ray and Singer conjectured that T_α was the classical Reidemeister torsion. This conjecture was proved (independently) by Cheeger and Müller. This is the first concrete encouragement for the Witten formula for $Z(M)$: the limit $k \to \infty$ can be regularised, and the result is metric independent. This observation relating Ray-Singer torsion to the abelian Chern-Simons theory was made by A. Schwarz about 10 years ago, in the Abelian theory (and it extends fairly easily to the non-Abelian case).

Phase factor

We now consider the phase factor as given by (ii). This is given by $(\operatorname{sgn} Q)$, which is related to $(\operatorname{sgn} L)$. This situation was studied by Atiyah-Patodi-Singer.

Consider the situation where L is a self-adjoint operator with both positive and negative eigenvalues, and:

$$\Delta = L^2 .$$

Define:

$$\eta(s) = \sum_{\lambda \neq 0} (|\lambda|^{-s} \operatorname{sgn} \lambda) .$$

Once again, η can be analytically continued, and $\eta(0)$ is well defined. Formally,

$\eta(0)$ = (no. of +ve eigenvalues) - (no. of -ve eigenvalues)

and it is thus natural to define $\operatorname{sgn} L = \eta(0)$. Note however that this quantity is a real number, not an integer. Thus the resulting phase in (ii) will not be a root of unity in general.

Then we have:

$$\operatorname{sgn} Q = \eta_\alpha(0)$$

where η_α is the η-function associated with L_α . We now have to investigate how $\operatorname{sgn} Q_\alpha$ depends on the metric.

Here α is a representation of $\pi_1(M)$, with no cohomology. Consider the trivial representation, and put

$$\tilde{\eta}_\alpha(0) = \eta_\alpha(0) - \eta_d(0)$$

where $\eta_d = d\,\eta_1$ and η_1 corresponds to ordinary differential forms, without group fibres; d is the dimension of our Lie group. Then:

1) $\tilde{\eta}_\alpha$ is independent of metric
2) $\tilde{\eta}_\alpha(0) = 4/\pi\ \delta(G)\, L(\alpha)$

where $\delta(G)$ is a numerical invariant of G (it is n for $SU(n)$: in general it depends on the value of the Casimir in the adjoint representation) and L is the Chern-Simons functional.

Thus we obtain from the stationary phase formula:

$$\sum_\alpha (\text{contrib. at } \alpha) = A(\tilde{\Sigma})$$

where A is a fixed multiplier, coming from η_d. A contains the only metric dependence in the formula; $\tilde{\Sigma}$ is metric independent. The phase factor is independent of G and the chosen representation, but depends on the choice of ground metric. The above formula for $\tilde{\eta}_\alpha(0)$ leads to a shift

$$e^{ikL(\alpha)} \longrightarrow e^{i(k+\delta)L(\alpha)}$$

in the exponential multiplier arising from the value of the action at the critical point α . Such a shift is well-known to physicists in various guises.

Comments

1) If we had succeeded in making an expression for $Z(M)$ independent of metric, we would have shown, for large k , how to make sense of the determinants and signatures by regularising. This is very nearly true, but not quite - we have a phase ambiguity.

All the techniques in this theory have counterparts in the Hamiltonian theory. The phase ambiguity corresponds to anomalies and central extensions in the Hamiltonian theory. This relation with the Hamiltonian theory was developed in some earlier ideas of Witten and rigorously proved by Bismut and Freed.

2) We have seen that one can carry out the stationary phase approximation, giving an answer in the large k limit. The question is: can we make sense of the functional integral for finite k , in a rigorous way? It is fairly clear that a direct attempt at the analysis is extremely difficult.

The only other way would seem to be to triangulate M , and then try to find $Z(M)$ by purely combinatorial methods. Such a programme would involve thinking of connections as assignments:

edge $\rightarrow$ group element

(discrete analogue of connection). Using a discrete analogue of gauge theory, one should express the Chern-Simons functional in a purely combinatorial framework. If this were possible, then the functional integral would make sense: one would be integrating over some power of G. One would then have to check that this was a topologically invariant definition etc.

Some encouragement comes from the fact the the Reidemeister torsion already has a combinatorial definition in terms of triangulation.

