Moore's Seminar 1957-8.

Outline:

1. Hopf Algebras
2. Constructions of Cartan
3. Borel's Theorem (classifying space)
4. Calculus of Variations
5. Bott's results on homotopy of Lie Groups.

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1. K, comm. ring with unit. A, alg. over K, graded, $A = \sum_{n \geq 0} A_n$ ∋

$\phi : A \otimes A \to A$, associative if $\phi(1 \otimes \phi) = \phi(\phi \otimes 1)$.

A has a base point (or unit) if given

$\eta : K \to A$ $\ni$
$$\begin{tikzcd}
 & K \otimes A \arrow[r, "\eta \otimes 1"] & A \otimes A \arrow[dr, "\phi"] & \\
A \arrow[ur] \arrow[dr] & & & A \\
 & A \otimes K \arrow[r, "1 \otimes \eta"] & A \otimes A \arrow[ur, "\phi"] &
\end{tikzcd}$$
is id.

$\varepsilon : A \to K$ is an augmentation of alg. with unit if homom. of alg. with unit.

Dual notions: A coalg., graded, $\phi : A \to A \otimes A$, assoc. if

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A \otimes A \arrow[d, "\phi \otimes 1"] \\
A \otimes A \arrow[r, "1 \otimes \phi"'] & A \otimes A \otimes A
\end{tikzcd}$$
is comm.. $\varepsilon : A \to K$ is a unit if

$$A \xrightarrow{\phi} A \otimes A \xrightarrow[\varepsilon \otimes 1]{1 \otimes \varepsilon} \begin{matrix} A \otimes K \\ K \otimes A \end{matrix} = A$$ is id.

An aug. is $\eta : K \to A$ of coalg. with unit.

A, $A^* = \text{Hom}(A, K)$, graded, i.e. $A^* = \sum_{n \geq 0} \text{Hom}(A_n, K)$. If A is alg.,

$A \otimes A \to A$, $\therefore A^* \to (A \otimes A)^* \overset{=}{\underset{\text{if}}{}} A^* \otimes A^*$, get coalg. & vice versa.

Hence suppose ~~K a principal ideal domain~~, A is free & locally finite dim'l.

A assoc. as alg $\Leftrightarrow$ A^* assoc. (as coalg).

A has unit $\Leftrightarrow$ A^* has a unit

A aug. $\Leftrightarrow$ A^* aug.

$T : A \otimes A \to A \otimes A$, $T(x \otimes y) = (-1)^{pq}\, y \otimes x$, $x \in A_p$, $y \in A_q$.

A comm. if $\phi T = \phi$, similar for coalg.

Def: A Hopf alg. is a graded module A,

$\phi : A \otimes A \to A$, aug. alg.

$\psi : A \to A \otimes A$, aug. coalg. $\ni$

$\eta : K \to A$ is a unit for alg. & a ~~suppl. for~~ aug. for coalg.

$\varepsilon : A \to K$ " " coalg " alg.

& ψ is a homom. of aug. alg.; i.e.

$$\begin{tikzcd}
A \otimes A \arrow[r, "\psi \otimes \psi"] \arrow[dd, "\phi"] & A \otimes A \otimes A \otimes A \arrow[d, "S"] \\
& A \otimes A \otimes A \otimes A \arrow[d, "\phi \otimes \phi"] \\
A \arrow[r, "\psi"] & A \otimes A
\end{tikzcd}$$

is comm.

(this also says ϕ is a homom of coalgs.)

If A is free, loc. finite dim Hopf alg, then so is A^*.

A Hopf alg. is connected if $\varepsilon : A_0 \xrightarrow{\approx} K$.

If A connected, dim $x > 0$, then $\psi(x) = x \otimes 1 + 1 \otimes x + \Sigma\, x_i' \otimes x_i''$

Suppose A is a Hopf alg., C a sub Hopf alg.

$A /\!/ C = A / \bar{C} A$, $\bar{C} = \operatorname{Ker} \varepsilon\ (C \to K)$. C is normal if

$\bar{C} A = A \bar{C}$.

Th: Suppose B is comm. Hopf alg, A a assoc. sub Hopf alg., $C = B//A$, A, B, C are free over K. Then B is a free module over A.

Proof: Let $\{x_i\}$ basis of C over K. $c_i \in B \ni c_i \to x_i$. Then $\{c_i\}$ is basis for B over A. Suppose $\sum a_i c_i = 0$, smallest top dim. $\overset{\nearrow}{(n)}$ of c_i.

$$\begin{tikzcd} B \arrow[r, tail] & B \otimes B \arrow[r] & B \otimes C \arrow[r] & B \otimes C \Big/ B \otimes \sum_{i=0}^{n-1} C_i \end{tikzcd}$$

$$a_i c_i \longrightarrow a_i \otimes \{x_i\}$$

$a_i = 0$ if $\dim x_i = n$, contrad.

$A \subset B$, $C = B//A = B/\bar{A}B$

$\pi : B \to C$, suppose $\exists$ $\alpha : C \to B$, K-module map $\ni$ $\pi\alpha = id$. $B = A \otimes C$ if B is assoc. connected

Proof:

$$\begin{tikzcd} A \otimes C \arrow[r, "i \otimes \alpha"] \arrow[rr, bend left] & B \otimes B \arrow[r] & B \arrow[r, "\psi"] & B \otimes B \arrow[r, "1 \otimes \pi"] & B \otimes C \end{tikzcd}$$

$b \in B_q$, $+ B_r = \operatorname{Im}(A \otimes C)_r$, $r < q$.

$\alpha\pi(b) - b = \sum a_i b_i$, $\dim a_i > 0$

$\therefore b_i = \sum a_{i,j} \alpha(c_j)$ $\therefore b = \alpha\pi(b) - \sum a_i a_{i,j} \alpha(c_j)$

$\therefore$ epi.

$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[rr, bend right, "f"'] & B \otimes B \arrow[r, "1 \otimes \pi"] & B \otimes C \end{tikzcd}$$

, f is map of left A-modules.

$a \in A_r$, $\psi(a) = a \otimes 1 + 1 \otimes a + \sum a_i' \otimes a_i''$

$(1 \otimes \pi)\, \psi(a) \cdot b_1 \otimes b_2 \overset{1 \otimes \pi}{=} (ab_1 \otimes b_2 + (-1)^{\alpha\beta_1} b_1 \otimes ab_2 + \sum (-1)\; a_i' b_1 \otimes a_i'' b_2)$

$= ab_1 \otimes \pi(b_2)$. If $b \in B_s$, $\psi(b) = b \otimes 1 + 1 \otimes b + \sum b' \otimes b''$

$\therefore f(ab) = a \cdot f(b)$. Now show $g : A \otimes C \to B$ is $\cong$.

Assume g mono on $A \otimes \sum_{r<n} C_r$.

Suppose $\sum a_i \otimes c_i \xrightarrow{g} 0$ & $\dim c_i \leq n$. $B \xrightarrow{f} B \otimes C \to B \otimes C / \sum_{r<n} C_r$ is map of left A-modules.

Assume $\dim c_i = \begin{cases} n & i > k \\ < n & i \leq k. \end{cases}$

$$1 \otimes c_i \longrightarrow 1 \otimes \alpha(c_i) \to \alpha(c_i) \longrightarrow \alpha(c_i) \otimes 1 + 1 \otimes \alpha(c_i) + \sum b'_{ij} \otimes b''_{ij}$$

If $\dim c_i < n$, all elts $\to 0$.

If $\dim c_i = n$, all but $1 \otimes \alpha(c_i)$ go to 0 &

$\therefore$ $1 \otimes c_i \to 1 \otimes c_i$ for $i > k$

0 for $i \leq k$.

$\therefore$ $\sum_{i \leq k} a_i \otimes c_i \to 0$ & $\sum_{i > k} a_i \otimes c_i \to \sum_{i > k} a_i \otimes c_i$ as map of A-modules

$\therefore$ if $g(\sum a_i \otimes c_i) = 0$, then $\sum_{i > k} a_i \otimes c_i = 0$; contradiction.

$\therefore$ $g : A \otimes C \xrightarrow{\cong} B$

Note: C is not nec. an alg., but is always a coalg.

$0 \to \bar{A} \to A \xrightarrow{\varepsilon} K \to 0.$ & $\bar{A} \approx A'$

$0 \to K \to A \to A' \to 0$

Define $P(A) = \mathrm{Ker}$ (primitive elts)

$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d] & \bar{A} \otimes \bar{A} \\
A \arrow[r] & A \otimes A \arrow[u]
\end{tikzcd}$$

i.e. $\psi(x) = x \otimes 1 + 1 \otimes x$, if A conn.

Define $Q(A) =$ Coker (indecomposible elts).

$$\begin{tikzcd}
\bar{A} \otimes \bar{A} \arrow[r] \arrow[d] & \bar{A} \\
A \otimes A \arrow[r] & A \arrow[u]
\end{tikzcd}$$

Th: $A \subset B$, $C = B/\!/A$, B comm, assoc., $\alpha : C \to B \ni$

$\pi\alpha = id.$, C torsion free. $\bar{A}B = B\bar{A}$. Then

$$0 \to P(A) \to P(B) \to P(C)$$

$$Q(A) \to Q(B) \to Q(C) \to 0.$$

Proof: $b \in B_q$, $b = \sum a_i\, \alpha(c_i)$

$[b] = [a + \alpha(c)]$. ~~Suppose~~ $[a + \alpha(c)] \longrightarrow [c] = 0$ means

$c = \sum c_i' \cdot c_i''$. $\left(\alpha(c_i')\right)\left(\alpha(c_i'')\right) = \alpha(c_i' c_i'') + \sum_{k} a_{ik} b_{ik}$

dim $a_i > 0$ because $\pi\alpha = id$.

$\therefore \sum_i \alpha(c_i')\alpha(c_i'') = \sum_i \alpha(c_i c_i'') + \sum_{i,k} a_{i,k} b_{i,k}$

$\therefore \alpha[0] = [\alpha(c) + a']$ ($b_{i,k}$ might be 1)

$\therefore [\alpha(c)] = -[a']$, $\therefore [b] = [a - a']$.

Suppose further that A, B, C are free + loc. finite dim'l.

Then $Q(A^*) = P(A)^*$ + $P(A^*) = Q(A)^*$. Use duality.

Otherwise: $b \in P(B)_q$, $b \to 0$ in $P(C)$. $b = \sum a_i\, \alpha(c_i)$, n = dim of highest c_i

dim $c_i = n$ $\quad i > k$

$< n$ $\quad i \leq k$

$$b \to b\otimes 1 + 1 \otimes b \to b \otimes 1 \longrightarrow 0$$

$$B \to B \otimes B \longrightarrow B \otimes C \longrightarrow B\otimes C \Big/ \sum_{r<n} C_r$$

$$\alpha(c_i) \longrightarrow 0 \quad i \leq k$$

$$1 \otimes c_i \quad i > k.$$

$\therefore \sum a_i\, \alpha(c_i) \longrightarrow \sum_{i>k} a_i \otimes c_i = 0$, $\therefore$ by induction, $b \in A$.

$$\begin{tikzcd} G \arrow[r] \arrow[rrr, bend right, "\varepsilon"'] & G \times G \arrow[r, "1\times -"] & G \times G \arrow[r] & G \end{tikzcd}$$, G a group.

A connected Hopf alg.; then $\exists!\ c : A \to A \ni$

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[rrr, bend right, "\eta\varepsilon"'] & A \otimes A \arrow[r, "1 \otimes c"] & A \otimes A \arrow[r, "\phi"] & A \end{tikzcd}$$

. c is uniquely defined in dim 0.

Inductively:

$x \in A_n$, $x \to x \otimes 1 + 1 \otimes x + \sum x_i' \otimes x_i''$

$c(x) = -x - \sum x_i' \, c(x_i'')$ + this gives inductive def.

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi"] \arrow[d, "T"] & A \arrow[d, "c"] \\ A \otimes A \arrow[d, "c \otimes c"] & A \\ A \otimes A \arrow[ru] & \end{tikzcd}$$

, i.e. c is anti-homom. or

$c(xy) = (-1)^{pq} c(y) c(x)$.

Proof by induction:

$x \to x \otimes 1 + 1 \otimes x + \sum x_i' \otimes x_i''$

$y \to y \otimes 1 + 1 \otimes y + \sum y_j' \otimes y_j''$.

$xy \to xy \otimes 1 + x \otimes y + \sum x y_j' \otimes y_j'' + (-1)^{pq} y \otimes x + 1 \otimes xy + \sum (-1)^{p q_j'} y_j' \otimes x y_j''$

$+ \sum (-1)^{q p_i''} x_i' y \otimes x_i'' + \sum x_i' \otimes x_i'' y + \sum (-1)^{q_j' p_i''} x_i' y_j' \otimes x_i'' y_j''$.

