Chapter 2: Modules, monoids, and rings of fractions.

In this chapter we wish to proceed further with the study of commutative rings and modules over commutative rings. Consequently we adopt the convention that throughout the chapter ring means commutative ring.

§1 . Monoids and prime ideals.

Definitions 1.1: A monoid consists of a set M , a function $\varphi: M \times M \longrightarrow M$, and an element $1 \in M$ such that if we denote $\varphi(x,y)$ by xy for $x,y \in M$, then

i) $x(yz) = xy)z$ for $x,y,z \in M$, and

ii) $1 \cdot x = x = x \cdot 1$ for $x \in M$.

If M and N are monoids, a function $f: M \longrightarrow N$ is a morphism of monoids if

i) $f(xy) = f(x)\, f(y)$ for $x,y \in M$, and

ii) $f(1) = 1$.

The monoid M is commutative if $xy = yx$ for $x,y \in M$.

Since in this chapter we are dealing only with commutative rings, we adopt a similar restriction concerning monoids. Thus monoid will mean commutative monoid.

If K is a ring, then K gives rise to a monoid also denoted by K . This monoid is obtained by forgetting about the addition operation in K . In other words if $\varphi: K \times K \longrightarrow K$ is the multiplication, and $1 \in K$ is the unit we have a monoid. Thus by a submonoid of K we mean a multiplicatively closed subset M of K such that $1 \in M$. More generally if M is a monoid when we say that $f: M \longrightarrow K$ is a morphism we mean that it is a morphism of monoids where K is endowed with the structure of a monoid as described above.

In recalling the definition of integral domain (Chapter 1, §2) we were a little careless. Our general definition of ring demands that the ring K must have a unit 1 . However, it is possible that $1 = 0$. In which case there is just one element in the ring, and in every module over the ring. In an integral domain this situation does not obtain, for an integral domain is a ring K such that $0 \neq 1$, and such that if $x,y \in K$ are elements such that $xy = 0$ then $x = 0$ or $y = 0$. This is the reason that for P to be a prime ideal in K we must have that $P \neq K$.

<u>Proposition 1.2</u>: An ideal I in K is a prime ideal if and only if $K - I$ is a submonoid of K .

The proof of the proposition is immediate from the definitions. Observe that since $K - I$ is a submonoid of K , we have $1 \in K - I$ and I is a proper ideal.

<u>Lemma 1.3</u>: Let I be an ideal in K , and let M be a submonoid of K such that $M \cap I = \emptyset$, then there is a prime ideal P in K

such that

1) $I \subset P$,

2) $M \cap P = \emptyset$, and

3) if P' is a prime ideal in K such that $I \subset P' \subset P$, then $P' = P$.

Proof: Let $\mathcal{M}$ be the set of all submonoids M_1 of K such that $M \subset M_1$, and $M_1 \cap I = \emptyset$. Since $M \in \mathcal{M}$, we have $\mathcal{M} \neq 0$. If $\{M_\alpha\}_{\alpha \in A}$ is a subset of $\mathcal{M}$ such that if $\alpha, \beta \in A$ then either $M_\alpha \subset M_\beta$ or $M_\beta \subset M_\alpha$, we have $\cup_{\alpha \in A} M_\alpha \in \mathcal{M}$. Therefore $\mathcal{M}$ has maximal elements. Suppose N is a maximal element of $\mathcal{M}$, and let $P = K - N$. Suppose $y_1, y_2 \in P$ then there exist integers r_1, r_2 and elements $m_1, m_2 \in N$ such that $y_1^{r_1} m_1 \in I$ and $y_2^{r_2} m_2 \in I$ for otherwise using the maximality of N we would have y_1 or $y_2 \in N$. Now if $r = r_1 + r_2$, and $m = m_1 m_2$, we see upon looking at the binomial expansion of $(y_1 + y_2)^r$ that $(y_1 + y_2)^r m \in I$, so it is not possible that $(y_1 + y_2) \in N$, and thus $(y_1 + y_2) \in P$.