The first question to be answered is how to define the Chern-Simons functional for flat connections (flat meaning product 1 around any triangle in the triangulation of M), in purely combinatorial terms.

These problems are likely to be related to problems in four dimensional manifold theory. In particular there is an interesting formula of Gelfand for the signature of a 4-manifold as a sum over the vertices of a triangulation of some fairly complicated expression involving angles of intersection of faces at vertices.

If such a combinatorial approach to the Chern-Simons theory is possible, it should be related to the 2 dimensional statistical mechanical models which have been used by Kauffman and others in relation to the Jones polynomial.

Seminar 7

Ruth Lawrence: Computing the Invariants

So far in these seminars, we have seen how the functional Z has been constructed. In this seminar we shall see how invariants of knots, and of 3-manifolds, can actually be calculated.

Recall that, in Witten's theory, we have functor Z :

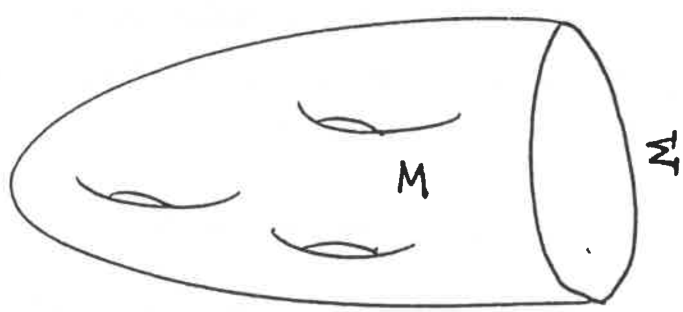
(framed) surface $\Sigma \to Z(\Sigma)$, a finite dimensional vector space

(framed 3-manifold $M \to Z(M)$, a vector in $Z(\Sigma)$ with $\partial M = \Sigma$

which request as initial data, a compact Lie group G , and an integer, k , the level. More generally, if (framed) marked points $P_1,\ldots,P_n$ are given on Σ , then we associate a finite dimensional vector space :

$$Z(\Sigma;P_1,\ldots,P_n)$$

so long as data of n representations of G are given, one representation R_i of G being given for each marked point P_i .

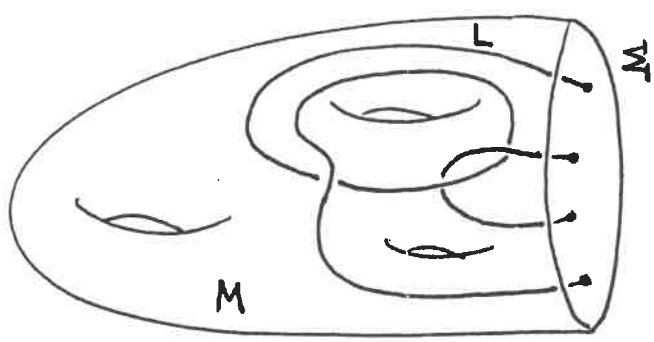


Also, if M is a 3-manifold containing an oriented (framed) curve, L, with:

$$\partial M = \Sigma$$

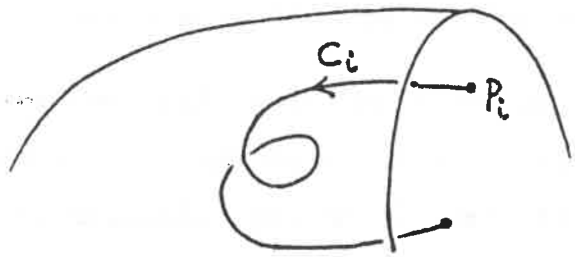
$$\partial L = \{P_1,\ldots,P_n\} \subseteq \Sigma$$

then we associate to M, L, a vector $Z(M,L)$ in the vector space $Z(\Sigma,P_1,\ldots,P_n)$, so long as initial data is given: a representation of G for each component of L. When such initial data is given, with the pair (M,L), the vector $Z(M,L)$ lies in the vector space associated with the surface Σ and marked points $P_1,\ldots,P_n$, where the representations of G associated with the P_i being given by those associated with the components of L.