$\therefore xy + x\,c(y) + \sum x y_j' \, c(y_j'') + (-1)^{pq} y\, c(x) + c(xy) + \sum (-1)^{p q_j'} y_j' \, c(x y_j'')$

$+ \sum (-1)^{q p_i''} x_i' y \, c(x_i'') + \sum x_i' \, c(x_i'' y) + \sum (-1)^{q_j' p_i''} x_i' y_j' \, c(x_i'' y_j'') = 0$

Use induction hyp. + collect terms etc.

(easier proof: use $\xi(x, y) = y^{-1} x^{-1}$)

Also,
$$\begin{tikzcd} & A \arrow[rr, "c"] \arrow[ld, "\psi"] & & A \arrow[d, "\psi"] \\ A\otimes A \arrow[r, "c\otimes c"] & A\otimes A \arrow[rr, "T"] & & A\otimes A \end{tikzcd}$$
, dual diagram is comm.

Suppose ϕ or ψ is comm. Then $c^2 = 1$ &

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend right, "\eta\varepsilon"'] & A\otimes A \arrow[r, "c\otimes 1"] & A\otimes A \arrow[r, "\phi"] & A \end{tikzcd}$$
is comm.

Suppose A is Hopf alg. over field K of char. p.

Define $\xi : A \to A$, by $\xi(x) = x^p$ (then ξ is sub Hopf. alg. if K perfect & A comm.)

<u>Thm</u> Suppose $Q(A)_r = 0$, $r > n$, $Q(A)_r$ is finite dim'l, A commu., conn.

$\xi^f(\bar{A}) = 0$. Then $P(A)_r = 0$ if $r > p^{f-1} n$.

<u>Proof:</u> Suppose C is Hopf alg of 1 gen.

x, dim $x \le n$.

$x \to x\otimes 1 + 1\otimes x$, $x^i \Longrightarrow \sum_{s+t=i} (s,t)\, x^s \otimes x^t$

$\therefore$ primitive are x^{p^j} & assertion is true.

Induction on # of gens.

x, gen. of highest dim, $A' =$ sub Hopf alg. on rest of gen. $\subset A$

$$\begin{tikzcd} 0 \arrow[r] & P(A') \arrow[r] \arrow[d, equal] & P(A) \arrow[r] & P(A/\!/A') \arrow[d, equal] \\ & 0 & & \gamma \end{tikzcd}$$

K arbit. ring

A, B Hopf alg., conn., $A \subset B$, $C = B/\!/A$, A, B, C proj. over K.

$$\begin{tikzcd}
B \arrow[r] \arrow[rrr, bend right, "f"'] & B\otimes B \arrow[r] & B\otimes C \arrow[r] & B\otimes\overline{C} \arrow[dr] & \\
& & & & B\otimes C \Big/ \sum_{i<n} C_i
\end{tikzcd}$$

Then $A = \operatorname{Ker} f$.

Proof: $B = A\otimes C$ as left A-modules.

$b = \sum a_i x_i \xrightarrow{f} 0$, $\Rightarrow \sum a_i \otimes x_i = 0$ i.e., $\dim x_i \geq n$.

$\therefore \sum_{\dim x_i = n} a_i x_i = 0$, & by induction, we see $A = \operatorname{Ker} f$.

K perfect field of char. p. B is a comm. $\overset{\text{comm. assoc (as alg)}}{\wedge}$ Hopf alg. over K. A is a sub Hopf alg. of B. $Q(B)_r = 0$ if $r > n$. $C = B/\!/A$, C has 1 gen. x, $\dim x = n$. Assume also that

$$0 \to Q(A) \to Q(B) \to Q(C) \to 0.$$

Also, either p odd & n even or $p = 2$ & $x^{p^f} = 0$ for some f, $x^{p^{f-1}} \neq 0$.

Then $B = A \otimes C$ as an alg.

Proof: $\xi^f(A)$ is sub Hopf alg of A. $A^{\#} = A/\!/\xi^f(A)$,

$B^{\#} = B/\!/\xi^f(A)$, $B^{\#} \twoheadrightarrow C$ & in fact $B^{\#}/A^{\#} \twoheadrightarrow C$

$$\begin{tikzcd}
0 \arrow[r] & Q(A) \arrow[r] \arrow[d, "\approx"] & Q(B) \arrow[r] \arrow[d, "\approx"] & Q(C) \arrow[r] \arrow[d, "\therefore \approx"] & 0 \\
0 \arrow[r] & Q(A^{\#}) \arrow[r] & Q(B^{\#}) \arrow[r] & Q(B^{\#}/A^{\#}) \arrow[r] & 0
\end{tikzcd}$$

$\therefore B^{\#}/A^{\#} \approx C$.

Now, let $z \in \underset{B}{\text{[illegible]}}{}^{\#}_{n} \ni z \to x$. $z \to z\otimes 1 + 1\otimes z + \sum a_i' \otimes a_i''$ as elts of dim $< n$ are in $A^{\#}$.

$$z^{p^f} \to z^{p^f} \otimes 1 + 1 \otimes z^{p^f} + \underbrace{\sum a_i'^{\,p^f} \otimes a_i''^{\,p^f}}_{\overset{\shortparallel}{0} \text{ in } B^{\#}}$$

$\therefore z^{p^f}$ is primitive

$$0 \to \underset{\overset{\shortparallel}{0}}{P(A^{\#})_{p^f n}} \to \underset{\therefore \overset{\shortparallel}{0}}{P(B^{\#})_{p^f n}} \to \underset{\overset{\shortparallel}{0}}{P(C)_{p^f n}}$$

$\therefore z^{p^f} = 0$, $\therefore x$ can be lifted to elt. of right height in $B^{\#}$
must lift yet to B.

$B \to B^{\#}$, $w \in B_n$, $w \to z$.

$B \to B \otimes B \to B \otimes B^{\#}$, $w \to w \otimes 1 + 1 \otimes z + \sum w_i' \otimes [w_i'']$.

$\therefore w^{p^f} \to w^{p^f} \otimes 1 + 1 \otimes \underset{\overset{\shortparallel}{0}}{z^{p^f}} + 0$, $B \otimes B^{\#} \to B \otimes \overline{B}^{\#}$
$\to 0$

$\therefore w^{p^f} \in \xi^{\#}(A)$

$\therefore w^{p^f} = a^{p^f}$ $\ni$ $w - a \to z$ + is of right height.

Borel's Thm:

p odd.

Canonical Hopf alg.

$E(x, n)$, Grassman, n odd.

$P(y, n)$, polyn. , n even.

$P^f(y, n) = P(y, n)/[y^{p^f}]$

If A is a comm., conn. assoc. Hopf alg. which is loc. finite diml., K perfect field char. p, then as an alg., $A = \bigotimes_{i \in I} A_i$, where A_i is a canonical Hopf alg.

Proof: By induction. $A', A'' \subset A$, gen. of A' of dim $\leq n$.

$A' \subset A''$ & $A''/\!/A'$ has 1 gen. of dim n.

Case 1, $A'' = A' \otimes A''/\!/A'$, if quotient is exterior

Case 2, also by case $n=1$. Take direct limit.

In case $p=2$, no restriction on dim. of generators.

In case $p=0$, $E(x,n)$ n odd & assertion is again the same.
$P(y,n)$ n even

Samelson - Leray thm.

$E(x,n)$, n odd, $x \xrightarrow{\psi} x\otimes 1 + 1 \otimes x$

$E(x_1,n_1) \otimes \dots \otimes E(x_k, n_k)$

Thm (Samelson - Leray): Suppose A is an (both) assoc. Hopf alg. ∋ the algebra is a Grassman alg. with odd generators, then

$0 \to P(A) \longrightarrow Q(A) \to 0$ (i.e. A is primitively gen.).

If $A = E(x,n)$, $P(A) \xrightarrow{\cong} Q(A)$. Prove by induction.

$A \subset B$, A & B satisfying conditions of thm. &

$C = B/\!/A$ is grassman on 1 gen, assume A, C have right property.

$$\begin{tikzcd}
0 \arrow[r] & P(A) \arrow[r] \arrow[d, "\cong"'] & P(B) \arrow[r] \arrow[d] & P(C) \arrow[d, "\cong"] & \\
0 \arrow[r] & Q(A) \arrow[r] & Q(B) \arrow[r] & Q(C) \arrow[r] & 0
\end{tikzcd}$$

Immediate that middle is mono. by chasing.

Yet to put in $P(C) \to 0$.

Take dual of all in sight.

$$B^*/\!/C^* = A^*$$

$$\begin{tikzcd}
0 \arrow[r] & P(C^*) \arrow[r] \arrow[d, "\approx"] & P(B^*) \arrow[r] \arrow[d, "\text{epi}"] & P(A^*) \arrow[r] & 0 \\
& Q(C^*) \arrow[r] & Q(B^*) \arrow[r] & Q(A^*) \arrow[r] \arrow[u, "\approx"'] & 0
\end{tikzcd}$$

Recall Tor: $\quad A$ alg.,

$$\cdots \to P_2 \to P_1 \to P_0 \to B \to 0 \quad \text{graded modules.}$$

$P = \Sigma P_q$, bigraded, total degree is sum of 2.

$H(P \otimes_A C) = \text{Tor}^A(B,C)$, $H_q(P \otimes_A C) = \text{Tor}^A_q(B,C)$, a graded module.

$A \xrightarrow{\varepsilon} K$, $\therefore K$ is A-module. $\left(A \otimes_K \bar{A} \otimes_K \bar{A} \to A \otimes_K \bar{A} \to A \to K \to 0 \text{ is proj. resol.} \right)$

$\text{Tor}^A_1(K,K) = \bar{A}/\bar{A}^2 = Q(A)$ $\;\therefore$ above sequence is part of

$$\text{Tor}_2^{B^*}(K,K) \to \text{Tor}_2^{A^*}(K,K) \to \text{Tor}_1^{C^*}(K,K) \to \text{Tor}_1^{B^*}(K,K) \to \text{Tor}_1^{A^*}(K,K) \to 0$$

show 0.

(A proj. over K)

Suppose now that $A \subset B$ is also alg over K which is proj over K.

$\overset{\text{proj. over } K}{=} C = B/\bar{A}B$, $\bar{A}B = B\bar{A}$, $B = A \otimes C$ as a left A-module.

P ~~proj~~. $\&K$ over B, $B \otimes \bar{P} = A \otimes C \otimes \bar{P}$, $\bar{P}$ is free over K.

$H(K \otimes_A P) = \text{Tor}^A(K,K)$, $H(K \otimes_B P) = \text{Tor}^B(K,K)$

over C.

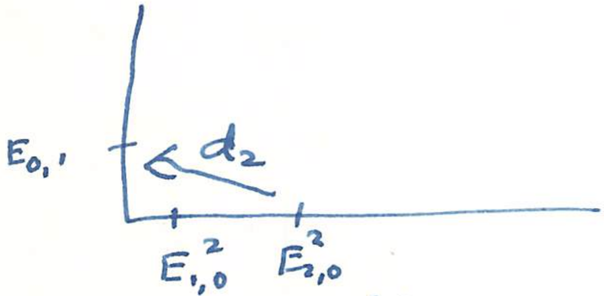
$R_1 \to R_0 \to K \to 0$

$R \otimes_B P$, double complex, filter 2 ways, 1 by R, $E^2 = \mathrm{Tor}^C(K, \mathrm{Tor}^A(K,K))$

other by P, $E^2 = \mathrm{Tor}^B(K,K) = E^\infty$,

$$d^r : E^r_{p,q,s} \longrightarrow E^r_{p-r,q+r-1,s}.$$

The exact sequence in low dim. is exactly what we want above.



If further, $A = E(x,n)$, $\mathrm{Tor}^A(K,K)$.

$$\Gamma(y, n+1)_q = \begin{cases} 0 & q \neq 0(n+1) \\ \text{free } K\text{-module on } \gamma_k(y) & \text{if } q = k(n+1) \end{cases}$$

$\gamma_0(y) = 1$, $\gamma_1(y) = y$, . . .

$E(x,n) \otimes \Gamma(y,n+1)$, $\quad d\gamma_i(y) = x\,\gamma_{i-1}(y)$, $dx = 0$

is a proj. resolution of K over $E(x,n)$.

Also, $\mathrm{Tor}^{A_1 \otimes A_2}(K,K) = \mathrm{Tor}^{A_1}(K,K) \otimes \mathrm{Tor}^{A_2}(K,K)$.

If A: grass. $x_1, \ldots, x_m$, dim odd

$\mathrm{Tor}^A(K,K) = \Gamma(y_1, n_1+1) \otimes \cdots \otimes \Gamma(y_m, n_m+1)$, all even dim'l.