Suppose $k \in K$, then $(k\, y_1)^{r_1} m_1 \in I$, so $k\, y_1 \notin N$, and $k\, y_1 \in P$, proving that P is an ideal in K . Since $P = K - N$ and N is a submonoid of K , we have that P is a prime ideal. The maximality of the submonoid N of K among the elements of $\mathcal{M}$ implies the minimality of the ideal P among prime ideals containing I , and the lemma is proved.

Proposition 1.3: If I is a proper ideal in K, then $\sqrt{I}$ is the intersection of those prime ideals in K which contain I and are minimal among the prime ideals containing I.

Proof: Proof if $x \notin \sqrt{I}$, then $1, x, x^2, \ldots$ is a submonoid of K which does not intersect I. Applying the preceding proposition there is a prime ideal P such that $x \notin P$ and this prime ideal is minimal among prime ideals of K which contain I, hence the lemma.

It is suggested that the reader compare this proposition with Proposition 2.14 of Chapter 1.

Lemma 1.4: Let M be a submonoid of K, and A a sub K-module of B. If $A(M) = \{b \mid b \in B$ and for some $m \in M$, $m\,b \in A\}$, then $A(M)$ is a sub K-module of B containing A.

The proof of the lemma is immediate from the definitions.

Proposition 1.5: Let A be a proper sub K-module of B such that B/A is finitely generated. If I is the annihilator of B/A and P is an ideal in K minimal among those prime ideals containing I, then

i) $A(K-P)$ is a primary submodule of B with associated prime ideal P, and

ii) if C is a primary submodule of B such that $A \subset C \subset A(K-P)$, then $C = A(K-P)$.

Proof: By the preceding lemma $A(K-P)$ is a submodule of B containing A. Suppose $x \in K-P$, $b \in B$ and $xb \in A(K-P)$, then for some $y \in K-P$, $xyb \in A$ so that $b \in A(K-P)$.

Suppose $b_1,\ldots,b_n$ are elements of B which generate B modulo A. If $A(K-P) = B$ there exist $v_1,\ldots,v_n \in K-P$ such that $v_i b_i \in A$. Let $v = v_1 \ldots v_n$. Now $v \in K-P$ and $v b_i \in A$ for $i = 1,\ldots,n$ so $v \in I$ which is impossible. Therefore $A(K-P)$ is a proper submodule of B.

Let J be the annihilator of $B/A(K-P)$, and note that $I \subset J$. Suppose $y \in J$. Choose $w_i \in K-P$ such that $w_i y b_i \in A$ for $i = 1,\ldots,n$, and let $w = w_1 \ldots w_n$. Since $wy \in I$ and $w \notin P$, it follows that $y \in P$ and $J \subset P$.

Since $K-P$ is maximal among the submonoids of K disjoint from I, it follows that if $y \in P$ there exists $x \in K-P$ and a positive integer m such that $x y^m \in I$. There $x y^m b_i \in A$ for $i = 1,\ldots,n$, and $y^m b_i \in A(K-P)$ for $i = 1,\ldots,n$, showing that $y^m \in J$ and $\sqrt{J} = P$.

Combining the preceding paragraphs we have that $A(K-P)$ is a primary submodule of B with associated prime ideal P. Let C be a primary submodule of B such that $A \subset C \subset A(K-P)$, and let I' be the annihilator of B/C. Since $I \subset I' \subset J$, and $\sqrt{I'}$ is a prime ideal containing I we have $\sqrt{I'} = P$, and that the associated prime ideal of C in B is P. Suppose that $b \in A(K-P)$, and choose $v \in K-P$ such that $v\,b \in A \subset C$. Since $v \notin P$ and C is primary with associated prime ideal P, we have $b \in C$, $C = A(K-P)$, and thus the proposition is proved.

