Chapter 4. Some well known algebras.

For convenience we will assume that we have chosen a fixed commutative ring K. All modules and algebras considered in this chapter will be assumed to be modules or algebras over K without further comment.

§1. Free algebras and free products of algebras.

Definition 1.1: If X is a set, the free algebra generated by X consists of an algebra Λ and a map $i: X \longrightarrow \Lambda$ such that if Γ is any algebra and $f: X \longrightarrow \Gamma$ is any map, then there is a unique morphism of algebras $\tilde{f}: \Lambda \longrightarrow \Gamma$ such that $\tilde{f}i = f$.

Note that the definition of algebra (Chapter 1, 3.1) requires that an algebra be associative. In the literature this requirement is not always made. Consequently the algebra defined above is sometimes referred to as the free associative algebra generated by X . Notice that since we have defined the free algebra generated by X by means of a universal property if it exists it will be unique. Thus as usual we are stuck with the problem of proving existence. In order to prove existence it is convenient to make a slight detour.

Definition 1.2: If X is a set, the free monoid generated by X consists of a monoid M and a map $i: X \longrightarrow M$ such that if N is a monoid, and $f: X \longrightarrow N$ is a map, then there is a unique morphism of monoids $\tilde{f}: M \longrightarrow N$ such that $\tilde{f}i = f$.

In the preceding definition we do not use the convention used in Chapter 3 that monoids are commutative. By convention if X is the empty set the free monoid generated by X is the monoid with 1-element. Notice that this monoid satisfies the desired universal property.

Proposition 1.3: If X is a set, the free monoid generated by X exists.

Proof: Let M be the set of finite sequences of elements of X, including the empty sequence which is denoted by 1. Now if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, let $xy = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Thus M becomes a monoid and its unit or identity element is 1.

Suppose N is a monoid, and $f\colon X \longrightarrow N$ is a map. Define $\tilde{f}\colon M \longrightarrow N$ by $\tilde{f}(1) = 1$, and $\tilde{f}(x_1, \ldots, x_n) = f(x_1) \ldots f(x_n)$. Clearly $\tilde{f}$ is a morphism of monoids. Let $i\colon X \longrightarrow M$ be the map such that for $x \in X$, $i(x)$ is the one termed sequence (x). Now $\tilde{f}i = f$, and it is clear that if $g\colon M \longrightarrow N$ is a morphism of monoids such that $gi = f$ then $g = \tilde{f}$. This proves the proposition.

Proposition 1.4: If X is a set, the free algebra generated by X exists.

Proof: Let Λ be the free module generated by the free monoid M generated by X, and let $i\colon X \longrightarrow \Lambda$ be the natural map. If Γ is an algebra, and $f\colon X \longrightarrow \Gamma$ is a mpa, let $f'\colon M \longrightarrow \Gamma$ be the morphism of monoids obtained by considering Γ as a monoid under multiplication.

Since Λ is the free module generated by M we have that f' induces a unique map of K-modules $\tilde{f}: \Lambda \longrightarrow \Gamma$. Further Λ is an algebra in a unique way so that $M \longrightarrow \Lambda$ is a morphism of monoids, and $\tilde{f}$ is a morphism of algebras. Certainly $\tilde{f}$ is unique, and the proposition is proved.

Proposition 1.5: If X is a set, and $i: X \longrightarrow \Lambda$ is the free algebra generated by X, then Λ as a module is the free module generated by M where M is the free monoid generated by X.

This proposition is just a restatement of some things we have observed previously and needs no proof.

Notation 1.6: If X is a set, we denote by $T[X]$ the free algebra generated by X. Further if $f: X \longrightarrow Y$ is a map of sets we denote by $T(f): T[X] \longrightarrow T[Y]$ the corresponding morphism of algebras.

Definition 1.7: If Λ and Γ are algebras, the free product of Λ and Γ is an algebra $\Lambda * \Gamma$ together with morphisms of algebras $i_\Lambda: \Lambda \longrightarrow \Lambda * \Gamma$ and $i_\Gamma: \Gamma \longrightarrow \Lambda * \Gamma$ such that if Σ is an algebra, and $f: \Lambda \longrightarrow \Sigma$, $g: \Gamma \longrightarrow \Sigma$ are morphisms of algebras, then there is a unique morphism of algebras $f * g: \Lambda * \Gamma \longrightarrow \Sigma$ such that $(f * g)i_\Lambda = f$ and $(f * g)i_\Gamma = g$.

Before proving the existence of the free product of algebras in general, we first prove a special case.

Proposition 1.8: If X and Y are sets, then

$$T[X] * T[Y] = T[X \cup Y] .$$

Proof: Corresponding to the natural maps $i_1: X \longrightarrow X \cup Y$ and $i_2: Y \longrightarrow X \cup Y$ we have morphisms of algebras $j_1: T[X] \longrightarrow T[X \cup Y]$ and $j_2: T[Y] \longrightarrow T[X \cup Y]$. If $f: T[X] \longrightarrow \Sigma$ and $g: T[Y] \longrightarrow \Sigma$ are morphisms of algebras, they correspond to maps $f_1: X \longrightarrow \Sigma$ and $g_1: Y \longrightarrow \Sigma$, and there results a unique map $f_1 \cup g_1: X \cup Y \longrightarrow \Sigma$ such that $(f_1 \cup g_1)i_1 = f_1$ and $(f_1 \cup g_1)i_2 = g_1$. There results a unique morphism of algebras $f * g: T[X \cup Y] \longrightarrow \Sigma$ such that $(f * g)i = f_1 \cup g_1$ where $i: X \cup Y \longrightarrow T[X \cup Y]$ is the natural map. Clearly $(f * g)j_1 = f$ and $(f * g)j_2 = g$ and $f * g$ is the only morphism of algebras having this property. Consequently the proposition is proved.