If $A \subset B$, $B/\!/A = C$; A, C grass.

$E^2 = \mathrm{Tor}^C(K, \mathrm{Tor}^A(K,K)) = \mathrm{Tor}^C(K,K) \otimes \mathrm{Tor}^A(K,K)$, even dim'l,

$\therefore E^2 = E^\infty$, but map of interest is d_2 in spectral seq., $\therefore 0$.

(note C^*, A^* are grass. alg with odd gen. because C, A are primitively gen.)

(all fuss is to show B^* is comm.)

Constructions.

Intuitively, G group, $$\begin{tikzcd} E \arrow[d, "G"'] \\ B \end{tikzcd}$$, E acyclic, principal bdle.

K fixed, DGA alg. (analog of G) A is graded, aug. K-alg. (assoc.)

$A \otimes A \xrightarrow{\phi} A$, diff. op. d in A of deg -1, ϕ is map of

D. G. modules.

DGA module M over DGA alg. A.

M graded, $A \otimes M \xrightarrow{\phi} M$, M has d of deg -1, ϕ commutes with d.

Bar Construction: $0 \to \bar{A} \to A \xrightarrow{\varepsilon} K \to 0$.

Define $\mathbb{B}^k(A) = A \otimes \underbrace{\bar{A} \otimes \cdots \otimes \bar{A}}_{k\text{-times}}$ as a K-module

$$\dim a[a_1, \ldots, a_k] = \alpha + \sum_{i=1}^{k} \alpha_i + k,$$

$\mathbb{B}(A) = \sum \mathbb{B}^k(A)$, an A-module (free if A is free K-module with 1 as part of basis)

$S\, a[a_1, \ldots, a_k] = [a - \varepsilon(a), a_1, \ldots, a_k]$ & want $d \ni$

$dS + Sd = 1 - \varepsilon$ $\quad (\varepsilon = \eta\varepsilon)$.

$(dS + Sd)a = a - \varepsilon(a)$ $\quad a \subset \mathbb{B}^0(A) = A$

$\overset{\shortparallel}{d}[a - \varepsilon a] + [da]$ $\therefore d[a - \varepsilon a] = a - \varepsilon(a) - [da]$.

If $\varepsilon(a_1) = 0$, $d[a_1] = a_1 - [da_1]$.

by commuting with ϕ, $\overset{a[a_1]}{\shortparallel} d(a[a_1]) = da[a_1] + (-1)^\alpha aa_1 - (-1)^{\alpha} a[da_1]$.

$(dS + Sd)\, a[a_1] = [a - \varepsilon a, a_1] + [da, a_1] - (-1)^\alpha [a - \varepsilon(a), da_1] + (-1)^\alpha [a, a_1]$

or $d[a_1, a_2] = a_1[a_2] + (-1)^{\alpha+1}[a_1 a_2] - [da_1, a_2] - (-1)^{\alpha_1+1}[a_1, da_2]$.

& in general

$$d[a_1, \dots, a_k] = a_1[a_2, \dots, a_k] + \sum_{i=0}^{k-1} (-1)^{\alpha_1+\dots+\alpha_i+i} [a_1, \dots, a_i a_{i+1}, \dots a_k]$$
$$- \sum_{i=1}^{k} (-1)^{\alpha_1+\dots+\alpha_{i-1}+i-1} [a_1, \dots, da_i, \dots, a_k]$$

& $H(\mathcal{B}(A)) = K$. Note 2 parts of d.

$d'[a_1, \dots, a_k] = a_1[\ \cdots\ ] + \sum(-1)$ . . . .

$- 0$. & $d''[a_1, \dots, a_k] = - \sum(-1)$ . . . .

d'' is $-$ ⊗ product d if dim like in $\mathcal{B}^k(A)$. d' only has mult. of A.

If $d(A) = 0$, then $d'S + Sd' = 1 - \varepsilon$; same as before gives that $d'S + Sd' = 1 - \varepsilon$ generally. $\therefore d''S + Sd'' = 0$.

$d' : \mathcal{B}^k(A) \to \mathcal{B}^{k-1}(A)$, $d'a = 0$, $d''a = da$

$a[a_1, \dots, a_k] \longrightarrow (-1)^{\alpha}\ a \cdot d'[a_1, \dots, a_k]$

or $d'\, ax = (-1)^{\alpha} a\, d'x$

[$f : M \to M'$, $f(M_n) \in M'_{n+r}$ & $f(ax) = (-1)^{\alpha r} a f(x)$
is map of deg r of A-modules, ($df = (-1)^r d f$ if ad)

$\therefore d'$ is map of deg -1

$d'S + Sd' = 1 - \varepsilon$ so ap we have an exact seq.

$$\cdots \to \mathcal{B}^2(A) \xrightarrow{d'} \mathcal{B}^1(A) \xrightarrow{d'} \mathcal{B}^0(A) \xrightarrow{\varepsilon} K$$

& is a resolution of K over A (if $\bar{A}$ proj. K-module, all these are proj. A-modules) $[K \otimes_A \mathcal{B}(A) = \overline{\mathcal{B}}(A)]$

$d'' : \mathcal{B}^{b}(A) \to \mathcal{B}^{a}(A)$, & d' is a map of diff. A-modules, anticommuting with d'' ($d'd'' = -d''d'$). ∴ above is proj. resol. of K by diff. modules.

By proj. resol. of DG A-module M

$$\cdots \to P_1(\cancel{M}) \xrightarrow{d'} P_0(\cancel{M}) \to M \to 0$$

d'' not aug. deg -1

$d = d' + d''$

$d^2 = 0$ as $(d')^2 = 0$, $(d'')^2 = 0$.

$A_n \xrightarrow{\phi_n} A_{n-1} \to \cdots$

$C_n = \operatorname{Im}\phi_n$, $K_n = \operatorname{Ker}\phi_n$

$0 \to K_n \to A_n \to C_n \to 0$

$0 \to K_{n-1} \to A_{n-1} \to C_{n-1} \to 0 \quad \cdots$

original is exact if $C_n = K_{n-1}$.

K comm. ring with unit

Λ, alg. with unit over K. $\mathcal{M}_\Lambda$ = category of left Λ-modules,

$\mathcal{M}_K$, $T: \mathcal{M}_\Lambda \to \mathcal{M}_K$, obvious functor.

New notion of exact sequences.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "T"] & C \\ T(A) \arrow[r] & T(B) \arrow[r] & T(C) \end{tikzcd}$$

in $\mathcal{M}_\Lambda$ is short exact if

— splits.

This gives special class $\mathcal{E}$ of short exact sequences in $\mathcal{M}_\Lambda$.

Long exact seq. defined as above.

Case $\Lambda = A =$ DGA alg. over K.

$\mathcal{M}_A$ = DG. [illegible]-modules over A.

$\mathcal{M}_K$ = DG " " K.

$$C' \to C \to C'' \quad \text{exact if}$$

$$T(C') \to T(C) \to T(C'') \quad \text{splits in } \mathcal{M}_K.$$

Proj.

$$\begin{tikzcd} & 0 \arrow[d] & \text{in } \mathcal{E} \\ & C' \arrow[d] & \\ & C \arrow[d] & \\ P \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & C'' \arrow[d] & \\ & 0 & \end{tikzcd}$$

$\hat{f}$ in $\mathcal{M}_A$.

Given $B \in \mathcal{M}_K$, get $A \otimes_K B \in \mathcal{M}_A$, extended module.

Anything of form $A \otimes_K B$ is proj. (in new sense)(obvious)

Suff. many proj:

$C \in \mathcal{M}_A$, $A \otimes_K T(C) \xrightarrow{\alpha} C$

$a \otimes c \longrightarrow c$, $A \otimes_K T(C)$ is proj.

$$\begin{tikzcd}
\bar{A} \otimes_K T(C) \arrow[r, shift left, "\beta"] & A \otimes_K T(C) \arrow[l, shift left, "\beta'"] \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\alpha'"]
\end{tikzcd}$$

$\beta(a \otimes c) = a \otimes c - 1 \otimes ac$, then $\alpha\beta = 0$

$\alpha'(c) = 1 \otimes c$, $\alpha\alpha' = 1_C$, $\beta'(a \otimes c) = (a - \varepsilon(a)) \otimes c$

$\beta'\beta_{(a\otimes c)} = \beta'(a \otimes c - 1 \otimes ac) = a \otimes c$ or $\beta'\beta = 1$, $\beta'\alpha' = 0$.

$\beta\beta'(a \otimes c) = (a - \varepsilon(a)) \otimes c - 1 \otimes (a - \varepsilon(a))c$

$\alpha'\alpha(a \otimes c) = 1 \otimes ac$

$\therefore \beta\beta' + \alpha'\alpha = 1$. α is map of diff. modules.

$\alpha'(c) = 1 \otimes c$, $\alpha'(dc) = 1 \otimes dc$; α' " " " "

$$\begin{tikzcd}
c \arrow[d, "d"] \\
1 \otimes dc \arrow[r, equal] & 1 \otimes dc
\end{tikzcd}$$

$$\begin{tikzcd}
a \otimes c \arrow[r, "\beta"] \arrow[d, "d"] & a \otimes c - 1 \otimes ac \arrow[d, "d"] \\
da \otimes c + (-1)^\alpha a \otimes dc \arrow[r, "\beta"] & da \otimes c - 1 \otimes da \cdot c + (-1)^\alpha a \otimes dc + (-1)^{\alpha+1} 1 \otimes adc
\end{tikzcd}$$

$\therefore \beta$ is " " " "

$\therefore$ C is quotient in strong sense of proj.

If $C \in \mathcal{M}_A$, take proj. resol.

$0 \to \bar{A} \otimes TC \to A \otimes_K T(C) \to C \to 0$

$0 \to \bar{A} \otimes \bar{A} \otimes C \to A \otimes \bar{A} \otimes C \to \bar{A} \otimes C \to 0$ (⊗ over K, leave out T)

$0 \to \bar{A}^{n+1} \otimes C \to A \otimes \bar{A}^n \otimes C \to \bar{A}^n \otimes C \to 0$ etc.

$$\cdots \to A\otimes\bar{A}^2\otimes C \to A\otimes\bar{A}\otimes C \to A\otimes C \to C \to 0$$

$$\mathbb{B}(A)\otimes C$$

All have deg 0, want -1 so can define total degree; ∴ jack up dims.

(Given B, define B^+, ∋ $B_q \xrightarrow[\cong]{\phi} B^+_{q+1}$, & $\phi(dx) = -d\phi(x)$

$x \to (-1)^q x$

$$(A\otimes\bar{A}^2\otimes C)^{++} \longrightarrow (A\otimes\bar{A}\otimes C)^+ \to A\otimes C \to C \to 0$$

$\dim a[a_1]c = \dim a + \dim a_1 + \dim c + 1$

(If $C = K$, just get $\mathbb{B}(A)$). Call $B_A(C)$

Let C be a rt. DG. module over A

D " left " " " " , can define

$\mathrm{Tor}_{p,q}(C,D)$, $H_p(P(C)\otimes_A D) = \mathrm{Tor}_p(C,D)$ w.r.t. total $d = d' + d''$ in $P(C)\otimes_A D$

$\cong$

$H(P(C)\otimes_A P(D))$

$\cong$

$H(C\otimes_A P(D))$

$(P(C)\otimes_A D)_{r,s} = \sum_{i+j=s} P(C)_{r,i} \otimes_A D_j$

+ similar proof as before due to def. of exactness. Get usual exact seq. for Tor if start one is in $\mathcal{E}$.

K, A, DGA alg over K.

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \bar{A}\otimes M \arrow[r, "\beta"] & A\otimes M \arrow[r, shift left, "\alpha"] & M \arrow[l, shift left, "\alpha'"] \end{tikzcd}$$

$\beta(a\otimes m) = a\otimes m - 1\otimes am$, operators of A on $\bar{A}\otimes M$ are strange

$a'(a\otimes m) = a'a\otimes m - (a'-\epsilon(a'))\otimes am$ & then β is A-module hom.

M, rt. A-module, N, left A-module,

$P(M)$ proj. resol. of M

$Q(N)$ " " N

$\mathrm{Tor}^A(M,N) = H(P(M)\otimes_A Q(N))$, using total diff. op.

$$\longrightarrow P_2(M) \xrightarrow{d''} P_1(M) \longrightarrow P_0(M) \longrightarrow M \longrightarrow 0$$

d'

$d = d'+d''$, in $P(M)\otimes_A Q(N)$, $d(x\otimes y) = dx\otimes y + (-1)^{\dim x} x\otimes dy$.