Thus, for each i, P_i must be an end-point of a component, C_i, of L. If R_i' is the representation associated to C_i, then the data given on Σ associates to P_i the representation R_i:

$$R_i = \begin{cases} R_i' & \text{if } P_i \text{ is at the end of } C_i \\ \overline{R}_i' & \text{if } P_i \text{ is at the beginning of } C_i \end{cases}$$



where $\overline{R}$ denotes the dual representation to R.

If L is a knot, it can be thought of as embedded in S^3, and then:

$$Z(S^3,L)$$

is a vector in:

$$Z(\partial S^3,\partial L) = Z(\phi,\phi) = \mathbb{C}$$

i.e. $Z(S^3,L)$ is a complex number, an invariant of the link $L \in S^3$. Similarly, at the other extreme, if M is a closed manifold,

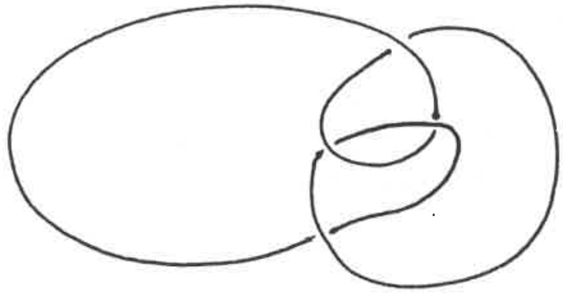
$$Z(M,\phi) = Z(M)$$

is a vector in $Z(\partial M) = Z(\phi) = \mathbb{C}$ i.e. a complex number.

In this lecture we shall investigate various properties of those invariants of closed manifolds, and of links, obtained by chopping up manifolds in different ways.

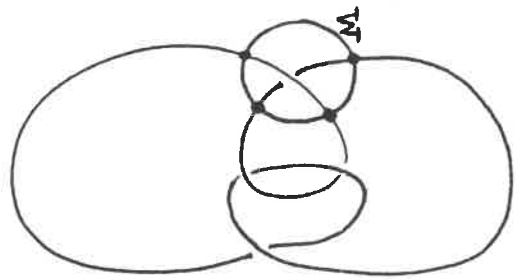
<u>Invariants of links</u>

Consider a link $L \subseteq S^3$.

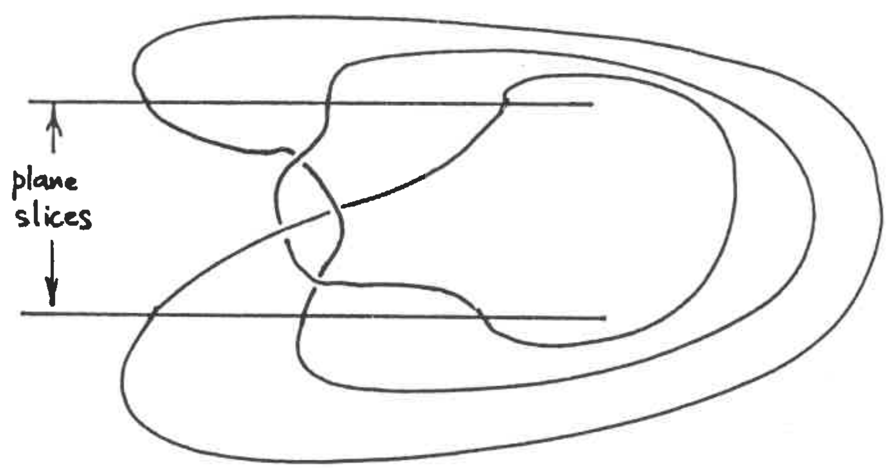


There are two ways of viewing this situation.

1) Consider L as specified by a plane projection, with over and under crossings. Pick one of these crossings, and consider cutting out a small disc around this crossing from the plane projection. This is equivalent to cutting the pair (S^3,L) with a small sphere Σ which cuts L in precisely four points.



2) Consider the link L as the closure of some braid, and then cut (L,S^3) by planes as shown below:



We shall now consider these two points of view in turn, and from each, we shall obtain interesting results about the knot invariant $Z(S^3,L)$.