Notice that the preceding proposition not only gives information concerning what the free product of free algebras look like but also proves the existence of the free product of such algebras.

Proposition 1.9: If Λ is an algebra, there exists a set X and an epimorphism of algebras $\tilde{f}: T[X] \longrightarrow \Lambda$.

Proof: Choose for example X to be Λ considered as a set, and $f: X \longrightarrow \Lambda$ to be the identity map. There results a morphism of algebra $\tilde{f}: T[X] \longrightarrow \Lambda$ such that $\tilde{f}i = f$. Certainly $\tilde{f}$ is an epimorphism of algebras.

Proposition 1.10: If Λ and Γ are algebras, then their free product $\Lambda * \Gamma$ exists.

Proof: Let X, Y be sets, and $\tilde{f}: T[X] \longrightarrow \Lambda$, $\tilde{g}: T[Y] \longrightarrow \Gamma$ be epimorphisms of algebras. Let $I = \ker \tilde{f}$ and $J = \ker \tilde{g}$. Further let $j_1: T[X] \longrightarrow T[X \cup Y]$ and $j_2: T[Y] \longrightarrow T[X \cup Y]$ be the natural morphism of algebras. Let $\Lambda * \Gamma$ be $T[X \cup Y]$ modulo the ideal generated by $j_1(I) \cup j_2(J)$. Letting $i_\Lambda: \Lambda \longrightarrow \Lambda * \Gamma$ be the morphism induced by j_1 and $i_\Gamma: \Gamma \longrightarrow \Lambda * \Gamma$ the morphism induced by j_2 the result follows easily.

Definition 1.11: If A is a module, the free algebra generated by A is an algebra $T(A)$ together with a morphism of modules $i: A \longrightarrow T(A)$ such that if $f: A \longrightarrow \Lambda$ is any morphism of modules where Λ is an algebra, then there is a unique morphism of algebras $\tilde{f}: T(A) \longrightarrow \Lambda$ such that $\tilde{f}i = f$. If $g: A \longrightarrow B$ is a morphism of modules, then $T(g): T(A) \longrightarrow T(B)$ is the corresponding morphism of algebras, i.e. $T(g)i_A = i_B g$.

Proposition 1.12: If A is a module, then the free algebra generated by A exists.

Proof: If X is a set and $F(X)$ is the free module generated by X it is easy to see that $T(F(X)) = T[X]$. Thus the free algebra generated by a free module exists.

Suppose now A is any module, let $\pi: F \longrightarrow A$ be an epimorphism where F is a free module. Let $N = \operatorname{Ker} \pi$, and let

$i_1: F \longrightarrow T(F)$ be the natural morphism. Let $T(A)$ be the quotient of $T(F)$ by the ideal generated by $i_1(N)$, and let $i: A \longrightarrow T(A)$ be the morphism induced by i_1. Clearly $T(A)$ together with i satisfies the desired universal property and the proposition is proved.

Notation 1.13: If A is a module, let $T_o(A) = K$ and $T_{n+1}(A) = A \otimes T_n(A)$ for n an integer greater than or equal to zero.

Note that $T_n(A) = A \otimes \ldots \otimes A$ the tensor product of A with itself n-times.

Proposition 1.14: If A is a module, then $T(A) = \oplus_{n \geq o} T_n(A)$ as a module.

Proof: Let $\Lambda = \oplus_{n \geq o} T_n(A)$. Now we want to map Λ into an algebra. In order to do this notice that $T_p(A) \otimes T_q(A)$ is naturally isomorphic with $T_{p+q}(A)$. The natural isomorphisms define a morphism of modules $\varphi: \Lambda \otimes \Lambda \longrightarrow \Lambda$ such that if $x = x_1 \otimes \ldots \otimes x_p \in T_p(A)$ and $y = y_1 \otimes \ldots \otimes y_q \in T_q(A)$, then $\varphi(x \otimes y) = x_1 \otimes \ldots \otimes x_p \otimes y_1 \otimes \ldots \otimes y_q \in T_{p+q}(A)$. Thus we have a bimorphism $\Lambda \times \Lambda \longrightarrow \Lambda$ such that (x,y) goes into $\varphi(x \otimes y)$ and Λ becomes an algebra. Further we have $A = T_1(A) \subset \Lambda$. Suppose now Γ is an algebra, and $f: A \longrightarrow \Gamma$ is a morphism of modules. For each integer n, we have a morphism of modules $\alpha_n: T_n(\Gamma) \longrightarrow \Gamma$ such that if $y = y_1 \otimes \ldots \otimes y_n \in T_n(\Gamma)$ then $\alpha_n(y) = y_1 \ldots y_n$. Further if $T_n(f): T_n(A) \longrightarrow T_n(\Gamma)$ is the tensor product of f with itself n-times, we have $\alpha_n \, T_n(f): T_n(A) \longrightarrow \Gamma$ and thus have defined a morphism of modules $\tilde{f}: \Lambda \longrightarrow \Gamma$ such that if $x \in T_n(A)$, then $\tilde{f}(x) = \alpha_n \, T_n(f)(x)$. Since there is no doubt that $\tilde{f}$ is a morphism of algebras this shows that $A \longrightarrow \Lambda$ satisfies the universal property needed to be the free algebra generated by A and proves the proposition.