Assume $P(M)$ is extended A-module, $P(M) = \tilde{P}(M)\otimes_K A$, not as d-mo

$$P(M)\otimes_A Q(N) = \tilde{P}(M)\otimes A\otimes \tilde{Q}(N)$$

Look at spectral sequences, $P(M) = \sum_{r,s} P(M)_{r,s}$, index r is low for loads

$$P(M)_{r,s}\otimes A_n \to P(M)_{r,s+n}$$

$Q(N) = \sum_{i,j} Q(N)_{i,j}$, index j is low for loads, $A_n\otimes Q_{i,j} \to Q_{i+n,j}$.

$(P \otimes_A Q)_{u,v,w} = \sum_{i+j=v} P_{v,i} \otimes Q_{j,w}$, trigraded

d has 3 parts, $d'' : P_{i,j} \to P_{i-1,j}$

$d' : P_{i,j} \to P_{i,j-1}$

$d' : Q_{r,s} \to Q_{r-1,s}$

$d'' : Q_{r,s} \to Q_{r,s-1}$

$d = d_1 + d_2 + d_3$

$d_1(x \otimes y) = d''x \otimes y$

$d_2(x \otimes y) = d'x \otimes y + (-1)^{\dim x} x \otimes d'y$

$d_3(x \otimes y) = (-1)^{\dim} x \otimes d''y$

all A-module homom.

$d_1 : (P \otimes_A Q)_{u,v,w} \to u-1, v, w$

$d_2 :$ " $\to u, v-1, w$

$d_3 :$ " $\to u, v, w-1$

$d_1(x \otimes ay) = d''x \otimes ay$

$d_1(xa \otimes y) = d''(xa) \otimes y =$

$(d''x)a \otimes y = d''x \otimes ay$.

$d_2(xa \otimes y) = d'(xa) \otimes y + \cdots$

$= d_2(x \otimes ay)$

6 spectral sequences.

I. Filtration deg = u.

$$\left(\hat{P}(M) \otimes A \otimes \hat{Q}(N)\right)_{u,v,w} = \sum_{i_1+i_2+i_3=v} \hat{P}(M)_{u,i_1} \otimes A_{i_2} \otimes \hat{Q}(N)_{i_3,w}$$

First diff. is $d_2 + d_3$

$$(d_2+d_3)(x \otimes a \otimes y) = d'x \otimes a \otimes y + (-1)^{\dim x} x \otimes d'a \otimes y + (-1)^{\dim x + \dim a} x \otimes a \otimes d'y$$

$+ (-1)^{\dim x + \dim a} x \otimes a \otimes d''y$. Can get further seq. to compute this one.

II. Filt. deg = u+v, first diff. op. is d_3.

$d_3(x \otimes a \otimes y) = (-1)^{\dim x + \dim a} x \otimes a \otimes d''y$

$E^1 = \hat{P}(M) \otimes A \otimes H(\tilde{Q}(N))$, if $\hat{P}(M), A$ are proj. over K.
w.r.t. d''

But $H(P(M)) \simeq H(M)$

$$\begin{tikzcd} P_2 \arrow[r, "d''"] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & M \end{tikzcd}$$

commutes with both d' + d''!

$H_{d''}(P(M)) = M \xrightarrow{\simeq} H_{d''}(M) = M$ + $H_{d'}(\quad) \simeq H_{d'}(M) = H(M)$
⊛ (use spectral seq., ≃ on E^1 level).

Also, $H(Q(N)) \simeq H(N)$.

So spectral sequences again.

I. Filter by $u+v$, $d^0 = d_3$, $E^1 = P(M) \otimes_A H^{d_0}(Q(N))$
$= P(M) \otimes_A N$ + $E^2 = E^\infty = H(P(M) \otimes_A N)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Tor}^A(M,N)$

II. $v+w$

$E^2 = E^\infty = H(M \otimes_A Q(N)) = \mathrm{Tor}^A(M,N)$
$\uparrow$
by degrees.

III $u+w$, $d^0 = d_2$, $E^1 = H^{d^2}(P(M) \otimes_A Q(N))$, but $P(M) = \hat{P}(M) \otimes_K A$
$\qquad\qquad\qquad\qquad\qquad \|$

$H^{d'}(\hat{P}(M) \otimes_K Q(N)) = H^{d'}(\hat{P}(M)) \otimes H^{d'}(Q(N)) = P(H(M)) \otimes_{H(A)} P(H(N))$
if K field (or $H(A)$ proj. over K) $\qquad$ + $d' = d_1 + d_3$

+ $E^2 = \mathrm{Tor}^{H(A)}(H(M), H(N))$

Z. B. $\mathrm{Tor}^{A}(K,K)$, ~~where~~ $= \mathrm{Tor}^{H(A)}(K,K)$ where

$H(A) = E(x_1, \dots, x_n)$, odd dim'l gen.

Proof: ~~If~~ $A = E(x,n)$, let $\Gamma(y,1,n)_q = \begin{cases} 0 & q \not\equiv 0\,(n+1) \\ \gamma_k(y), & q = k(n+1) \end{cases}$

$$0 \leftarrow K \leftarrow E(x,n) \leftarrow E(x,n) \otimes \Gamma(y,1,n)_{n+1} \leftarrow E(x,n) \otimes (\Gamma(y,1,n))_{2(n+1)} \leftarrow \cdots$$

or $E(x,n) \otimes \Gamma(y,1,n) = P(K)$.

$$K \otimes_{E(x,n)} \big(E(x,n) \otimes \Gamma(y,1,n)\big) = \overset{\text{,, } \Gamma(y,1,n)}{\mathrm{Tor}^{E(x,n)}(K,K)} \text{ as } d = 0.$$

If more than 1 gen., take $\otimes$ product

+ $\mathrm{Tor}^{E}(K,K) = \Gamma(y_1, \dots, y_n)$ + everything is even

dim. $\therefore E^2 = E^\infty$, $\therefore \mathrm{Tor}^{H(A)}(H(K), H(K)) \simeq \mathrm{Tor}^{A}(K,K)$

$$\overset{\text{,,}}{\mathrm{Tor}^{H(A)}(K,K)}.$$

$$\begin{tikzcd} P(K) \arrow[d] \\ \tilde{P}(K) \end{tikzcd}$$

is "fibre map" with fibre A.

$P(K) = A \otimes \hat{P}(K)$, filter by degree in $\hat{P}(K)$.

+ $d^0 =$ diff. in A., $E^1 = H(A) \otimes \hat{P}(K)$, if $\hat{P}(K)$ is projective.

+ $E^2 = H(H(A) \otimes \hat{P}(K))$.

Also, $A \otimes P(K) \to P(K)$, filtered as above.

$$\begin{tikzcd} H(A) \otimes E'(P(K)) \arrow[r] & E'(A \otimes P(K)) \arrow[r] & E'(P(K)) \arrow[d, "d'"] \\ & & \overset{d}{\hat{P}(K)} \end{tikzcd}$$

is map of spectral seq.

map.

$E^1_{*,0} = H_0(A) \otimes \tilde{P}(K)$. If $H_0(A) = K$ (connected),

then $E^1_{*,0} \cong \tilde{P}(K)$. $d^1(a\otimes x) = d^1(a(1\otimes x)) =$

$(-1)^{\dim a} a\, d^1(1\otimes x)$ + we get usual ~~of~~ diff. in $E^1(P(K))$.

If ~~$H_0(A)$~~ A is not connected, get local coeff. in $E^1(P(K))$,

$E^1 = H(A) \otimes_{H_0} E^1_{*,0}$ (as in universal covering spaces).

+ $d^1(a\otimes x) = d^1(a(1\otimes x)) = (-1)^{\dim a} a\, d^1(1\otimes x) = a \otimes d^1(1\otimes x)$ $\left(\otimes_{H_0(A)}\right)$.

When can you use N instead of $P(K)$?

Want: $P(K) \to N \ni H(\tilde{P}(K)) \cong H(K\otimes_A N)$.

1) $H(N) = K$, N a left A-module.

2) N has spectral sq. with ${}'E^1 = H(A) \otimes_{H_0(A)} {}'E^1_{*,0}$

+ filtration preserving map $P(K) \to N$

Then $\mathrm{Tor}^A(K,K) = H(K \otimes_{H_0(A)} {}'E^1_{*,0})$ $\left(= H(\tilde{N}) \text{ if } N = A\otimes \tilde{N}\right)$

Borel's thm. on transg.

E_r, $E_2 = H^*(A) \otimes H^*(B)$ $\ni$

fibre

$H^*(A) = E(x_1, \dots, x_n)$, $\dim x_i$ odd.

$d_r : E_r^{p,q} \to E_r^{p+r,\, q-r+1}$ (column).

+ $d_r(xy) = d_r x \cdot y + (-1)^{\dim x} x\, d_r y$.

+ $E_\infty = K$, then

$H^*(B) = P(y_1, \dots, y_n)$ & $\sigma(y_i) = x_i$.

(Last time, A DGA alg. $\ni H(A) = E(x_1, \dots, x_n)$

$$E^2 = \operatorname{Tor}^{H(A)}(K,K), \quad E^\infty = E^0(\operatorname{Tor}^A(K,K)) \quad \& \quad E^2 = E^\infty.$$

& $\operatorname{Tor}^{H(A)}(K,K) = \Gamma(y_1, \dots, y_n)$. $\therefore$ right additive structure.

(Must study diag. map.

$E^0 H^*(\bar{B}) = P(y_1, \dots, y_n)$ & if $H^*(\bar{B})$ is comm., then

$H^*(B)$ is a poly. ring.

Proof: $F^p \supset F^{p+1} \dots$

$E^0 = P(y_1, \dots, y_n)$

$\ni F^0/F^1 = M(y_1, \dots, y_n)$, free over K.

$$H^*(B) \underset{\leftarrow \cdots ?}{\longrightarrow} M(y_1, \dots, y_n) \to 0$$

$\therefore$ extend to map $P(y_1, \dots, y_n) \to H^*(B)$ & is $\cong$.

because $H^*(B)$ is commutative.

But A is Hopf alg., $A \to A \otimes A \to A$ $\quad H(A) = E(x_1, \dots, x_n)$

assoc. up to homotopy,

$H(A) \to H(A) \otimes H(A) \to H(A)$ & apply Samelson Leray,

$\updownarrow$

assoc.

$$P(H(A)) \xrightarrow{\cong} Q(H(A))$$

& $\therefore H^*(A) = E(x_1, \dots, x_n)^*$ (can choose $x_i \ni x_i \to x_i \otimes 1 + 1 \otimes x_i$)

Explicitly, $E(x,n) \otimes \Gamma(y,1,n)$, $d\gamma_i(y) = x\gamma_{i-1}(y)$, $dx = 0$.

$\Gamma(y,1,n) = \operatorname{Tor}^{E(x,n)}(K,K)$

Put in diag. map.

$$E(x)\otimes\Gamma(y) \to (E(x)\otimes\Gamma(y))\otimes(E(x)\otimes\Gamma(y))$$

if

$$\begin{tikzcd} y \arrow[r] \arrow[d, "d"] & y\otimes 1 + 1\otimes y \arrow[d, "d"] \\ x \arrow[r] & x\otimes 1 + 1\otimes x \end{tikzcd}$$

& send

$$\gamma_k(y) \to \sum_{i+j=k} \gamma_i(y)\otimes\gamma_j(y)$$

& is a diff. map.

$$x\gamma_k(y) = \sum x\,\gamma_i\otimes\gamma_j + \sum \gamma_i\otimes x\,\gamma_j$$

Collapse, $\Gamma(y) \to \Gamma(y)\otimes\Gamma(y)$ is usual & dual is poly. ring.

Do same for n generators, get dual a polyn. ring; moreover, in $\Gamma(y_1, \dots, y_n)$, each gen. is of filtration 1 & can use remarks above.

Now note that this Δ is a uniquely given map for DGA algs. & gives cup product in dual.

if $A \to A\otimes A$, then

$$\begin{matrix} \mathrm{Tor}^{A}(K,K) & \longrightarrow & \mathrm{Tor}^{A\otimes A}(K,K) \\ \| & & \| \\ H(\overline{\mathbb{B}}(A)) & & H(\overline{\mathbb{B}}(A))\otimes H(\overline{\mathbb{B}}(A)) \end{matrix}$$

if not torsion

K, comm. ring with unit,

A = DGA alg. over K.

M a DGA module over A, M filtered, $F_p M \subset F_{p+1} M \subset \dots$,

$F_p M$ is an A-submodule of M, $\cup F_p M = M$.