Skein relation

In the general theory, if Σ cuts (M,L) into (M_1,L_1) and (M_2,L_2), with:

$$\partial L_1 = \partial L_2 = \{P_1,\ldots,P_n\} \subseteq \Sigma$$

then (M_1,L_1) distinguishes a vector $Z(M_1,L_1)$ in

$$Z(\Sigma,P_1,\ldots,P_n) .$$

also $Z(M_2,L_2)$ is a distinguished vector in:

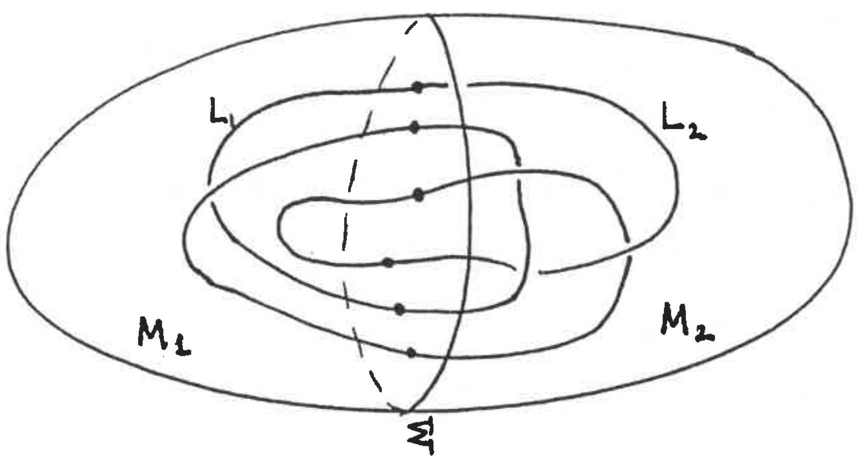
$$Z(\overline{\Sigma},P_1,\ldots,P_n) = Z(\Sigma,P_1,\ldots,P_n)^*$$

where $\overline{\Sigma}$ is Σ with opposite orientation.

Then $Z(M,L)$ can be calculated by:

$$Z(M,L) = \langle Z(M_1,L_1),Z(M_2,L_2)\rangle$$

where $\langle,\rangle$ is the natural contraction.



However, in the case of :

$$\Sigma = S^2$$

with marked points $P_1,\ldots,P_n$, and associated representations $R_1,\ldots,R_n$, the associated vector space is given by:

$$Z(\Sigma) = (R_1 \otimes \ldots \otimes R_n)^G$$

(where the superfix G denotes the invariant part) for sufficiently large k. When k is smaller, $Z(\Sigma)$ may be smaller than the above space.

Thus, in particular, for two marked points P_1, P_2,

$$Z(\Sigma) = \begin{cases} 0 & \text{if } R_1 \not\cong R_2^* \\ \mathbb{C} & \text{if } R_1 \cong R_2^* \end{cases} .$$

For four marked points, with associated representations R_1, R_2, R_1^*, R_2^*, we have:

$$Z(\Sigma) = (R_1 \otimes R_2 \otimes R_1^* \otimes R_2^*)^G .$$

Thus, if $R_1 \otimes R_2$ is expanded as a sum of irreducible reprentations:

$$R_1 \otimes R_2 = \bigoplus_{i=1}^{s} (R^{(i)})$$

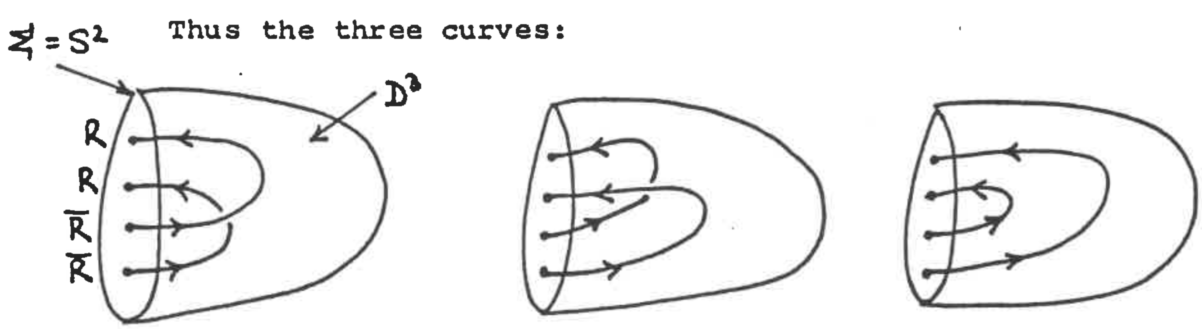
then (if the $R^{(i)}$ are distinct) there are s ways to contract $R_1 \otimes R_2 \otimes R_1^* \otimes R_2^*$, and thus:

$$\dim(Z(\Sigma)) = s$$

for large k.

In the case of $G = SU(n)$, with R being the standard representation, $R \otimes R$ is a sum of two irreducible representations (symmetric and anti-symmetric parts), and thus $Z(\Sigma)$ is 2-dimensional, for large k.

Thus the three curves:

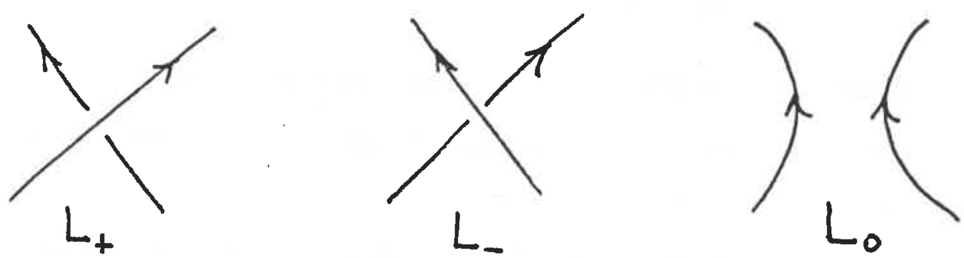


specify three vectors, V_+ , V_- , V_0 say, in the vector space $Z(\Sigma;$ 4 points). Since this vector space is two dimensional, these vectors are linearly dependent, i.e.

$$\alpha V_+ + \beta V_0 + \gamma V_- = 0$$

some α, β, γ , not all zero.

Consider three links L_+, L_0, L_- which are identical, except in a neighbourhood of one crossing, where they look like:



Then by cutting out the part of the link within a small sphere Σ around the crossing, we obtain identical curves, so that:

$$Z(S^3, L_+) = \langle V, V_+ \rangle$$

$$Z(S^3, L_-) = \langle V, V_- \rangle$$

$$Z(S^3, L_0) = \langle V, V_0 \rangle$$

where V is the vector in $Z(\bar{\Sigma};$ 4 points) corresponding to the part of the link outside Σ. Thus:

$$\alpha Z(S^3, L_+) + \beta Z(S^3, L_0) + \gamma Z(S^3, L_-) = 0 \; .$$

This relation is called a Skein relation.

The coefficients α, β, γ can be calculated by using conformal field theory, and the result obtained for $SU(N)$ is:

$$-q^{N/2} \, Z(L_+) + (q^{\frac{1}{2}} - q^{\frac{1}{2}}) Z(L_0) + q^{-N/2} \, Z(L_-) = 0$$

where:

$$q = \exp(2\pi i/(N+k)) .$$

This relation is similar to the skein relation satisfied by the 1-variable Jones polynomial, namely that

$$-tV_{L_+} + (t^{\frac{1}{2}} - t^{\frac{1}{2}})V_{L_0} + t^{-1}V_{L_-} = 0 .$$

Hence, it is clear that, up to a factor, $Z(L)$ for $SU(2)$ is simply $V_L(q)$ evaluated at:

$$q = \exp(2\pi i/(k+2)) .$$

Note that shift '+2' in the denominotor (or '+N' in the general case) : this corresponds to the shift $\delta(G)$ mentioned in the previous seminar.

In the case of $G = SU(N)$, α, β, γ can still be calculated. They do not correspond to values of the 1-variable Jones polynomial, but this time correspond to values of the 2-variable generalization.