<u>Comments 1.15</u>: In view of the structure of $T(A)$ as a module, the algebra $T(A)$ is frequently called the <u>tensor algebra of</u> A.

Notice that now that we have tensor products at hand the definition of algebra (1, 3.1) may be rephrased. Thus an algebra Λ is a module Λ together with morphisms of modules $\varphi: \Lambda \otimes \Lambda \longrightarrow \Lambda$ and $\eta\varphi \; K \longrightarrow \Lambda$ such that

1) the diagram

$$\begin{array}{ccc} \Lambda \otimes \Lambda \otimes \Lambda & \xrightarrow{\varphi \otimes i} & \Lambda \otimes \Lambda \\ \downarrow{\scriptstyle i \otimes \varphi} & & \downarrow{\scriptstyle \varphi} \\ \Lambda \otimes \Lambda & \xrightarrow{\varphi} & \Lambda \end{array}$$

is commutative, and

2) the diagram

$$\begin{array}{ccc} \Lambda = K \otimes \Lambda & \xrightarrow{\eta \otimes i} & \Lambda \otimes \Lambda \\ \| & \searrow{\scriptstyle i} & \downarrow{\scriptstyle \varphi} \\ \Lambda \otimes K & & \\ \downarrow{\scriptstyle i \otimes \eta} & & \\ \Lambda \otimes \Lambda & \xrightarrow{\varphi} & \Lambda \end{array}$$

is commutative, where i is the identity morphism of Λ.

Note that condition 1) says that Λ is associative and 2) that it has a unit namely $\eta(1)$. The other conditions used in defining a ring are included in the fact that $\Lambda \times \Lambda \longrightarrow \Lambda \otimes \Lambda$ is a bimorphism, or that the composite $\Lambda \times \Lambda \longrightarrow \Lambda \otimes \Lambda \xrightarrow{\varphi} \Lambda$ is a bimorphism.

Proposition 1.16: If A is a projective module, then $T(A)$ is a projective module.

Proof: If A is projective there exists a free module F, and a morphism $A \xrightarrow{i} F \xrightarrow{\pi} A$ such that πi is the identity, i.e. A is a direct summand of A free module. Thus $T(A) \xrightarrow{T(i)} T(F) \xrightarrow{T(\pi)} T(A)$ and $T(\pi)\, T(i)$ is the identity. Since $T(F)$ is free the proposition is proved.

Proposition 1.17: If A is a flat module, then $T(A)$ is a flat module.

Proof: This follows immediately from three facts. First the tensor product of flat module is flat, secondly the direct sum of flat modules is flat, and thirdly proposition 1.14.

Proposition 1.18: If A and B are modules, then $T(A \oplus B) = T(A)*T(B)$.

The proof of this proposition duplicates that of 1.8.

§2. Tensor products of algebras and free commutative algebras.

Recall that in 1.15 we observed that an algebra Λ consisted of a module Λ together with a morphism of modules $\varphi: \Lambda \otimes \Lambda \longrightarrow \Lambda$ and a morphism $\eta: K \longrightarrow \Lambda$ satisfying certain conditions. The morphism φ is called the multiplication morphism of Λ and η is called the unit of Λ. More properly $\eta(1)$ should be the unit of Λ and η should be called by some slightly different name.

Definition 2.1: Let Λ be an algebra with multiplication morphism $\varphi_\Lambda: \Lambda \otimes \Lambda \longrightarrow \Lambda$ and unit $\eta_\Lambda: K \longrightarrow \Lambda$, and let Γ be an algebra with multiplication $\varphi_\Gamma: \Gamma \otimes \Gamma \longrightarrow \Gamma$ and unit $\eta_\Gamma: K \longrightarrow \Gamma$. Let i_Λ and i_Γ denote the identity morphism of Λ and Γ respectively. The tensor product of Λ and Γ is the algebra $\Lambda \otimes \Gamma$ with multiplication the composite morphism

$$\Lambda \otimes \Gamma \otimes \Lambda \otimes \Gamma \xrightarrow{i_\Lambda \otimes T \otimes i_\Gamma} \Lambda \otimes \Lambda \otimes \Gamma \otimes \Gamma \xrightarrow{\varphi_\Lambda \otimes \varphi_\Gamma} \Lambda \otimes \Gamma$$ and unit $\eta_\Lambda \otimes \eta_\Gamma: K = K \otimes K \longrightarrow \Lambda \otimes \Gamma$ where $T: \Gamma \otimes \Lambda \longrightarrow \Lambda \otimes \Gamma$ is the twisting isomorphism.

The preceding says that if $x, x' \in \Lambda$, $y, y' \in \Gamma$ then $(x \otimes y)(x' \otimes y') = xx' \otimes yy'$, and the unit of $\Lambda \otimes \Gamma$ is the element $1 \otimes 1$. Properly we should verify that the multiplication in $\Lambda \otimes \Gamma$ is associative and that $1 \otimes 1$ is a unit before making the above definition. These verifications are left to the reader.

We have canonical morphisms of algebras $j_\Lambda: \Lambda \longrightarrow \Lambda \otimes \Gamma$ defined by $\Lambda = \Lambda \otimes K \xrightarrow{i_\Lambda \otimes \eta_\Gamma} \Lambda \otimes \Gamma$ and $\Gamma = K \otimes \Gamma \xrightarrow{\eta_\Lambda \otimes i_\Gamma} \Lambda \otimes \Gamma$.