$E^0_{p,q} = \left(F_p M / F_{p-1} M\right)_{p+q}$, $E^1_{p,0}$ is an $H_0(A)$-module.

$$A \otimes M \to M$$

$F_p(A \otimes M) = A \otimes F_p(M)$, filtr. preserving, + $E^1(A \otimes M) \to E^1(M)$ does it.

$$\begin{tikzcd} E^1(A\otimes M) \arrow[r] & E^1(M) \\ H(A)\otimes E^1(M) \arrow[u] & \end{tikzcd}$$

Define $B_p = K \otimes_{H_0(A)} E^1_{p,0}$, $B = \Sigma B_p$ (and. of base).

$$B = K \otimes_{H_0(A)} E^1_{*,0} = E^1_{*,0} \Big/ \overline{H(A)}\, E^1_{*,0}.$$

Suppose B is proj. over K.

$$d^0 : E^0_{p,q} \to E^0_{p,q-1}, \qquad \begin{tikzcd} E^0_{p,1} \arrow[r, "d^0"] & E^0_{p,0} \arrow[r] \arrow[dr, dotted, "F"'] & E^1_{p,0} \arrow[r] \arrow[d] & 0 \\ & & B_p \arrow[d] & \\ & & 0 & \end{tikzcd}$$

diff map.

$\therefore$ map $B \to E^0$ $\therefore$ get map (diff.)

$$\begin{tikzcd} A\otimes B \arrow[r] \arrow[dr] \arrow[ddr, bend right] \arrow[dd] & E^0(A\otimes M) \arrow[r] & E^0 \arrow[d] \\ & A\otimes E^0 \arrow[u] & B \\ K\otimes B \arrow[urr] & & \end{tikzcd}$$

is commutative.

Def: A *construction* *over A* is PGF module over A $\ni$

1) B is proj. over K

2) $\exists$ a lifting $j : B \to E^0(M) \ni$

if $\tilde{j}: A \otimes B \to E^0(M)$ is induced map, then

$\tilde{j}_*: H(A) \otimes B \to E^1(M)$ is $\approx$.

A, A' are DGA algs. over K.

$f: A \to A'$, then if M is const. over A, M' over A';

$g: M \to M'$ is a <u>map of construction</u> if g is a map DGF modules.

$$\left(\begin{tikzcd} A \otimes M \arrow[r] \arrow[d, "f \otimes g"] & M \arrow[d, "g"] \\ A' \otimes M' \arrow[r] & M' \end{tikzcd}\right)$$

<u>Thm</u>: If $f_*: H(A) \xrightarrow{\approx} H(A')$ + $g_*: H(M) \xrightarrow{\approx} H(M')$, then

$g'_*: H(B) \xrightarrow{\approx} H(B')$.

<u>Proof</u>: By mapping cylinder.

Define $M''_q = M_{q-1} + M'_q$, $F_pM'' = F_{p-1}M + F_pM'$;

$d(x,y) = (-dx, g(x) + dy)$ + $\exists$ exact seq:

$$0 \to M' \xrightarrow{i} M'' \xrightarrow{j} M \to 0$$

$y \to (0, y)$ $\quad$ i filtr. + deg. pres.

$(x, y) \to x$ $\quad$ j lowers by 1

$0 \to E^0(M') \to E^0(M'') \to E^0(M) \to 0$ splits as Φ-modules.

$d^0(x,y) = (-d^0x, d^0y)$

$\therefore 0 \to E^1(M') \to E^1(M'') \to E^1(M) \to 0$ + $E^1(M'')$ is the mapp. cylinder of $E^1(M) \to E^1(M')$.

But $H(A) \otimes B \xrightarrow{\approx} E^1(M)$. Defining': $B_p = K \otimes_{H_0(A)} E^1_{p,0}(M) \to B'_p$

$$\begin{tikzcd}
B \arrow[r] & E^0(M) \arrow[r] \arrow[d] & B \arrow[d] \\
B' \arrow[r] & E^0(M') \arrow[r] & B' \\
\boxed{H(A) \otimes B} \arrow[r, "\approx"] \arrow[d, dashed] & E^1(M) \arrow[r] \arrow[d] & B \arrow[d] \\
H(A') \otimes B' \arrow[r, "\approx"] & E^1(M') \arrow[r] & B'
\end{tikzcd}$$

replace by $H(A) \otimes_{H_0(A)} E^1_{*,0}(M)$

$E^1(M'') = H(A) \otimes B''$, $H(A) \overset{f_*}{\approx} H(A')$, identified.

$(\cdots \to$ is essentially ~~HTA~~ $f_* \otimes g')$ (??).

+ $E^1(M'') \to H(A) \otimes_{H_0(A)} E^1_{*,0}(M'')$

$C = E^1_{*,0}(M'')$, proj. $H_0(A)$-module + $E^2(M'') = H(C; H(A))$

$E^\infty(M'') \equiv 0$

$\Rightarrow H(C) = 0$

Proof: $E^\infty_{0,0} = 0$, $\therefore E^2_{0,0} = 0 = H_0(C; H_0(A)) = 0 = H_0(C)$

$\therefore H_0(C ; \cancel{E}) = 0 \therefore E^2_{0,q} = H_0(C; H_q(A)) = 0$

$\therefore E^2_{1,0} = 0$, etc. by induction.

But $K \otimes_{H_0(A)} C$ is mapping cyl. of $B \to B'$ + $H(C; K) = 0$

$\therefore H(B) \xrightarrow{\approx} H(B')$.

Simplified Def. of construction:

K comm ring with 1

A DGA alg., over K.

M filtered graded diff A-module.

$$N = E^1_{*,0}(M) = \sum_p E^1_{p,0}(M)$$

$$N \xrightarrow{i} E^1(M)$$

a DGF A-module M is a <u>construction</u> if

1) N is proj. over $H_0(A)$

2)
$$\begin{tikzcd}
H(A)\otimes N \arrow[r] \arrow[d] & H(A)\otimes E^1(M) \arrow[r] & E^1(M) \\
H(A)\underset{H_0(A)}{\otimes} N \arrow[urr, "\approx"'] & &
\end{tikzcd}$$

Let M be a constr. over A which is an extended A · module; i.e.

$M = A \otimes B$ ∋ (given d)

1) B is proj over K

2) $F_p M = A \otimes F_p B$ ($F_p B$ is fin dim). M is a <u>relatively free</u> constr.

<u>Th:</u> If M is a rel. free constr. over A, M' is an acyclic constr. over A ($H(M')=K$), $\varepsilon : H_0(M) \to K$, then

∃ map f of constr., $f : M \to M'$ ∋ $f_* : H(M) \to H(M')$ is ε.

If g is another such map, there is a homotopy $D : M \to M'$ ∋ $D(am) = a\,Dm$, $D F_p M \subset F_{p+1} M'$, $D M_q \subset M'_{q+1}$, $dD + Dd = f - g$. <u>Proof:</u> usual.

Some Explicit Constructions

K comm. ring with unit.

Case 1 $A = E(x, n)$

$\mathrm{Tor}^A(K,K) = \Gamma(y,1,n)$

$\Gamma(y,1,n)$ has basis $1 = \gamma_0(y)$, $y = \gamma_1(y)$ (dim $n+1$, filtr. 1)

$\gamma_i(y)$

filtr. $\gamma_i(y) = i$

dim $\gamma_i(y) = i(n+1)$

$E(x,n) \otimes \Gamma(y,1,n)$,

$d(\gamma_i(y)) = x\gamma_{i-1}(y)$, is acyclic (proj. resol.)

suspension: $x \leftarrow dy = x$, $y \rightarrow y$ proj. into base.

$\sigma(x) = y$. Also, $\gamma_i(y)\cdot\gamma_j(y) = (i,j)\gamma_{i+j}(y)$.

More explicitly, if $K = Z_p$.

$y, y^2, \ldots, y^{p-1}$ are $\neq 0$ + basis elts.

$E(x,n) \xrightarrow{\sigma} \mathrm{Tor}_1^{\Gamma(y,1,n)}(K,K)$ = factor out decom. stuff.

+ here σ is $\cong$ if $q < p(n+1)$

$\left[\mathrm{Tor}_1^A(K,K)_q \xrightarrow{\sigma} \mathrm{Tor}_1^{\Gamma(y,1,n)}(K,K)_{q+1}\right]$

$\left\{ \cong \text{ if } q+1 < p(n+1) \right\}$

Case 2 $A = P(x,n)/[x^{p^f}]$ $K = Z_p$

$A \otimes E(y,n+1)$, $dy = x$, then $d\,x^i y = x^{i+1}$

If $i = p^f - 1$, then $x^{p^f-1}y$ is cycle, dim $p^f n + 1$ & filtr. $H(A \otimes E(y,n+1)) = E(x^{p^f-1}y, \ldots)$

+ ∴ to complete, we know how to kill $E(\ \ )$.

$A \otimes E(y, n+1) \otimes \Gamma(z, p^f n+2)$, $dz = x^{p^f-1} y$, filts. $z = 2$.

or $d\gamma_i(z) = \gamma_{i-1}(z) x^{p^f-1} y$ & constr. is acyclic.

$\therefore \operatorname{Tor}^A(K,K) = E(y, n+1) \otimes \Gamma(z, p^f n+2)$

$$\begin{tikzcd} x & \underset{dy=x}{y} \arrow[l] \arrow[r] & y \end{tikzcd}$$

$\sigma(x) = y$. Products $\to 0$,

$\therefore \operatorname{Tor}_1^A(K,K)_q \xrightarrow{\sigma} \operatorname{Tor}_1^B(K,K)_{q+1}$ where $B = \operatorname{Tor}^A(K,K)$.

& σ is $\approx$ for $q+1 < p^f n+2$

Case 3. $A = P(x, n)$

$A \otimes E(y, n+1)$, $dy = x$ is acyclic already.

$(\operatorname{Tor}_1^A(K,K))_q \xrightarrow{\sigma} \operatorname{Tor}_1^B(K,K)_{q+1}$ is $\approx$.

More generally,

$K = Z_p$, $A = $ D.G.A Hopf alg.

$\therefore$ so is $H(A)$ & $\therefore H(A) = \otimes A^i$, where $A^i = E(x,n)$,

$= P(y, 2n)/[y^{p^f}]$

$= P(y, 2n)$

$\exists$ spectral seq.

$$E^2 = \operatorname{Tor}^{H(A)}(K,K), \quad E^\infty = E^0(\operatorname{Tor}^A(K,K))$$

$$\| $$

$$\otimes_{\substack{\uparrow}} \operatorname{Tor}^{A^i}(K,K)$$

Now calculate:

(as alg, not just as modules)

	A	$B = \mathrm{Tor}^A(K,K)$	$\mathrm{Tor}_1^A(K,K)_q \xrightarrow{\tau} \mathrm{Tor}_1^B(K,K)_{q+1}$
I	$E(x,n)$	$\Gamma(\sigma(x), n+1)$	$q < p(n+1) - 1$
II	$P(x,n)/[x^{p^f}]$	$E(\sigma(x), n+1) \otimes \Gamma(z, p^f n+2)$	$q < p^f n + 1$
III	$P(x,n)$	$E(\sigma(x), n+1)$	all q.

In general, if P = proj. resol. is mult.

+ $A \to P$ is mult.

$P \to \bar{P} = K \otimes_A P$ is mult. + if mult. in P + $\bar{P}$ behaves properly w.r.t. filtration (filtration is # of terms back in $\cdots \to P_1 \to P_0 \to K \to 0$)

+ d's in spectral seq. are derivations w.r.t. mult.

($\therefore\ \mathrm{Tor}^{H(A)}(K,K) = \otimes_i \mathrm{Tor}^{A^{n}}(K,K)$ as alg.)

all stuff of filtn 1, 2 go into 0 under all d^r

$\therefore$ all goes well until $\sigma(x_i)^p$, x_i a gen. of type I

or z_i^p, z_i of type II.

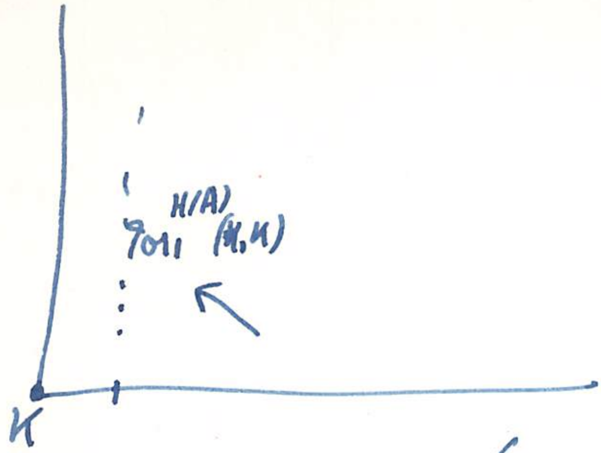
$\therefore$ if A is $(n-1)$ connected, if n odd, first $\neq 0$ d^r is on

$\gamma_p(\sigma(x))$ + its diff. is in dim $p(n+1)-1$.