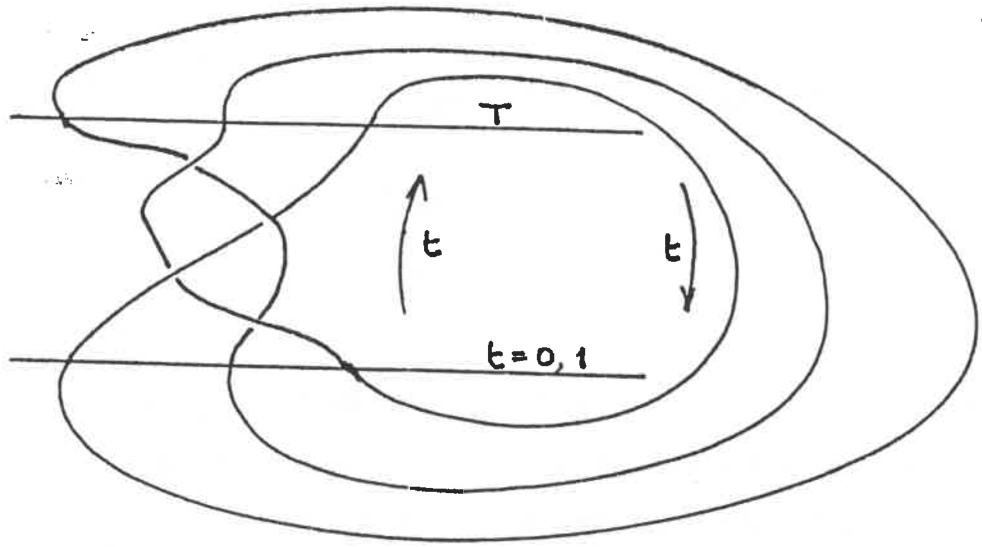
Braid approach

Alexander's theorem states that any link, L, is isotopically equivalent to the closure of a braid $\beta \in B_n$, for sufficiently large n. A braid is, by definition, a map:

$$\beta : [0,1] \to (\text{configuration space of } n \text{ points in } \mathbb{C})$$

with $\beta(0) = \beta(1)$ i.e. β is a loop in the configuration space of n (unordered) points in $\mathbb{C}$.

Let V be the vector space associated with $\Sigma = S^2$ and n marked points. Since the braid is going to be closed, we shall consider only the case in which the same representation R_i is associated with all the marked points on the same component of L.



To each pair of horizontal slices through the braid we have associated a map:

$$V \to V$$

given by Z . In particular $\beta \in B_n$ induces a map $\beta^* : V \to V'$. The process of closure also gives a map $V \to V$. Let $t \in S$ be a parameter on the closed braid extending the natural parameter on the braid. Then we can think of a time evolution given by the braid; starting from $\underline{v}_0 \in V$ at $t = 0$, as t increased, we assign

$$\underline{v}_t = \beta_t^*(\underline{v}_0)$$

where $\beta_t^* : V \to V$ corresponds to the section of the braid between 0 and t .

The time-evaluation is, in the Hamiltonian framework governed by a zero Hamiltonian. Thus for $t > T$,

$$\underline{v}_t = \underline{v}_T$$

and hence the invariant is :

$$\sum_{\underline{v}_0} \langle \underline{v}_0 | \underline{v}_1 \rangle = \sum_{\underline{v}_0} \langle \underline{v}_0 | \underline{v}_T \rangle$$

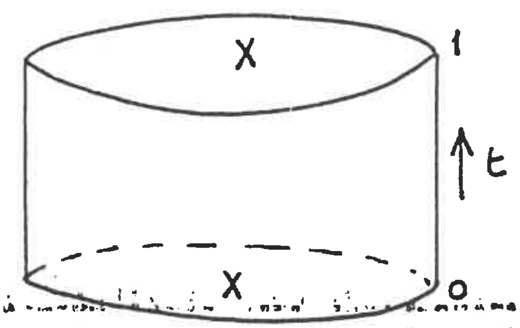
$$= \sum_{\underline{v}_0} \langle \underline{v}_0 | \beta^* (\underline{v}_0) \rangle$$

$$= \operatorname{Tr} \beta^* .$$

Note that in the skein theory it is most natural to think of L as embedded in S^3, whereas in the braid theory, it is most natural for L to be embedded in $S^1 \times S^2$.