Proposition 2.2: If $f\colon \Lambda \longrightarrow \Sigma$ and $g\colon \Gamma \longrightarrow \Sigma$ are morphisms of algebras such that the diagram

$$\begin{array}{ccccc} \Lambda \otimes \Gamma & \xrightarrow{f \otimes g} & \Sigma \otimes \Sigma & & \\ \big\downarrow T & & & \searrow \varphi_\Sigma & \\ & & & & \Sigma \\ & & & \nearrow \varphi_\Sigma & \\ \Gamma \otimes \Lambda & \xrightarrow{g \otimes f} & \Sigma \otimes \Sigma & & \end{array}$$

is commutative, then $\varphi_\Sigma \circ (f \otimes g)$ is the unique morphism of algebras $h\colon \Lambda \otimes \Gamma \longrightarrow \Sigma$ such that $hj_\Lambda = f$ and $hj_\Gamma = g$.

The proof of this proposition is immediate from the definitions.

In view of the preceding proposition it is easily seen that if Λ, Γ are commutative algebras then $\Lambda \otimes \Gamma$ is a commutative algebra such that if Σ is a commutative algebra and $f\colon \Lambda \longrightarrow \Sigma$ $g\colon \Gamma \longrightarrow \Sigma$ are morphisms of algebras then there is a unique morphism of algebras $h\colon \Lambda \otimes \Gamma \longrightarrow \Sigma$ such that $hj_\Lambda = f$ and $hj_\Gamma = g$. This means that when we are working with commutative algebras the tensor product of algebras plays exactly the same role as does the free product when we are working with all algebras.

Definition 2.3: If X is a set, <u>the free commutative algebra generated by</u> X consists of a commutative algebra Λ and a map $i\colon X \longrightarrow \Lambda$ such that if Γ is any commutative algebra and $f\colon X \longrightarrow \Gamma$ any map, then there is a unique morphism of algebras $\tilde{f}\colon \Lambda \longrightarrow \Gamma$ such that $\tilde{f}i = f$.

<u>Notation and recollections 2.4</u>: In Chapter 1, 3.2, we have already defined the notion of the free commutative algebra generated by a set X. It is just the polynomial algebra $K[X]$. Recall that $K[X]$ is a free K-module.

<u>Proposition 2.4</u>: If X and Y are sets, then

$$K[X] \otimes K[Y] = K[X \cup Y]$$

Proof: This proposition has a proof completely analogous to that of 1.8. It is left to the reader. The reader is urged to compare this proposition with 3.4 of Chapter 1.

<u>Definition 2.5</u>: If A is a module, the <u>free commutative algebra generated by</u> A is a commutative algebra $L(A)$ together with a morphism of modules $i: A \longrightarrow L(A)$ such that if Λ is a commutative algebra and $f: A \longrightarrow \Lambda$ is any morphism of modules, then there is a unique morphism of algebras $\tilde{f}: L(A) \longrightarrow \Lambda$ such that $\tilde{f}i = f$. If $g: A \longrightarrow B$ is a morphism of modules, then $L(g): L(A) \longrightarrow L(B)$ is the corresponding morphism of algebras.

<u>Proposition 2.6</u>: If A is a module, then the free commutative algebra generated by A exists.

<u>Proof</u>: There are two easy proofs of this proposition. One is an imitation of the proof of proposition 1.12 using $K[X]$. The other is made by letting $L(A)$ be the quotient of $T(A)$ by the ideal I generated by those elements of the form $i(x)\,i(y) - i(y)\,i(x)$ for $x,y \in A$. The ideal I is <u>the commutator ideal</u> of $T(A)$.

Proposition 2.7: If A and B are modules, then

$$L(A \oplus B) = L(A) \otimes L(B) .$$

This proposition follows at once from 2.4. It is the analogue of 1.18.

Proposition 2.8: If A is a projective module, then L(A) is a projective module.

Proof: If A is projective there exists a free module F , and morphism $A \xrightarrow{i} F \xrightarrow{\pi} A$ such that πi is the identity of A . Thus $L(A) \xrightarrow{L(i)} L(F) \xrightarrow{L(\pi)} L(A)$ and $L(\pi)\, L(i)$ is the identity of L(A) . If the set X is a basis for F , then $L(F) = K[X]$, so L(F) is free, and consequently L(A) is projective.

§3. Graded modules and graded algebras.

Definitions 3.1: A graded module A is a sequence $A_o, A_1, \ldots$ of modules indexed on the non-negative integers. The module A_n is called the component of A in degree n. If A and B are graded modules a morphism $f: A \longrightarrow B$ is a sequence of morphism $f_n: A_n \longrightarrow B_n$ indexed on the non-negative integers. If $f: A \longrightarrow B$ is a morphism of graded modules, then Ker f is the graded module whose component in degree n is Ker f_n, Im f is the graded module whose component in degree n is Im f_n, Coim f is the graded module whose component in degree n is Coim f_n, and Coker f is the graded module whose component in degree n is Coker f_n.

Saying that a morphism $f: A \longrightarrow B$ of graded modules is a monomorphism is equivalent to saying Ker $f = 0$. Similarly f is an epimorphism if and only if Coker $f = 0$. The sequence of morphisms of graded modules $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Notice that this is the same as saying that for every n, the sequence $A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$ is exact.