Proposition 1.6: Suppose that A is a proper sub K-module of B, $A_1,\ldots,A_n$ is a reduced primary decomposition of A in B, the associated prime ideal of A_i in B is P_i for $i = 1,\ldots,n$, I is the annihilator of B/A, P is an ideal in K minimal among those prime ideals containing I, and B/A is finitely generated, then

i) for some i between 1 and n, $P = P_i$, and

ii) $A_i = A(K-P)$.

Proof: Since $P \supset I$, and $\sqrt{I} = P_1 \cap \ldots \cap P_n$, we have $P = P_i$ for some i between 1 and n. Consequently $A_i \cap A(K-P)$ is a primary submodule of B with associated prime ideal P. Applying the preceding proposition, we have $A_i \cap A(K-P) = A(K-P)$, and thus $A(K-P) \subset A_i$.

Using the fact that P is a minimal prime containing I, choose $x_j \in P_j - P$ for $j \neq i$. Choose an integer m such that x_j^m belongs to the annihilator of B/A_j for $j \neq i$, and let $x = x_1^m \ldots x_{i-1}^m\, x_{i+1}^m \ldots x_n^m$. Now $x \notin P$, and if $b \in A_i$, then $b\,x \in A_1 \cap \ldots \cap A_n = A$ so that $b \in A(K-P)$ and the proposition is proved.

Notice that the preceding proposition may be considered as an addendum to Theorem 2.11 of Chapter 1.

§2. Rings of fractions and modules over rings of fractions.

Recall that element x of a ring K is a unit if there exists an element $y \in K$ such that $xy = 1$. The element y is unique and is usually denoted by x^{-1}. Notice that the set of all units of K is a group, the group operation being induced by multiplication in K. This group is called the group of units of K.

Given a ring K and a submonoid M of K, there is interest in finding a ring Λ and a morphism $\pi: K \longrightarrow \Lambda$ such that if $m \in M$, then $\pi(m)$ is a unit in Λ. The most classical example of this procedure is when K is an integral domain, $M = K - \{0\}$, Λ is the field of fractions of K, and $\pi: K \longrightarrow \Lambda$ is the imbedding of K in its field of fractions. Generalizations of the procedure of constructing the field of fractions of an integral domain were introduced and studied by Grell and Krull in the 1920's and 1930's. In this paragraph, we begin the study of such generalizations. The classical case will be included.

Definition 2.1: Let M be a submonoid of the ring K. A ring of fractions of K relative to M is a ring Λ and a morphism $\pi: K \longrightarrow \Lambda$ such that

i) $\pi(m)$ is a unit of for $m \in M$, and

ii) if $f: K \longrightarrow \Gamma$ is a morphism such that $f(m)$ is a unit of Γ for $m \in N$, then there exists a unique morphism $\tilde{f}: \Lambda \longrightarrow \Gamma$ such that $\tilde{f}\pi = f$.

<u>Proposition 2.2</u>: If M is a submonoid of the ring K, there exists $\pi: K \longrightarrow \Lambda$, a ring of fractions of K relative to M, and if $\pi': K \longrightarrow \Lambda'$ in another such ring of fractions there is a unique isomorphism $\theta: \Lambda \longrightarrow \Lambda'$ such that $\theta \circ \pi = \pi'$.

<u>Proof</u>: Let $\tilde{\Lambda}$ be the set of pairs (k,m) such that $k \in K$ and $m \in M$. Define addition and multiplication in $\tilde{\Lambda}$ by letting $(k,m) + (k',m') = (m'k + mk', mm')$, and $(k,m)(k',m') = (kk', mm')$. Observe that $\tilde{\Lambda}$ is an abelian group with zero element $(0,1)$ under the operation of addition. Further the multiplication in $\tilde{\Lambda}$ is associative and has a unit, $(1,1)$. However, $\tilde{\Lambda}$ is not a ring since the multiplication is not distributive with respect to addition.