<u>Relations between S^3 and $S^1 \times S^2$</u>

Consider a manifold $S^1 \times X$. Let H_X be the Hilbert space corresponding to X. Then on $[0,1] \times X$, we can propagate vectors from H_X to itself; however, since the Hamiltonian vanishes, this map is the identity.



Thus $Z(X \times S^1) = \operatorname{Tr}(\mathrm{Id}_{H_X})$

$$= \dim H_X .$$

In particular, $H_{S^2} = \mathbb{C}$, and thus:

$$\underline{\underline{Z(S^2 \times S^1) = 1}} .$$

Since it is so easy to evaluate $Z(S^2 \times S^1)$, we shall evaluate $Z(S^3)$ by obtaining S^3 from $S^2 \times S^1$ using surgery. Now S^3 is given by:

$$\{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

Let $T = \{(z_1,z_2) \in S^3 : |z_1| \leq 1/\sqrt{2}\}$

$T' = \{(z_1,z_2) \in S^3 : |z_2| \leq 1/\sqrt{2}\}$.

Clearly $T \cup T' = S^3$, and the boundary of T is:

$$\{(z_1,z_2) \in S^3 : |z_1| = 1/\sqrt{2}\} = S^1 \times S^1$$

a 2-torus. Thus T is a solid torus; similarly for T', the map identifying T with T' being

$$(z_1,z_2) \rightarrow (z_2,z_1) .$$

However, topologically, $T = D^2 \times S^1$, and thus gluing two such solid tori together, identifying boundary points one obtains

$$S^2 \times S^1 .$$

Let v, w be the vectors in $Z(T^2)^*$, $Z(T^2)$ associated with the solid tori making up $S^2 \times S^1$, whose boundaries are T^2, respectively. Then:

$$Z(S^2 \times S^1) = \langle v|w\rangle .$$

However, as we have seen, S^3 is obtained by gluing two solid tori together, with a non-trivial diffeomorphism $S^1 \times S^1$. Thus:

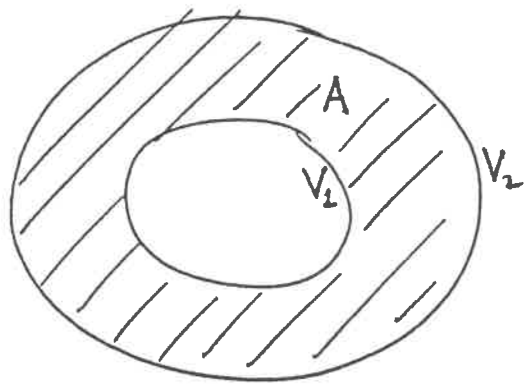
$$Z(S^3) = \langle v|S|w\rangle$$

where $S ; Z(T^2) \to Z(T^2)$ is the map corresponding to this diffeomorphism.

The vector space for a torus is obtained from that for an annulus:

$$Z(T^2) = \bigoplus_{V} Z(A, V^*, V)$$

since T^2 is obtained from an annulus by gluing the boundaries together.



However, $Z(A, V_1, V_2) = \begin{cases} 0 & \text{if } V_1^* \neq V_2 \\ \mathbb{C} & \text{if } V_1^* = V_2 \end{cases}$

as was seen in seminar 4. Thus:

$$Z(\text{torus}) = \bigoplus_{V} (\mathbb{C})$$

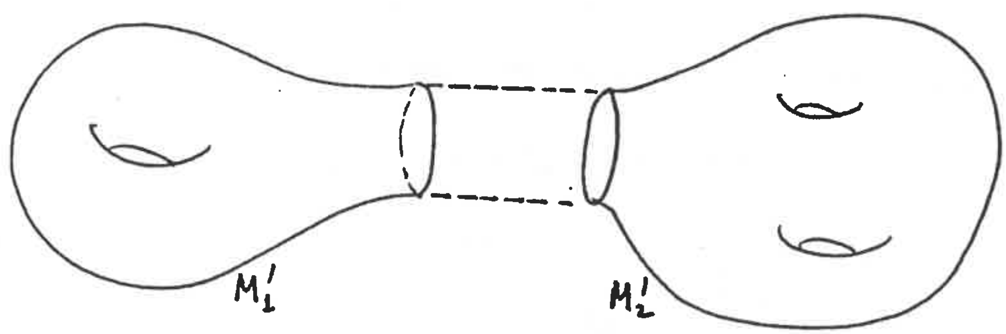
i.e. its dimension is the number of labels allowed (limited by the level k) .