$\therefore (E^2)_q = (E^\infty)_q$ if $q < p(n+1)-1$

Applications:

$0 \to (\mathrm{Tor}_1^{H(A)}(K,K))_q \xrightarrow{\sigma} (\mathrm{Tor}^A(K,K))_{q+1}$ $q < p(n+1) - 2$



Z.B. $A = C$ (loops on a group) (as $H(A)$ is comm.)

$$0 \longrightarrow \bar{H}(\text{loops}) \big/ \bar{H}^2 \longrightarrow H(\text{base})$$

Dual: every primitive elt. is a suspension for $q < \dots$

$$\begin{tikzcd} E \arrow[d, "\Omega(G)"'] \\ G \end{tikzcd}$$

$$H^{q+1}(G; Z_p) \longrightarrow P(H^q(\Omega G; Z_p)) \longrightarrow 0$$

for $q < p(n+1) - 2$

if $H_q(G; Z_p) = 0$ if $q \leq n$.

Th: G = simply conn. top. group.

$H_q(G; Z_p) = 0$ for $q > n$ $\Rightarrow$

$H_*(\Omega G; Z_p)$ is finitely gen. as a ring by elts. of dim $\leq n$.

Proof: $A = C(\Omega G)$ (always coeff. Z_p)

$\mathrm{Tor}^A(Z_p, Z_p) = H_*(G; Z_p)$ + spectral seq. $\ni$

$E^2 = \mathrm{Tor}^{H(A)}(Z_p, Z_p)$ as Hopf alg. +

$E^\infty = E^0 \mathrm{Tor}^A(Z_p, Z_p)$. By Borel, $H_*(G; Z_p)$ is finitely gen, $\therefore$

also $H_*(\Omega G; Z_p)$ (as groups) + comm. up to homotopy.

by Borel

$\therefore H_*(\Omega G) = E(M) \otimes P(N) \otimes P(O) \Big/ [x_1^{p^{i_1}}, x_2^{p^{i_2}}, \dots]$

Now calculate E^2.

But $\mathrm{Tor}^{E(x,n)}(Z_p, Z_p) = \Gamma(\sigma(x), 1, n)$

$\mathrm{Tor}^{P(y,n)}(Z_p, Z_p) = E(\sigma(y), 1, n)$

$\mathrm{Tor}^{P(y,n)/[y^{p^k}]}(Z_p, Z_p) = E(\sigma(y), 1, n) \otimes \Gamma(z, 2, p^{k_n})$

$\therefore$ in cohom., $E_2 = E(A_2) \otimes P(B_2)$ (duals of above)

A_2 has filt. 1, fibre deg. even

B_2 has filt. $\begin{cases} 1 & \text{" " odd} \\ 2 & \text{" " even.} \end{cases}$

In a diff. Hopf alg., ~~the~~ d(prim. elt.) is prim.

E_2 is prim. generated.

Prim: $A_2, B_2, \xi B_2, \xi^2 B_2, \dots, \xi^k B_2, \dots$ $(\xi(x) = x^p)$.

$d_r : E_r^{m,n} \to E_r^{m+r,\, n-r+1}$, find first non-trivial d_r. Suppose $r < p$.

in B_2 $E_r^{1,n} \to E_r^{r+1,\, n-r+1}$, all $\to 0$ by odd-even arg.

$E_r^{2,n} \to 0$ also.

in A_2 $E_r^{1,n} \to E_r^{r+1,\, n-r+1}$, lowest is $r+1 = p$

$\therefore d_r = 0,\ r < p-1$

$E_{p-1}^{1,n} \to E_{p-1}^{p,\, n-p+2}$ $\therefore A_2 \xrightarrow{d_{p-1}} \xi(B_2)$

Define $A_p = \mathrm{Ker}(d_{p-1} | A_2)$

$B_2 = B_p + B_p'$, where $B_p' \xrightarrow[\cong]{\xi} d_{p-1}(A_2)$

$$\therefore E_{p-1} = \underbrace{E(A_p) \otimes P(B_p)}_{d_{p-1}^{\,\| 0}} \otimes E(A_2/A_p) \otimes P(B'_p)$$

$$d_{p-1} : A_2/A_p \xrightarrow{\cong} \xi(B'_p)$$

$$\therefore E_p = E(A_p) \otimes P(B_p) \otimes P(B'_p) \big/\!\!\big/ \underbrace{\xi P(B'_p)}.$$

all elts have height p & + filt. 1.

& back to same ^type situation,

Inductive hyp.

$E_r = E(A_r) \otimes P(B_r) \otimes C_r$, $\Rightarrow$ $A_r \subset A_{r-1} \subset \cdots$

$B_r \subset B_{r-1} \subset \cdots$

& C_r is prim. gen. Hopf alg by elts. of filtn. 1 + 2 & even total degree.

& all prim. elts. in C_r have filtn $< r+1$.

verify Inductive hyp. like above

at end, haven't changed # of gen: in B_2, only height.

$\therefore \dim B_2 < \infty$.

also dim $A_2 < \infty$ because they must kill off heights

of finitely many gen. in B_2.

From last time $\boxed{\text{podd}}$: $E_2 = E(N_1^*) \otimes P(N_2^*) \otimes P(N_3^*)$

$r \geq 2$, $E_r = E(N_{1,r}^*) \otimes P(N_{2,r}^*) \otimes P(N_{3,r}^*) \big/ [x_i^{p^{k_i}}, \ldots)$

& # of gen. of even dim same as in E_2.

	filt.	fibre degree	total deg
gen. in N_1^*	1	even	odd
" in N_2^*	1	odd	even
" in N_3^*	2	even	even

Only changes when $n = p^k - 1$ or $n = 2p^k - 1$

$n = p^k - 1$

$$\begin{tikzcd}
0 \arrow[r] & N^*_{1,n+1} \arrow[r] & N^*_{1,n} \arrow[r] & \xi^k N^*_{2,n} \arrow[r] \arrow[d, phantom, "\wr"] & N^*_{2,n+1} \arrow[r] & 0 \\
& & & N^*_{3,n} & &
\end{tikzcd}$$

$N^*_{3,n+1} = N^*_{3,n} + {}'N^*_{3,n}$

+ $\dim {}'N^*_{3,n} = \dim N^*_{1,n} - \dim N^*_{1,n+1}$

or $\dim N^*_{1,n} = \dim {}'N^*_{3,n} + \dim N^*_{1,n+1}$

But can't keep going forever, ∴ N_2^* is finite dim'l, ∴ so are $N^*_{3,n}$

~~$+ N^*_{2,n} +$~~ $\dim N^*_{2,n} + \dim N^*_{3,n} = \dim N_2^* + \dim N_3^*$.

Also N_1^* is finite dim'l. because subtracting finite # gives $\dim N^*_{1,n}$ = finite.

$(2p-2)\dim N_2 + (4p-4)\dim N_3 +{}^3\dim N^*_{1,\infty} \le n$ (= height of $\operatorname{Tor}^A(Z_p, Z_p)$).

∴ If $\dim N_2 > 0$, then $2p-2 \le n$ or $p \le \frac{n+2}{2}$ (Grassmann in $H^*(A)$)

If $\dim N_3 > 0$, then $4p-4 \le n$ or $p \le \frac{n+4}{4}$

(Say group of dim n, then loops has no p-torsion for $p > \frac{n+2}{2}$

Q – look at Bocksteins)

Z.B. Let G be simply con. top. group ∋ $H_q(G; Z_p) = 0$ for $q > n$. Then, if $p > \frac{n+2}{2}$, then $H_*(\Omega G; Z_p)$ is poly. alg. with even dim gen.; i.e. $H_*(\Omega G; Z_p) = P(B)$ + moreover, $H_*(G; Z_p) = E(\sigma B)$.

Furthermore, every Grassman alg. gen. has dim $< \frac{n}{p} - 1$.

Also, every truncated poly. alg. gen. has dim $\le \dfrac{\frac{n}{p-1} - 2}{p^k}$.

---

an examples.

n even, $C_*(\Omega^2(S^{n+1})) \otimes Z_p = A$ $\qquad E^2 = \mathrm{Tor}^{H(A)}(Z_p, Z_p)$

$E^\infty = E^0\, \mathrm{Tor}^A(\,\cdot\cdot\,,\,\cdot\cdot\,)$

$$\begin{tikzcd} E \arrow[d, "\Omega^2 S^{n+1}"'] \\ {}^{1}S^{n+1} \end{tikzcd}$$

$H_*(\mathfrak{J}^{n+1}; Z_p) = P(x, n) = \mathrm{Tor}^A(Z_p, Z_p)$

In $E^0\, \mathrm{Tor}^A(Z_p, Z_p)$, must have height at most p.

(also is Hopf alg.)

∴

elts	filt.	fibre deg.
x	1	$n-1$
x^p	1 (is susp.)	$pn-1$
$\vdots$		
x^{p^k}	1	$p^k n - 1$
$\vdots$		

$\therefore E^0\mathrm{Tor}^A = \bigotimes_{k \ge 0} Q(x^{p^k}, p^k n)$.

Now try to find E^2 & from that, $H(A)$.

Answer

$$H_*({}^{2}S^{n+1}; Z_p) = \bigotimes_{k \ge 0} E(x_k, p^k n - 1) \otimes \bigotimes_{k > 0} P(y_k, p^k n - 2)$$

$$+\ \delta^* y_k = x_k,\ k > 0.$$

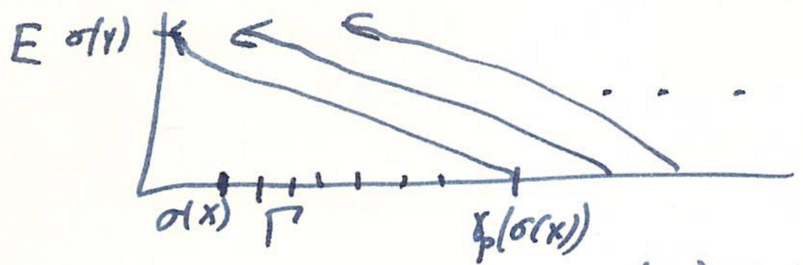
$$E^2 = \mathrm{Tor}^{H(A)}(Z_p, Z_p) = \bigotimes_{k \ge 0} \Gamma(\sigma(x_k), p^k n) \otimes \bigotimes_{k > 0} E(\sigma(y_k), p^k n - 1)$$

To get to E^∞, must kill all $E(\quad)$. ~~This is all done~~ This is all done by d_{p-1} & $E^p = E^\infty$.

~~by d_{p-1} & in fact $E^p = E^\infty$.~~

$d_{p-1}\gamma_p(\sigma(x_k)) = \sigma(y_{k+1})$

$\therefore \Gamma(\sigma(x_k), p^k n) \otimes E(\sigma(y_{k+1}), p^{k+1}n - 1)$



& $H_* = Q(\sigma(x_k), p^k n)$ & $E^p = E^\infty$.

Can go other way using things like before.

$Z_p, (A, N, M)$

$H(A) = E(B^{-1}) \otimes \bigotimes_{i \geq 0} P(B^i) /\!/ \xi^i P(B^i)$, where $\xi^0 = 1$ (g ring)

$\boxed{\text{To prove } B^i = 0 \text{ for } i > 0}$

$L, L^+ =$ ~~odd dim stuff~~ raise dim by 1.

E^2 for $\mathrm{Tor}^A(Z_p, Z_p)$

Let $C^{-1} = \Sigma\, (B^i)^+$

$C^0 = (B^{-1})^+$

$C^1 = \Sigma_i\, (\xi^i B^i)^{++}$

Then $E^2 = E(C^{-1}) \otimes \Gamma(C^0) \otimes \Gamma(C^1)$

<u>Assume</u> $\mathrm{Tor}^A(Z_p, Z_p) = E(D^{-1}) \otimes P(D^0)$ as Hopf alg.

Then $\mathrm{Ext}_A(Z_p, Z_p) = E(D^{-1*}) \otimes \Gamma(D^{0*})$, all height p.

& here $E_2 = E(C^{-1*}) \otimes P(C^{0*}) \otimes P(C^{1*})$ $\therefore$ must truncate off

$\therefore \neq 0$ diff. ops. are d_{p-1} & d_{2p-1}

$$\therefore \quad 0 \to {}'C \to C^{-1*} \xrightarrow{d_{p-1}} \xi(C^{0*}) \to 0$$

$$\therefore E_p = E({}'C) \otimes P(C^{0*}) /\!/ \xi P(C^{0*}) \otimes P(C^{1*})$$

$$\therefore \quad 0 \to {}''C \to {}'C \xrightarrow{d_{2p-1}} \xi(C^{1*}) \to 0$$

& hence $E_{2p} = E({}''C) \otimes P(C^{0*}) /\!/ \xi P(C^{0*}) \otimes P(C^{1*}) /\!/ \xi P(C^{1*})$

is the only way to truncate everything.