Introduce an equivalence relation in $\tilde{\Lambda}$ so that (k,m) is equivalent to (k',m') $((k,m) \sim (k',m'))$ if and only if there exists $m'' \in M$ such that $m''m'k = m''mk'$. Denote the set of equivalence classes by Λ, and observe that the addition and multiplication in $\tilde{\Lambda}$ induce operations of addition and multiplication in Λ which make Λ into a ring. Denote the equivalence class of (k,m) by k/m. Note that the zero of Λ is the element $^0/_1$ and the unit, the element $^1/_1$.

Define $\pi: K \longrightarrow \Lambda$ by letting $\pi(k) = k/1$, and notice that if $m \in M$, then $\pi(m) = m/1$, and $m/1 \cdot 1/m = {}^1/_1$ so that $\pi(m)$ is a unit in Λ.

Suppose $f: K \longrightarrow \Gamma$ is a morphism such that $f(m)$ is a unit in Γ for $m \in M$. Observe that if $(k,m) \sim (k',m')$, then $f(k)\, f(m)^{-1} = f(k')\, f(m')^{-1}$. Define $\tilde{f}: \Lambda \longrightarrow \Gamma$ by letting $\tilde{f}(k/m) = f(k)\, f(m)^{-1}$. Certainly $\tilde{f}\,\pi = f$, and the existence part of the proposition is proved. The proof of uniqueness is standard, and it is left for the reader.

Since $\pi: K \longrightarrow \Lambda$ is a morphism, we see that Λ is a K-algebra. The concept of ring of fractions of K could easily have been formulated in terms of K-algebras.

Observe that it may happen that $0/1 = 1/1$, and that this is the case if and only if $0 \in M$. Most usually this case will be avoided.

Now that we have proved the existence and uniqueness of rings of fractions of K relative to M, we will refer to the ring of fractions of K relative to M instead of a ring of fractions of K relative to M.

Proposition 2.3: Let M be a submonoid of K, and $\pi: K \longrightarrow \Lambda$ the ring of fractions of K relative to M.

i) If J is an ideal in Λ, and $I = \pi^{-1}(J)$, then $J = \Lambda\, \pi(I)$.

ii) If I is an ideal in K, then $\pi^{-1}(\Lambda\, \pi(I)) = I(M)$.

Proof: Under the conditions of the first part of the proposition, $\Lambda\,\pi(I) \subset J$. If $k/m \in J$, then $k/1 \in \Lambda(\pi(I))$, and $k/m = {}^1/_m \cdot k/_1 \in \Lambda\,\pi(I)$ which proves the first part of the proposition.

If I is an ideal in Λ, and $x \in \pi^{-1}(\Lambda(\pi\, I))$, then $\pi(x) = y/m$ where $y \in I$, $m \in M$. Therefore, there exists $m' \in M$ such that $m'mx = m'y$. Since $m'm \in M$, and $m'y \in I$, we have $x \in I(M)$.

If $x \in I(M)$ there exists $m \in M$ such that $mx \in I$. Since $\pi(m)$ is a unit in Λ this implies $\pi(x) \in \Lambda\pi(I)$, and the proposition is proved.

Corollary 2.4: If K is Noetherian, then Λ is Noetherian.

Proof: If J is an ideal in Λ, there is a finite subset X of $\pi^{-1}(J)$ which generates $\pi^{-1}(J)$ as a K-module since K is Noetherian. Applying the first part of the preceding proposition, $\pi(X)$ generates J as a Λ-module, and so any ideal in Λ is finitely generated which proves the corollary.

Corollary 2.5: Λ is an integral domain if and only if Ker π is a prime ideal in K.

Proof: Suppose Ker π is a prime ideal in K, then $0 \neq 1$ in $K/\text{Ker}\,\pi$, so $0 \neq 1$ in Λ, and $M \subset K - \text{Ker}\,\pi$. If $k/m \cdot k'/m' = 0$, there exists $m'' \in M$ such that $m''kk' = 0$. Since $m'' \in M$, this implies $k,' \in \text{Ker}\,\pi$, and either $k \in \text{Ker}\,\pi$ in which case $k/m = 0$, or $k' \in \operatorname{Ker} \pi$ in which case $k'/m' = 0$.