Definition 3.2: If A and B are graded modules, then $A \otimes B$ is the graded module such that $(A \otimes B)_n = \oplus_{i+j=n} A_i \otimes B_j$. The standard twisting isomorphism $T: A \otimes B \longrightarrow B \otimes A$ is defined by letting $T(a \otimes b) = (-1)^{rs}\, b \otimes a$ for $a \in A_r$, $b \in B_s$.

A graded module A is projective if every exact sequence $0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$ of graded modules is split exact.

This is equivalent to saying that each A_n is projective. Similar considerations apply to flat modules, or injective modules. We take these to be understood.

Notice that in the preceding the fact that our ground ring K is commutative entered only into definition 3.2. Otherwise all of these considerations apply equally well to modules over an arbitrary ring.

Definitions and remark 3.3: The graded module A is *concentrated in degree* n if $A_q = 0$ for $q \neq n$. If B is any module we can associate with B a graded module $B(n)$ concentrated in degree n for any n, by letting $B(n)_q = 0$ for $q \neq n$, and $B(n)_n = B$. By an abuse of notation we write K instead of $K(0)$. With this convention we have that if A is any graded module, then $K \otimes A = A = A \otimes K$.

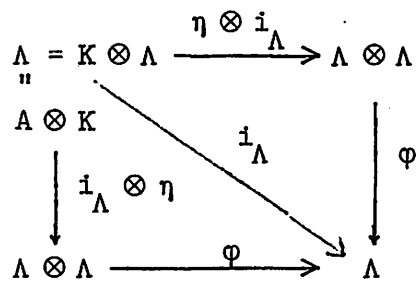
Definition 3.4: A *graded algebra* Λ is a graded module Λ together with morphism of graded modules $\varphi \colon \Lambda \otimes \Lambda \longrightarrow \Lambda$ and $\eta \colon K \longrightarrow \Lambda$ such that

1) the diagram

$$\begin{array}{ccc} \Lambda \otimes \Lambda \otimes \Lambda & \xrightarrow{i_\Lambda \otimes \varphi} & \Lambda \otimes \Lambda \\ \Big\downarrow {\scriptstyle \varphi \otimes i_\Lambda} & & \Big\downarrow {\scriptstyle \varphi} \\ \Lambda \otimes \Lambda & \xrightarrow{\varphi} & \Lambda \end{array}$$

is commutative where i_Λ is the identity morphism of Λ, and

2) the diagram



is commutative.

If $x \in \Lambda_p$, $y \in \Lambda_q$ we frequently denote by xy the element $\varphi(x \otimes y)$ belonging to Λ_{p+q}. Notice that condition 1) says that graded algebras are associative, and conditon 2) that they have a unit. Morphisms of graded algebras are defined in an evident manner.

Definition 3.5: If A is a graded module, the free graded algebra generated by A is a graded algebra $T(A)$ together with a morphism of graded modules $i\colon A \longrightarrow T(A)$ such that if Λ is any graded algebra and $f\colon A \longrightarrow \Lambda$ any morphism of modules, there is a unique morphism of graded algebras $\tilde{f}\colon T(A) \longrightarrow \Lambda$ such that $\tilde{f}i = f$.

Proposition 3.6: If A is a graded module the free graded algebra generated by A exists.

Proof: The proof is essentially the same as the proof of 1.14. We let $T_o(A) = K$, and $T_{n+1}(A) = A \otimes T_n(A)$ for n a non-negative integer. We then let $T(A) = \oplus_{n \geq o} T_n(A)$, and define a multiplication in $T(A)$ such that if $x = x_1 \otimes \ldots \otimes x_p \in T_p(A)$, and $y = y_1 \otimes \ldots \otimes y_q \in T_q(A)$ then $(x_1 \otimes \ldots \otimes x_p)(y_1 \otimes \ldots \otimes y_q) = (x_1 \otimes \ldots \otimes x_p \otimes y_1 \otimes \ldots \otimes y_q)$. Note that $xy \in T_{p+q}(A)$. Further observe that $T_n(A)$ is not the component of $T(A)$ in degree n which we denote by $T(A)_n$.

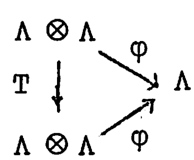
If Λ is a graded algebra, and $f\colon A \longrightarrow \Lambda$ is a morphism of graded modules then $\tilde{f}\colon T(A) \longrightarrow \Lambda$ is defined by letting $\tilde{f}(x) = f(x_1)\ldots f(x_p)$ if $x = x_1 \otimes\ldots\otimes \in T_p(A)$.

Proposition 3.7: If A is a graded module, then

1) $T(A)$ is projective if A is projective, and

2) $T(A)$ is flat if A is flat.

Definitions 3.8: The graded algebra Λ is commutative if the diagram



is commutative where T is the twisting isomorphism and φ is the multiplication of Λ .

The graded algebra Λ is strictly commutative if it is cummutative and if further for $x \in \Lambda_n$ where n is odd $x^2 = 0$.

Definition 3.9: If A is a graded module, the free strictly commutative graded algebra generated by A is a strictly commutative graded algebra $L(A)$ together with a morphism of graded modules $i\colon A \longrightarrow L(A)$ such that if Λ is a strictly commutative graded algebra and $f\colon A \longrightarrow \Lambda$ is a morphism of graded modules then there is a unique morphism of graded algebras $\tilde{f}\colon L(A) \longrightarrow \Lambda$ such that $\tilde{f}i = f$.

Proposition 3.10: If A is a graded module, the free strictly commutative graded algebra generated by A exists.