& $E_\infty = E_{2p} = E({}''C) \otimes Q(C^{0*}) \otimes Q(C^{1*})$

But $\Gamma(D^{0*}) = \bigotimes_{k \geq 0} Q(\gamma_{p^k}(D^{0*}))$

$$\therefore E_\infty \simeq E(D^{-1*}) \otimes \bigotimes_{k \geq 0} Q(\gamma_{p^k}(D^{0*}))$$

$$\therefore D^{-1*} \simeq {}''C$$

$$\& \quad C^{0*} + C^{1*} = \sum_{k \geq 0} \gamma_{p^k}(D^{0*})$$

Dim of stuff in C^{1*} is $p^i n + 2$ & $\therefore$ must be gen. of D^{0*}

(trying to show $C^{1*} = 0$) $\quad \therefore \sum_{k > 0} \gamma_{p^k}(D^{0*}) \subset C^{0*}$

$D^0 = {}'D^0 + {}''D^0$ & ${}'D^0$ are elts. in D^0 which are suspensions

& then $C^{0*} = \sum_{k > 0} \gamma_{p^k}(D^{0*}) + {}'D^{0*}$

$C^{1*} = {}''D^{0*}$

An elt. in ${}''D^0$ has filt. 2 & even complementary deg.

If $\operatorname{Tor}^{A}(Z_p, Z_p)$ is gen. by suspensions (as ring), then
"$D^0 = 0$, & $C^1 = 0$ & $\therefore$ $\xi^{0i} B^{i} = 0$ if $i > 0$
& $\therefore$ $\beta^{i} = 0$ for $i > 0$. &

$H(A) = E(B^{-1}) \otimes P(B^0)$.

z.B.

$$H(\Omega S^2 X; Z_p) = \otimes(H(SX; Z_p))$$

DGA alg. A

define $[x, y] = xy - (-1)^{pq} yx$

$d[x, y] = [dx, y] + (-1)^{p}[x, dy]$

$$\begin{tikzcd} \Omega G & E \arrow[d] \\ & G \text{ (group)} \end{tikzcd}$$

$H_*(G; K)$ is DGA-alg.
field

$\sigma(H_*(\Omega G; K)) \subset H_*(G; K)$

To show $[\sigma H_*(\Omega), \sigma H_*(\Omega)] \subset \sigma H_*(\Omega)$.

In E, x_t a path $\ni$ $x_t(s) = x(st)$.

$S: I \times E \to E$, $S(t, x) = x_t$ is contr. homotopy

Define $D_1, D_2 : I \times E \times E \to E$

$D_1(t, x, y) = [x_t, y] = x_t y x_t^{-1} y^{-1}$ ([ ] in G)

$D_2(t, x, y) = [x, y_t] = x y_t x^{-1} y_t^{-1}$

$\pi : E \to G$, $\pi(x) = x(1)$.

~~Rory ε Ω~~

$D_1 : I \times E \times \Omega \longrightarrow \Omega$

$D_2 : I \times \Omega \times E \longrightarrow \Omega$

Now make incorrect calc.

$D_1' : C(E) \otimes C(\Omega) \to C(\Omega)$ & $dD_1' + D_1'd = [x,y]$ (not quite)

also D_2', $S' : C(E) \to C(E)$, $ds' + s'd = 1 - \varepsilon$

$C(\Omega) \xrightarrow{\sigma} C(G)$ is defined by $\sigma(x) = \pi S'(x)$

& $d\sigma(x) = \pi d s'(x) = \pi(x - \varepsilon(x) - s'dx) = -\varepsilon(x) - \sigma dx$

$= \sigma d x$ if aug. $x = 0$.

$$[\sigma x, \sigma y] = \pi([sx, sy]) = \pi(Sd + ds)([Sx, Sy])$$

$$= \underbrace{d\pi S[Sx, Sy]}_{\text{bdry}} + \pi S([\underset{\underset{x}{\shortparallel}}{dsx}, sy] + (-1)^{p+1}[Sx, \underset{\underset{y}{\shortparallel}}{dsy}])$$

because $dx = 0$, $dy = 0$

$$\{[\sigma x, \sigma y]\} = \pi S([x, sy] + (-1)^{p+1}[Sx, y])$$

$$= \sigma\{[x, sy] + (-1)^{p+1}[Sx, y]\}$$ ($[x, sy]$ is still in the fibre)

What is [ ] in G?

$$\begin{tikzcd}
G \times G \arrow[r, "\psi \times \psi"] \arrow[ddrrr, bend right] & G \times G \times G \times G \arrow[r, "1 \times T \times 1"] & G \times G \times G \times G \arrow[r, "1 \times 1 \times c \times c"] & G \times G \times G \times G \\
x \quad y & x \quad x \quad y \quad y & x \quad y \quad x \quad y & x \quad y \quad x^{-1} \quad y^{-1} \arrow[d] \\
 & & & G
\end{tikzcd}$$

[ ] as DGA alg.

$$C(G)\otimes C(G) \to C(G\times G) \xrightarrow{\text{commut.}} C(G)$$

$$\lambda : \pi(G) \to H(G)$$

$\lambda\langle f, g\rangle = [\lambda f, \lambda g]$, $\langle\ \rangle$ is commutator of $\pi_p \otimes \pi_q \to \pi_{p+q}$.

To show for primitive elts, two [ ]'s are same.

$$x\otimes y \longrightarrow x\otimes 1\otimes y\otimes 1 + 1\otimes x\otimes y\otimes 1 + x\otimes 1\otimes 1\otimes y + 1\otimes x\otimes 1\otimes y$$

(because primitive elts)

$$\xrightarrow{1\times T\times 1} x\otimes y\otimes 1\otimes 1 + (-1)^{pq}\, 1\otimes y\otimes x\otimes 1 + x\otimes 1\otimes 1\otimes y + 1\otimes 1\otimes x\otimes y$$

$$\longrightarrow xy + (-1)^{pq}\, y\cdot c(x) + x\cdot c(y) + c(x)\cdot c(y)$$

but $c(x) = -x$ for prim. elts

$$= xy - (-1)^{pq}\, yx - xy + xy = xy - (-1)^{pq}\, yx$$

Then $dD_1' + D_1'd \overset{(x\otimes y)}{=} [x, y]_E$ ($[\cdot]_E$ means [ ] induced by mult. in G)

$$dD_2' + D_2'd\,(x\otimes y) = [x, y]_E$$

Again: G group

$$\begin{tikzcd}
C(G)\otimes C(G) \arrow[r, "\nabla"] \arrow[rrrdd, "{[\ ,\ ]}"'] & C(G\times G) \arrow[r, "\nabla'\times\nabla'"] & C(G\times G\times G\times G) \arrow[r, "1\times T\times 1"] & C(G\times G\times G\times G) \arrow[d, "1\times 1\times c\times c"] \\
 & & & C(G\times G\times G\times G) \arrow[d] \\
 & & & C(G)
\end{tikzcd}$$

Obviously, $d[x,y] = [dx,y] + (-1)^{\dim x}[x,dy]$

If x & y are primitive homology elts, $[x,y] = xy - (-1)^{pq}yx$ (in general, more complicated)

Now consider

$$\begin{tikzcd} E \arrow[d, "\Omega"'] \\ G \end{tikzcd}$$

, $S : I \times E \to E$ by

$S(t,f)(s) = f(st)$.

$d\, dS + Sd = 1 - \varepsilon$

$\Omega \times E \xrightarrow{[\,,\,]} \Omega$

$E \times \Omega \xrightarrow{[\,,\,]} \Omega$

gives rise to

$C(\Omega) \otimes C(E) \to C(\Omega)$

$C(E) \otimes C(\Omega) \to C(\Omega)$

$\bar{H}_*(\Omega) \xrightarrow{\sigma} \bar{H}_*(G)$

$x \xrightarrow{\sigma} \pi S x$ is chain map giving susp.

Now show
If $x, y \in \sigma \bar{H}_*(\Omega) \subset \bar{H}_*(G)$, then $[x,y] \in \sigma \bar{H}_*(\Omega)$.

(But a susp. elt is primitive, $\therefore$ $[x,y] = xy - (-1)^{pq}yx$)

Proof: Let $x, y \in \bar{C}_*(\Omega)$, $dx = 0, dy = 0$ consider $[\sigma x, \sigma y] = [\pi S x, \pi S y]$

$= \pi[Sx, Sy] = \pi\, dS[Sx,Sy] + \pi S d[Sx,Sy] =$

$d\pi S[Sx,Sy] + \pi S\left([\underset{x}{\underset{\|}{dSx}}, Sy] + (-1)^{\dim x + 1}[Sx, \underset{y}{\underset{\|}{dSy}}]\right)$.

$= d\pi S[Sx,Sy] + \pi S\underbrace{\left([x,Sy] + (-1)[Sx,y]\right)}_{\in C_*(\Omega)\ \&\text{ is cycle}}$

$= \text{bdry} + \sigma(\text{cycle in } \bar{C}_*(\Omega))$

$Y, \ \Omega(sY) \supset Y$

$$\begin{tikzcd}
H_*(Y) \arrow[r] \arrow[dr, "\approx"] & H_*(\Omega s Y) \arrow[d, "\sigma"] \\
 & H_*(sY)
\end{tikzcd}$$

& in fact, if K = field, $H_*(\Omega s Y; K) = T(\bar{H}_*(Y;K))$

(K, M vector space, $T(M)$, $[x,y] \underset{df}{=} xy - (-1)^{pq} yx$

$\nearrow$ no elts dim 0

is defined in $T(M)$. M generates Lie alg, $L(M) \subset T(M)$.

if $Y = sX$, all coeff. in K.

$$\begin{tikzcd}
H_*(\Omega s^2 X) \arrow[r, phantom, "="] & T(\bar{H}_*(sX)) \\
H_*(\Omega^2 s^2 X) \arrow[u, "\sigma"] & H_*(X). \arrow[l] \arrow[u]
\end{tikzcd}$$

$\bar{H}_*(sX)$ is susp. (σ)

∴ $\Sigma\,]$ ~~comm.~~ are also susp. elts

∴ $L(\bar{H}_*(sX)) \subset \sigma \, \bar{H}_*(\Omega^2 s^2 X)$

Define filtration on $T(M)$.

$F_0(T(M)) = K$

$F_1 \ T(M) = L(M) + K$

$F_2 \ T(M) = F_1 + L^2$

$F_{p+1} = F_p + L^{p+1} = (K+L)^{p+1}$

$\cup F_p = T(M)$, $E^0_p T = F_p/F_{p-1}$ ∴ a ring, L gen. E^0T.

If $x, y \in L$, $[x,y] \in L$ + is of filtr. 1, ∴ in E^0 is 0.

$A(L)$ = free comm. alg. on L

$A(L) \longrightarrow E^0T \longrightarrow 0$ + in fact is ≈ (Poincaré-Witt thm)

∴ topol. there is filtr. on $H_*(\Omega(S^2X)) \ni$

$E^0 \approx A(L)$. Now ∃ filt. on " + E^0

has even elts of height p. ∴

<u>Proof of Poincaré-Witt thm</u>: char 0. , $A(L) \to E^0T \to 0$

$T(M)$ is Hopf alg by $m \to 1\otimes m + m \otimes 1$.

$x, y \in T(M)$ + primitive, then $[x,y] = xy - (-1)^{pq} yx$

$x \to x\otimes 1 + 1 \otimes x$

$y \to y \otimes 1 + 1 \otimes y$, $xy \to xy \otimes 1 + x \otimes y + (-1)^{pq} y \otimes x + 1 \otimes xy$

∴ $[x,y] \to [x,y]\otimes 1 + 1 \otimes [x,y]$

E^0T is Hopf alg + L is primitive. Make $A(L)$ into Hopf alg.

∴ above $A(L) \to E^0T \to 0$ is map of Hopf alg.

Mono. if primitive elts are mono. But char 0, prim. elts in $A(L)$ are L + they go by mono.

In char p, if L is loc. finite dim.

$$A(L)^* \leftarrow E^0T^* \leftarrow 0$$

$\shortparallel$

$V(L^*)$ (divided powers) + ∴ must put def. of div. powers in E^0T^*.

If divided powers in fibre of acyclic construction, then can also in base (all alg. are comm.)