The converse part of the corollary is immediate.

Proposition 2.6: Let M, N be submonoids of K such that $M \subset N$, $\pi': K \longrightarrow K'$ the ring of fractions of K relative to M , and $\pi'': K \longrightarrow K''$ the ring of fractions of K relative to N. If $\theta: K' \longrightarrow K''$ is the unique morphism such that $\theta \pi' = \pi''$, then $\theta: K' \longrightarrow K''$ is the ring of fractions of K' relative to $\pi'(N)$.

Proof: Suppose $f: K' \longrightarrow \Gamma$ is a morphism and $f \pi'(x)$ is a unit in Γ for $x \in N$, then there is a unique morphism $f_1: K'' \longrightarrow \Gamma$ such that $f_1 \pi'' = f \pi'$. Since $\theta \pi' = \pi'$, $f_1 \theta \pi' = f \pi'$, and this implies $f_1 \theta = f$. Checking that if $f_2 \theta = f$ then $f_2 = f_1$, we have that $\theta: K' \longrightarrow K''$ satisfies the universal property necessary for it to be the ring of fractions of K' relative to $\pi'(N)$, and the proposition is proved.

Definition 2.7: Let $\pi: K \longrightarrow \Lambda$ be a morphism. If A is a K-module, the extended module of A relative to π is

i) a Λ-module $T(A)$, and

ii) a morphism of K-modules $\pi_A: A \longrightarrow T(A)$, such that if B is a Λ-module, and $f: A \longrightarrow B$ is a morphism of K-modules, then there is a unique morphism of Λ-modules $\tilde{f}: T(A) \longrightarrow B$ such that $\tilde{f} \pi_A = f$.

Since the extended module of A is defined by a universal property if it exists it is unique. In a later chapter we will see that extended modules exist relative to any morphism rings. In this chapter we will content ourselves with showing existence in the special case that $\pi\colon K \longrightarrow \Lambda$ is the ring of fractions of K relative to a submonoid M of K.

Observe that in the preceding definition we used the fact that any Λ module may be considered as a K-module via the morphism π.

<u>Theorem 2.8</u>: Let M be a submonoid of K, and $\pi\colon K \longrightarrow \Lambda$ the ring of fractions of K relative to M.

i) If A is a K-module, then the extended module of A relative to π, $\pi_A\colon A \longrightarrow T(A)$ exists.

ii) If $f\colon A \longrightarrow A'$ is a morphism of K-modules there is a unique morphism of Λ-modules, $T(f)\colon T(A) \longrightarrow T(A')$ such that $T(f)\,\pi_A = \pi_{A'} f$.

iii) If $A'' \xrightarrow{f} A \xrightarrow{g} A'$ is an exact sequence of K-modules, then $T(A'') \xrightarrow{T(g)} T(A) \xrightarrow{T(g)} T(A')$ is an exact sequence of Λ-modules.

<u>Proof</u>: Let $\tilde{T}(A)$ be the set of pairs (a,m) such that $a \in A$, $m \in M$. Define an equivalence relation in $\tilde{T}(A)$ by saying that (a,m) is equivalent to (a',m') if there exists $m'' \in M$ such that

$m''m'a = m''ma'$, and let $T(A)$ denote the set of equivalence classes. Making the appropriate verifications define addition in $T(A)$ so that if a/m denotes the equivalence class of (a,m) , then $a/m + a'/m' = m'a + ma'/mm'$. Similarly define the operation of Λ on $T(A)$ by $k/m' \; a/m' = ka/mm'$ for $k \in K$, $m,m' \in M$, and $a \in A$. Define $\pi_A: A \longrightarrow T(A)$ by $\pi_A(a) = a/1$.