The value of $Z(S^3)$ can then be calculated from the matrix of the map S . When this is done for $G = SU(2)$, one obtains:

$$Z(S^3) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right)$$

Note, once again the shift by 2 in the 'effective' value of k. Also observe that Z is normalised with respect to $S^1 \times S^2$, but not with respect to S^3.

If M_1, M_k are two manifolds whose connected sum is M, then we have the following diagram.



Let Σ_1, Σ_2 be two copies of S^2 in M_1, M_2 respectively, so that they cut off 'caps' D^3:

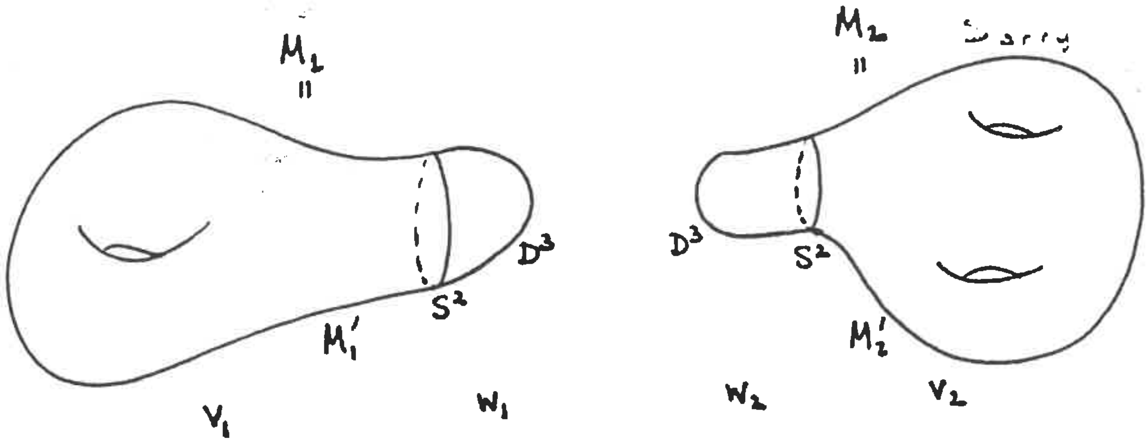
$$M_1 = M_1' \cup D^3$$

$$M_2 = M_2' \cup D^3 .$$

Then M is the union of M_1' and M_2'.

Let v_1, v_2 be vectors in $Z(\overline{S}^2) = Z(S^2)^*$, $Z(S^2)$ corresponding to M_1', M_2'; and let w_1, w_2 be vectors in $Z(S2)$, $Z(S^2)$ corresponding to the caps D^3 of M_1, M_2 respectively. Then:

$$Z(M_1) = \langle v_1 | w_1 \rangle$$
$$Z(M_2) = \langle w_2 | v_2 \rangle$$
$$Z(M) = \langle v_1 | v_2 \rangle$$
$$Z(S^3) = \langle w_2 | w_1 \rangle \quad \text{(since } S^3 \text{ is two copies of } D^3 \text{ glued together).}$$



However $Z(S^2) = \mathbb{C}$ is one dimensional, and thus:

$$\langle v_1 | w_1 \rangle \langle w_2 | v_2 \rangle = \langle v_1 | v_2 \rangle \langle w_2 | w_1 \rangle$$

i.e. $Z(M_1) \times Z(M_2) = Z(M) \times Z(S^3)$

$$= \frac{Z(M_1)}{Z(S^3)} \cdot \frac{Z(M_2)}{Z(S^3)} = \frac{Z(M)}{Z(S^3)}$$

i.e. it is not $Z(M)$ that is multiplicative under connected sum, but it is:

$$\frac{Z(M)}{Z(S^3)} .$$

Once again, normalisation with respect to S^3 does not come naturally, but has to be specifically introduced.

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