Proof: Let I be the ideal in $T(A)$ generated by the elements $xy - (-1)^{pq} yx$ for $x \in A_p$, $y \in A_q$ and the elements x^2 for $x \in A_p$ where p is odd. Let $L(A) = T(A)/I$, and let $i\colon A \longrightarrow L(A)$ be the natural morphism. Observing that the algebra $L(A)$ is strictly commutative and that it satisfies the desired universal property the proposition follows.

Definition 3.11: If Λ and Γ are graded algebras then $\Lambda \otimes \Gamma$ is the graded algebra with multiplication the composite $\Lambda \otimes \Gamma \otimes \Lambda \otimes \Gamma \xrightarrow{i_\Lambda \otimes T \otimes i_\Gamma} \Lambda \otimes \Lambda \otimes \Gamma \otimes \Gamma \xrightarrow{\varphi_\Lambda \otimes \varphi_\Gamma} \Lambda \otimes \Gamma$ where φ_Λ is the multiplication of Λ and φ_Γ the multiplication of Γ, and unit $K = K \otimes K \xrightarrow{\eta_\Lambda \otimes \eta_\Gamma} \Lambda \otimes \Gamma$ where η_Λ is the unit of Λ and η_Γ is the unit of Γ.

Let $j_\Lambda\colon \Lambda \longrightarrow \Lambda \otimes \Gamma$, and $j_\Gamma\colon \Gamma \longrightarrow \Lambda \otimes \Gamma$ be the natural morphisms of graded algebras.

Observe that if $x \otimes y \in \Lambda_p \otimes \Gamma_q$, $x' \otimes y' \in \Lambda_m \otimes \Gamma_n$ then $(x \otimes y)(x' \otimes y') = (-1)^{qm} xx' \otimes yy'$ is an element of $\Lambda_{p+m} \otimes \Gamma_{q+n} \subset (\Lambda \otimes \Gamma)_{p+m+q+n}$.

Proposition 3.12: If $f\colon \Lambda \longrightarrow \Sigma$ and $g\colon \Gamma \longrightarrow \Sigma$ are morphisms of graded algebras such that the diagram

$$\begin{array}{ccccc} \Lambda \otimes \Gamma & \xrightarrow{f \otimes g} & \Sigma \otimes \Sigma & & \\ \Big\downarrow T & & & \searrow \varphi_\Sigma & \\ & & & & \Sigma \\ & & & \nearrow \varphi_\Sigma & \\ \Gamma \otimes \Lambda & \xrightarrow{g \otimes f} & \Sigma \otimes \Sigma & & \end{array}$$

is commutative where φ_Σ is the multiplication of Σ, then $\varphi_\Sigma \circ (f \otimes g)$ is the unique morphism of graded algebras $h: \Lambda \otimes \Gamma \longrightarrow \Sigma$ such that $hj_\Lambda = f$ and $hj_\Gamma = g$.

<u>Proposition 3.13</u>: If A and B are graded modules, then

$$L(A \oplus B) = L(A) \otimes L(B) .$$

The proofs of the preceding proposition are immediate from the definition and they are left to the reader.

<u>Definition 3.14</u>: If A is a graded module A^e is the graded module such that $A_q^e = 0$ for q odd, and $A_q^e = A_q$ for q even. Similarly A^o is the graded module such that $A_q^o = 0$ for q even and $A_q^o = A_q$ for q odd. The graded module A^e is called the even part of A, and A^o is called the odd part of A. The module A is <u>even</u> if $A = A^e$, and odd if $A = A^o$.

Notice that if A is any graded module, then $A = A^e \oplus A^o$.

<u>Definitions 3.15</u>: If F is a free graded module a <u>basis for</u> F is a graded set X, i.e. a set for each non-negative integer n, such that X_n is a basis for F_n.

If X is a graded set, let $\overline{X}$ be the graded set such that an element of $\overline{X}_n$ is a function $h: \cup_q X_q \longrightarrow \mathbb{Z}$ such that

1) $h(x) \geq 0$ for $x \in \cup_q X_q$,
2) $\{x | x \in \cup_q X_q$ and $h(x) \neq 0\}$ is a finite set, and
3) if for q a non-negative integer we let

   $h_q = q(\Sigma_{x \in X_q} h(x))$, then $\Sigma_q h_q = n$.

Note that in 3) above though we are taking infinite sums in each such sum only a finite number of elements are not zero, so that we are in reality dealing with finite sums.

If $h_1 \in \overline{X}_m$, $h_2 \in \overline{X}_n$, define $h_1 h_2 \in \overline{X}_{m+n}$ by $(h_1 h_2)(x) = h_1(x) + h_2(x)$ for $x \in \cup_q X_q$. With this definition we have defined a function $\overline{X}_m \times \overline{X}_n \longrightarrow \overline{X}_{m+n}$. Consider $X \subset \overline{X}$ by $X_n \subset \overline{X}_n$ the element x of X_n being the function which takes the value 1 on x and vanishes on all other elements of $\cup_q X_q$.

Let $K[X]$ be the graded algebra over K such that $\overline{X}_n$ is a basis for $K[X]_n$ and such that the diagram

$$\begin{array}{ccc} \overline{X}_p \times \overline{X}_q & \longrightarrow & \overline{X}_{p+q} \\ \downarrow & & \downarrow \\ K[X]_p \times K[X]_q & \longrightarrow & K[X]_{p+q} \end{array}$$

is commutative for every pair of non-negative integers p and q . The inclusion $X \subset \overline{X}$ induces a morphism $i: F \longrightarrow K[X]$ if X is a basis for F .