If $f: A \longrightarrow B$ is a morphism of K-modules, and B is a Λ-module, one verifies that a morphism of Λ-modules $\tilde{f} : T(A) \longrightarrow T(B)$ is defined by $\tilde{f}\,(a/m) = \pi(m)^{-1}\, f(a)$, and that $\tilde{f}\,\pi_A = \pi_B f$. Checking that if $g: T(A) \longrightarrow B$ is a morphism of Λ-modules such that $g\pi_A = \pi_B f$ implies that $g = \tilde{f}$, the proof of the first part of the theorem is complete.

The second part of the theorem follows immediately from the definition. Notice that $T(f)\; a/m = f(a)/m$.

In order to verify the third part of the theorem first observe that since $gf = 0$, $T(g)\,T(f) = 0$, and $\operatorname{Ker} T(g) \supset \operatorname{Im} T(f)$. Now if $a/m \in \ker T(g)$, there exists $m' \in M$ such that $0 = m'g(a) = g(m'a)$. Consequently there exists $a' \in A'$ such that $f(a') = m'a$, and since $T(f)\; a'/m'm = m'a/m'm = a/m$, $\operatorname{Ker} T(g) \subset \operatorname{Im} T(f)$, and the proof of the theorem is complete.

<u>Proposition 2.9</u>: Let M be a submonoid of K , and $\pi: K \longrightarrow \Lambda$ the ring of fractions of K relative to M . If A is a Λ-module,

then the identity map $i_A: A \longrightarrow A$ is the extended module of A relative to π .

Proof: Suppose B is a Λ-module, and $f: A \longrightarrow B$ is a morphism of K-modules. In order to prove the proposition it suffices to show that f is a map of Λ-modules. If $k/m \in \Lambda$ and $a \in A$, then $mf(k/m\ a) = f(m \cdot k/m \cdot a) = f(k \cdot a) = kf(a)$. Since $\pi(m)$ is a unit in Λ , we may multiply both sides of the preceding equation by $\pi(m)^{-1}$, and thus obtain $f(k/m\ a) = k/m\ f(a)$. This proves f is a morphism of Λ-modules, and hence the proposition.

Proposition 2.10: Let M be a submonoid of K , $\pi: K \longrightarrow \Lambda$ the ring of fractions of K relative to M , I a primary ideal in K such that $M \cap I = \emptyset$, and $P = \sqrt{I}$, then

i) $M \cap P = \emptyset$,

ii) $T(I)$ is a primary ideal in Λ ,

iii) $\sqrt{T(I)} = T(P)$,

iv) $I = I(M)$, and

v) $P = P(M)$.

Proof: Suppose $x \in M \cap P$, then for some positive integer n , $x^n \in M \cap I$ which is impossible so $M \cap P = \emptyset$.

Suppose $k/m \cdot k'/m' \in T(I)$, and $k'/m' \notin T(I)$. Now for some $m'' \in M$, $kk'm'' \in I$ and $k'm'' \notin I$ so that for some positive integer n ,

$k^n \in I$. This implies $({}^k/_m)^n \in T(I)$ and $T(I)$ is primary. Further it implies $^k/_m \in T(P)$ so $\sqrt{T(I)} \subset T(P)$. Clearly $T(P) \supset \sqrt{T(I)}$ and so $\sqrt{T(I)} = T(P)$.

Suppose $x \in I(M)$, then there exists $m \in M$ such that $mx \in I$. Since $m^n \notin I$ for any positve integer n , and I is primary, we have $x \in I$, and $I = I(M)$. The preceding also implies $P = P(M)$ and hence the proposition.

Proposition 2.11: Let M be a submonoid of K , $\pi: K \longrightarrow \Lambda$ the ring of fractions of K relative to M , and I an ideal in K , then $\pi_{K/I}: {}^K/_I \longrightarrow T({}^K/_I)$ is the ring of fractions of K/I relative to the image of M in K/I .