Let N be graded module, $A = N + K$, product $= 0$

Apply Bar constr. $\bar{A} = N$, $\mathcal{B}(A) = A + A \otimes N + \cdots$

$\bar{\mathcal{B}}(A) = K + N + N \otimes N + \cdots \underset{\text{add.}}{=} T(N^+)$ (N^+ = dim up by 1)

$S(x[x_1, \dots, x_n]) = [x - \epsilon x, x_1, \dots, x_n]$ is contr. homotopy, $S^2 = 0$, $dS + Sd = 1 - \epsilon$. To define div. powers upstairs so get it in $T(N^+)$. $\gamma_1(x)$ defined. $d\gamma_2(x) = x\,dx$, but $d(x\,dx) = (dx)^2 = 0$ (comm.)

$\therefore$ can define $\gamma_2(x) = S(x\,dx)$, $\dots$ $\gamma_{n+1}(x) = S(dx\,\gamma_n(x))$, $\dots$.

Collapsing gives divided powers in $\bar{\mathcal{B}}(A) = T(N^+)$. To prove identities in $\mathcal{B}(A)$. $d(2\gamma_2(y) - y^2) = 0$, $\therefore dS(2\gamma_2(y) - y^2) =$ $2\gamma_2(y) - y^2$. But $\gamma_2(y) \in \text{Im}\, S$, $\therefore$ " $-dS(y^2)$

Projecting, $-d\pi S y^2 = 2\gamma_2(x) - x^2$. But in $\bar{\mathcal{B}}(A)$, $d = 0$ because mult. was 0. $\therefore$ $x^2 = 2\gamma_2(x)$.

$T^* = K + M^* + \cdots$ + above gives div. powers in T^*. Must verify that this is compatible with mult. in $E \circ T^*$. This finishes proof of Poincaré-Witt thm for a stage p.

M, no 0 dim'l elts., module over K.

$\psi: T(M) \to T(M) \otimes T(M)$

$\lambda(m) = m \otimes 1$

$\lambda(m_1 \otimes \cdots \otimes m_n) = \lambda(m_1)\, \psi(m_2 \otimes \cdots \otimes m_n)$

& $\lambda(xy) = \lambda(x)\, \psi(y)$. Also, $T : T(M) \otimes T(M) \circlearrowleft$

$\psi = \lambda + T\lambda$

$\pi_1 \lambda = id$

$\pi_2 \lambda = \varepsilon$. Assume loc. finite dim'l.

$$T(M)^* \otimes T(M)^* \xrightarrow[\lambda^*,\, \psi^*]{} T(M)^*$$

Define $a \downarrow b = \lambda^*(a \otimes b)$ & we have $a \cdot b = a \downarrow b + (-1)^{pq}\, b \downarrow a$

If dim a even, $2\, a \downarrow a = a^2$ (or $a \downarrow a = \gamma_2(a)$)

[this is all to put in divided powers in a simpler way into $T(M)$].

Define $\gamma_n(a) = a \downarrow (a \downarrow (\cdots \downarrow a) \cdots)$ works.

X space, consider $H_*(\Omega s^2 X; \underset{\substack{\uparrow \\ \text{field}}}{K}) = T(M)$ where $M = \bar{H}_*(sX; K)$.

$sX \to sX \vee sX \to sX \times sX$ extend to

$\Omega s^2 X \to \Omega s^2 X \times \Omega s^2 X$ & this splits diagonal map

as above, & gives divided powers in cohom. $H^*(\Omega s^2 X)$.

Want to compute $H_*(\Omega^2 s^2 X; Z_p) = A$. (p odd)

$IE^0(TM) = A(L)$ (= free com. gen by L, L is Lie in $T(M)$ gen by M).

Another filtration ∍

$$E^2 = \mathrm{Tor}^A(Z_p, Z_p)$$

$$E^\infty = E^0(T(M)) = \underset{E(L^-)\otimes Q(L^+)}{} \otimes \bigotimes_i \underset{\underset{\text{filt.}1}{\uparrow}}{Q(A_i)} \otimes \bigotimes_i \underset{\underset{\text{filt.}2}{\uparrow}}{Q(B_i)}$$

L filt. one.

By Borel, $A = E(C_{-1}) \otimes \bigotimes_{k\geq 0} P(C_k) /\!/ \xi^k C_k$

$\therefore \mathrm{Tor}^A(Z_p, Z_p) = \Gamma(C_{-1}^+) \otimes E(C_0^+) \otimes E(C_1^+) \otimes \Gamma(\xi(C_1)^{++}) \otimes$

$\bigotimes_{k>1} \left(E(C_k^+) \otimes \Gamma(\xi^k (C_k)^{++})\right)$ or dually

$$\underset{\overset{\shortparallel}{E_2}}{\mathrm{Ext}_A(Z_p, Z_p)} = P(C_{-1}^+) \otimes E(C_0^+) \otimes \bigotimes_{k\geq 1} \left(E(C_1^+) \otimes P(\xi^k (C_k)^{++})\right)$$

By E^∞, everything has height p, $\therefore$ must look at d_{p^k-1}, or d_{2p^k-1}.

(truncate things at p^k) $\therefore$ look at d_{p-1} & d_{2p-1}.

$$E_p = Q(C_{-1}^+) \otimes E(\tilde{C}_0^+) \otimes \bigotimes_{k>0} \left(E(C_k^+) \otimes P(\xi^k (C_k)^{++})\right)$$

$$0 \to \xi(C_{-1}^+) \xrightarrow{d_{p-1}} C_0^+ \to \tilde{C}_0^+ \to 0$$

$$E_\infty = E_{2p} = Q(C_{-1}^+) \otimes E(\tilde{C}_0^+) \otimes \bigotimes_{k>0} E(\tilde{C}_k^+) \otimes Q(\xi^k (C_k)^{++})$$

But by dim'l reasons ($2np^k \neq p^k m+2$), $C_k = 0$ if $k>0$.

$\therefore A = E(C_{-1}) \otimes P(C_0)$.

Now $E^0 T(M) = E(L^-) \otimes Q(L^+) \otimes \bigotimes_{k>0} Q(\xi^k L^+)$

& $\tilde{C}_0^+ = L^-$ & $C_{-1}^+ = \sum_{k\geq 0} \xi^k (L^+)$ & $C_0^+ = (L^-) + \sum_{k\geq 0} \xi^{k+1} (L^+)^-$

Z. B. $X = S^{n-1}$, n even, $L = {}^{*}x$, dim $x = n$.

$\therefore\ C^{+}_{-1} = \{ x, x^{p}, x^{p^2}, \dots \}$

$C^{+}_{0} = \{ \xi(x)^{-}, \xi^{2}(x)^{-}, \dots$

& this checks with known Pontry. alg. of the loops in odd sphere.

Restatement: $M = \bar{H}_*(sX; Z_p)$, L = Lie alg. gen. by M in $T(M)$.

$H_*(\Omega(s^2X); Z_p) = T(M)$ as Hopf alg.

$H_*(\Omega^2(s^2X); Z_p) = A(C)$

where $L = \overset{\text{odd}}{L_1} + \overset{\text{even}}{L_2}$; $C = \underset{\text{even}}{\bar{L}_1} + \underset{\text{odd}}{\sum_{k \geq 0} (\xi^k L_2)^{-}} + \underset{\text{even}}{\sum_{k > 0} (\xi^k L_2)^{=}}$

$\bar{A}(C) \xrightarrow{\sigma} T(M)$, susp., 0 on products,

$\bar{A}(C) / \bar{A}(C)^2 = C$, $\therefore C \xrightarrow{\sigma} T(M)$.

$\sigma(x^{-}) = x$, $\sigma(x^{=}) = 0$. If x is a susp, so is $\xi(x) = x^p$.

Want to use this as univ. example to show

Th: $\sigma x \in H_*(G; Z_p)$, then $\exists\ y \in H_*(\Omega G; Z_p) \ni \sigma y = (\sigma x)^p$.

Proof:
$$\begin{tikzcd} & E \arrow[d] \\ s\Omega \arrow[r] & G \end{tikzcd}$$

$(c\Omega \to E$ & project$)$.

& by same argument

$$\begin{tikzcd} & E_G \arrow[d, "G"] \\ s^2\Omega \arrow[r] & B_G \end{tikzcd}$$

$\therefore\ \Omega s^2 \Omega \to G$

& finally $\Omega^2 s^2 \underset{\underset{\Omega(G)}{\uparrow}}{\Omega(G)} \to \Omega G$ & is $\simeq$ id.

$$\begin{tikzcd} H_*(\overset{x}{\Omega G})^{Z_p} \arrow[r, "i"] \arrow[d, phantom, "\cancel{\downarrow \sigma}"] & H_*(\Omega^2 S^2 \Omega G) \arrow[r, "j"] \arrow[d, "\sigma"] & H_*(\Omega G) \arrow[d, "\sigma"] \\ \cancel{H_* T(G)} \arrow[r, phantom, "\cancel{\to}"] & H_*(\Omega S^2 \Omega G) \arrow[r, "j"] & H_*(G) \end{tikzcd}$$

~~[illegible]~~ $\sigma(ix)$, $\therefore\ \exists\ y \in H_*(\Omega^2 S^2 \Omega G) \ni \sigma y = (\sigma i x)^p$

$\& \therefore \sigma j(y) = (\sigma j i x)^p = (\sigma x)^p$.

Some cleaning up.

X, consider $H_*(\Omega S X)$.

$$\begin{tikzcd} E \arrow[d, "\Omega"'] \\ SX \end{tikzcd}$$

Singular chains give construction.

$$\begin{tikzcd} C(\Omega) \arrow[r] & C(E) \arrow[d] \\ & C(SX) \end{tikzcd}$$

Now give easier construction which we can calculate.

$C(X) \subset C(\Omega S X)$. $M = C(X)_N$ (of augm. 0)

$M \to C(\Omega)$, $\therefore$ unique mult. $T(M) \to C(\Omega)$
T assoc.

& this is chain map with usual d on $T(M)$.

Let ~~[illegible]~~ $B = Z + M^+$ ($C(SX)$ essentially).

Make acyclic const. $\ni T(M)$ is fibre & B base.

$T(M) \otimes B$ has natural diff. op. d'.

Define $d''(1 \otimes m^+) \overset{(-1)^{\dim}}{\underset{\wedge}{=}} m \otimes 1$ & $d''(\text{base pt.}) = 0$.

$d'd''(1 \otimes m^+) \overset{(-1)^{\dim}}{\underset{\wedge}{=}} dm \otimes 1$, $d''d'(1 \otimes m^+) = d''(1 \otimes (dm)^+) = (-1)^{\dim m + 1} dm \otimes 1$

$\therefore -d''d' = d'd''$.

Total diff. $d' + d''$. Projecting gives right diff. in B.

Contracting homotopy $S(m \otimes 1) = (-1)^{\dim m}(1 \otimes m^+)$

$d''S + Sd'' = 1$ on such elts. $S(m_1 \otimes \cdots \otimes m_k \otimes 1) = m_1 \otimes \cdots \otimes m_{k-1}$

$\otimes m_k^+ (-1)^{\cdots}$ & again as above $\therefore$ acyclic under d''.

Also, $d'S + Sd' = 0$, $\therefore$ acyclic under $d' + d''$.

Moreover, filtering by deg. in B, $E^1(T(M) \otimes B) = H(T(M)) \otimes B$.

$\therefore$ a construction.

$\therefore$ a map (unique up to homotopy)

$$\begin{tikzcd}
T(M) \otimes B \arrow[r] \arrow[d] & C(E) \arrow[d] \\
B \arrow[r] & C(SX)
\end{tikzcd}$$

& base induces hom. $\approx$.

$\therefore H_*(\Omega S X) \overset{\approx}{\leftarrow} H(T(M))$, & map was mult; $\therefore$ right ring also.

If field of coeff., $H(T(M) \otimes Z_p) = H_*(\Omega; Z_p)$

$$\overset{\shortparallel}{T(H_*(M; Z_p))}$$

Diff. coalg. with base pt. $B \xrightarrow{\psi} B \otimes B$, assume no 1-dim'l elts.

Describe cobar const. of B. Let $B' = B/pt.$, $F = (B')^-$

$B + B \otimes F + B \otimes F \otimes F + \cdots = B \otimes T(F)$. Put in internal d &

external d (to do with only coalg. structure)

$B \xrightarrow{\psi} B \otimes B \to B \otimes B' \overset{\approx}{\to} B \otimes F$ gives external d on elt. of B.

$$\begin{tikzcd}
B \otimes B \arrow[r, "\psi \otimes 1 \pm 1 \otimes \psi"] \arrow[d] & B \otimes B \otimes B \arrow[d] \\
B \otimes F \arrow[r, dashed] & B \otimes F \otimes F
\end{tikzcd}$$

& likewise.

Then whole thing is acyclic.