<u>Proposition 3.16</u>: If F is an even free graded module with basis X, then $K[X] = L(F)$.

<u>Proof</u>: Let Λ be a strictly commutative graded algebra and $f: F \longrightarrow \Lambda$ a morphism of graded modules. Define $f': \bar{X} \longrightarrow \Lambda$ in the following way. Suppose $h \in \bar{X}_n$, and $x_1,\ldots,x_t$ are the elements of $\cup X_q$ on which h is not zero. Let $f'(h) = f(x_1)^{h(x_1)}\ldots f(x_t)^{h(x_t)}$. Let $\tilde{f}: K[X] \longrightarrow \Lambda$ be an induced morphism of graded modules. Observe that $\tilde{f}$ is a morphism of graded algebras, and in fact it is the only morphism of graded algebras such that $\tilde{f}i = f$ which proves the proposition.

The reader is urged to compare the preceding with Chapter 1, 3.3. The algebra $K[X]$ is the polynomial algebra generated by the graded set X . Note that we needed that X be even, i.e. $X_q = \varphi$ the empty set, in order that $K[X]$ be strictly commutative.

<u>Definition 3.17</u>: If X is a graded set, let $\tilde{X}$ be the graded set such that an element of $\tilde{X}_n$ in a function $h: \cup_q X_q \longrightarrow \{0,1\}$ such that

1) $\{x \mid x \in \cup_q X_q$ and $h(x) \neq 0\}$ is finite, and
2) if $h_q = q(\Sigma_{x \in X_q} h(x))$, then $\Sigma_q h_q = n$.

Let $E[X]$ be the algebra which is as a K-module the free graded module with basis $\tilde{X}$. Consider that $\tilde{X} \subset E[X]$, and let the multiplication in $E[X]$ be such that if $h_1 \in \tilde{X}_m$ and $h_2 \in \tilde{X}_n$ then h_1h_2 is zero if there exists $x \in \cup_q X_q$ such that $h_1(x) = h_2(x) = 1$ and such that if there is no such x then h_1h_2 is the element of

$\tilde{X}_{m+n}$ such that $(h_1 h_2)(x) = h_1(x) + h_2(x)$ for $x \in U_q X_q$.

Notice that we consider that the empty set is the basis for the zero module. Further in the preceding when we say $h_1 h_2$ is zero we mean the zero of the module $E[X]_{m+n}$.

If F is an odd free graded module, and X is a basis for F , then X is an odd set. Using the imbedding $X \longrightarrow \tilde{X}$ defined as in 3.15 we have $i: F \longrightarrow E[X]$. Since the set X is odd we have that the algebra $E[X]$ is strictly commutative.

Proposition 3.18: If F is an odd free graded module with basis X , then $E[X] = L(F)$.

The proof is as the proof of 3.16.

Proposition 3.19: If F is a free graded module, then $L(F)$ is a free graded module.

Proof: W have $F = F^e \oplus F^o$, and thus $L(F) = L(F^e) \otimes L(F^o)$. The proposition now follows from 3.16, 3.18, and the fact that the tensor product of free modules is free.

Suppose that X is a basis for F . Observe that a basis for $L(F)_n$ consists of the mononomials $x_1^{r_1} \cdots x_q^{r_q}$ where $x_i \in X_{n_i}$ for $i = 1,\ldots,q$; $r_i = 1$ if n_i is odd, and r_i is positive if n_i is even, and $\Sigma r_i n_i = n$. In the special case $n = 0$, the element $1 \in L(F)_o$ must also be included.

Proposition 3.20: If A is a projective graded module, then $L(A)$ is a projective graded module.

Proof: Choose F free and morphisms i and π $A \xrightarrow{i} F \xrightarrow{\pi} A$ such that πi is the identity morphism of A. Now $L(A) \xrightarrow{L(i)} L(F) \xrightarrow{L(\pi)} L(A)$ and $L(\pi)\, L(i)$ is the identity morphism of $L(A)$. In view of 3.19 this proves the proposition.

Definition 3.21: If A is a module, the exterior algebra of A is the graded algebra $L(A(1))$. The exterior algebra of A is denoted by $E(A)$. If $f: A \longrightarrow B$ is a morphism of modules, then $E(f): E(A) \longrightarrow E(B)$ is the corresponding morphism of graded algebras.

Recall that $A(1)$ is the graded module such that $A(1)_q = 0$ for $q \neq 1$, and $A(1)_1 = A$. Thus we have $E(A)_0 = K$, $E(A)_1 = A$. The module $E(A)_q$ is called the q-th exterior power of A. Note that if A is projective, so is $A(1)$, and thus $E(A)$ is projective and $E(A)_q$ is projective for any q.

Exercises.

1. Show that if Λ, Γ and Σ are algebras, then $(\Lambda * \Gamma) * \Sigma = \Lambda * (\Gamma * \Sigma)$.

2. Give an example of a ring K and algebras Λ and Γ over K such that $\Lambda * \Gamma = 0$.

3. Show that if G and H are groups (not necessarily abelian), there exists a group $G * H$ and morphisms of groups $i_G\colon G \longrightarrow G * H$, and $i_H\colon H \longrightarrow G * H$ such that if π is a group and $g\colon G \longrightarrow \pi$, $h\colon H \longrightarrow \pi$ are morphisms, then there is a unique morphism $f\colon G * H \longrightarrow \pi$ such that $f_{i_G} = g$ and $f_{i_H} = h$.

4. Show that if M is a monoid, there exists an algebra $K(M)$ and morphism of monoids $i\colon M \longrightarrow K(M)$ such that if Λ is an algebra and $f\colon M \longrightarrow \Lambda$ is a morphism of monoids then there is a unique morphism of algebras $\tilde{f}\colon K(M) \longrightarrow \Lambda$ such that $\tilde{f}i = f$. (Recall that when considering a ring Λ as a monoid it is the multiplication in the ring which is used to give it the monoid structure (Chapter 2) and the unit of the monoid is the element $1 \in \Lambda$. Compare this problem with 11 of Chapter 2.)

5. If G is a group the algebra $K(G)$ of the preceding problem is called the group algebra of G. Show that if G and H are groups, then $K(G * H) = K(G) * K(H)$.

6. Show that if Λ and Γ are algebras, there is an epimorphism $f\colon \Lambda * \Gamma \longrightarrow \Lambda \otimes \Gamma$. Describe the generators of $\operatorname{Ker} f$.

7. Show that if Λ, Γ and Σ are algebras, then $(\Lambda \otimes \Gamma) \otimes \Sigma = \Lambda \otimes (\Gamma \otimes \Sigma)$.

8. Show that if G and H are groups $K(G \times H) = K(G) \otimes K(H)$.

9. Show that if H is a subgroup of G, then $K(G)$ is a free $K(H)$ module.

10. Let I be a partially ordered set, and let A be a system of algebras indexed on I, i.e. for $i \in I$, A_i is an algebra, if $i \leq j$, $\alpha_{i,j}\colon A_i \longrightarrow A_i$ is a morphism of algebras, and if $i \leq j$, $j \leq k$, then $\alpha_{j,k}\,\alpha_{i,j} = \alpha_{i,k}$. Define $\varinjlim /A$ and $\varprojlim /A$ using algebras in a fashion similar to that used to define direct limits and inverse limits of modules in Chapter 3, §5. Show that $\varprojlim /A$ always exists. Give an example to show that $\varinjlim /A$ may not exist. Show that if I is direct, then $\varinjlim /A$ exists. (Recall that algebras have units and that if $f\colon \Lambda \longrightarrow \Gamma$ is a morphism of algebras then $f(1) = 1$.)

11. Let I be a direct set and $\mathbb{A}$ a system of modules indexed on I. Show that there is a direct system of algebras $T(\mathbb{A})$ indexed on I such that if $i \in I$ the corresponding algebra is $T(A_i)$, and if $\alpha_{i,j}: A_i \longrightarrow A_j$ for $i \leq j$, then $T(\alpha_{i,j}): T(A_i) \longrightarrow T(A_j)$. Show that $\varinjlim T(\mathbb{A}) = T(\varinjlim \mathbb{A})$.

12. Let I be a direct set and $\mathbb{A}$ a system of modules indexed on I. Let $L(\mathbb{A})$ be the corresponding system of free commutative algebras. Show that $\varinjlim L(\mathbb{A}) = L(\varinjlim \mathbb{A})$. Show that if for each $i \in I$, A_i is projective, then $L(\varinjlim \mathbb{A})$ is a flat K-module.

13. Define the notions of direct and inverse limits for graded modules and graded algebras. Derive the elementary properties of such limits and carry out the analogues of exercises 10, 11 and 12.

14. If A is a graded module the free commutative algebra generated by A is a commutative graded algebra Λ together with a morphism of graded modules $i: A \longrightarrow \Lambda$ such that if Γ is any commutative graded algebra and $f: A \longrightarrow \Gamma$ is any morphism of graded modules, then there is a unique morphism of graded algebras $\tilde{f}: \Lambda \longrightarrow \Gamma$ such that $\tilde{f}i = f$. Prove that if A is any graded

module the free commutative graded algebra generated by A exists. Show that if A is even, then $\Lambda = L(A)$. Give an example to show that it may happen that $\Lambda \neq L(A)$. Prove that there is always an epimorphism $\Lambda \longrightarrow L(A)$.

15. Give an example of a free graded module A over some ring K such that the free commutative graded algebra generated by A is not projective over K .

16. Show that if K is an integral domain of characteristic different from two (i.e. such that $1 + 1 \neq 0$), and Λ is a graded commutative algebra over K which is a flat K-module, then Λ is strictly commutative.

17. If A is a graded module, let $\oplus A = \oplus_n A_n$. Thus $\oplus A$ is a module (not graded). Show that if A and B are graded modules, then $(\oplus A) \otimes (\oplus B) = \oplus (A \otimes B)$.

18. Show that if Λ is a graded algebra, then $\oplus \Lambda$ is an algebra. Show that if Λ is commutative and even, then $\oplus \Lambda$ is commutative.

19. Let A be a module, and let $i: A \longrightarrow \oplus E(A)$ be the natural morphism of modules. Show that if Λ is any algebra and $f: A \longrightarrow \Lambda$ a morphism of modules such that $f(a)^2 = 0$ for $a \in A$, then there is a unique morphism of algebras $\tilde{f}: \oplus E(A) \longrightarrow \Lambda$ such that $\tilde{f}i = f$.

Suggested Reading

Bourbaki, N.

Algèbre, Livre II, Chapitre III, Algèbre multilinéaire, Paris, 1948.

Chevalley, C.

Fundamental Concepts of Algebra, New York, 1956. Particularly read Chapters IV and V.