Proof: Note that Λ itself is the extended module of K relative to $\pi: K \longrightarrow \Lambda$, so we could write $T(K)$ instead of Λ . By 2.8, iii), the sequence $0 \longrightarrow T(I) \longrightarrow T(K) \longrightarrow T(K/I) \longrightarrow 0$ is exact, so $T(I)$ is an ideal in $T(K)$, and $T(K/I)$ is a ring. Certainly $\pi_{K/I}: K/I \longrightarrow T(K/I)$ is a morphism of rings. Verifying that it satisfies the universal property for it to be the ring of fractions of K/I relative to the image of M in K/I , the proposition follows.

Notations and remarks 2.12: If P is a prime ideal in K , the ring of fractions of K relative to $K-P$ is denoted by $\pi: K \longrightarrow K_P$. If A is a K-module the extended module of A relative to π is denoted by $\pi_A: A \longrightarrow A_P$. Further if $f: A \longrightarrow B$ is a morphism of K-modules, $f_P: A_P \longrightarrow B_P$ denotes the corresponding morphism of K_P modules.

In one special case a notation different from the preceding is used. If K is an integral domain, then the ring of fractions of K relative to $K - \{0\}$ is denoted by $Q(K)$. The field $Q(K)$ is called the <u>field of fractions of K</u> . Since $\pi: K \longrightarrow Q(K)$ is a monomorphism, we may consider that K is a subring of $Q(K)$. If M is any submonoid of $K - \{0\}$, the ring of fractions of K relative to M , is just the sub K-algebra of $Q(K)$ generated by the elements m^{-1} for $m \in M$. In particular if P is a prime ideal in K , then $K \subset K_P \subset Q(K)$, and K_P is the sub K-algebra of $Q(K)$ generated by the elements x^{-1} for $x \in K-P$.

<u>Theorem 2.13</u>: Let P be a prime ideal in K , and $\pi: K \longrightarrow K_P$ the ring of fractions of K relative to $K - P$, then

i) every element of $K_P - P_P$ is a unit in K_P ,

ii) P_P is the only maximal ideal in K_P ,

iii) $K_P/P_P = Q(K/P)$,

iv) if I is an ideal K , then $I_P \neq K_P$ if and only if $I \subset P$,

v) J is a primary ideal in K_P if and only if $\pi^{-1}(J)$ is a primary ideal in K ,

vi) J is a prime ideal in K_P if and only if $\pi^{-1}(J)$ is a prime ideal in K, and

vii) if I, I' are primary ideals in K such that $I_P = I'_P$, then $I = I'$.

Proof: Part ii) of 2.8 implies that if $I \subset K$, then $I_P \subset K_P$. If $x \in K_P - P_P$, $x = k/m$ where $k, m \in K\text{-}P$, and since $x^{-1} = m \in K_P$ part i) follows. Further parts ii) and iii), iv) of the theorem follow from part i), and 2.11.

Suppose J is a primary ideal in K_P, and $I = \pi^{-1}(J)$. By 2.3, $I = I(K\text{-}P)$. If $x\,y \in I$ and $y \notin I$, then $x/1 \cdot y/1 \in J$, and $y/1 \notin J$ for $y/1 \in J$ would imply that there exists $m \in K\text{-}P$ such that $m\,y \in I$ so $y \in I(K\text{-}P) = I$. Therefore there is a positive integer n such that $(x/1)^n \in J$, and an element $m' \in K\text{-}P$ such that $m'\,x^n \in I$. Thus $x^n \in I$, and half of part v) is proved. The converse part of v) follows from 2.10, and vi) follows from v) and 2.10.

Part vii) follows from 2.3 and 2.10, and the proof of the theorem is complete.

<u>Definitions and comments 2.14</u>: A ring K is called a <u>local ring</u> if the non units in K form an ideal, and $0 \neq 1$ in K. Clearly K is a local ring if there exists a unique maximal ideal in K and conversely. One of the assertions of the preceding theorem says that

if K is a ring and P is a prime ideal in K, then K_P is a local ring. The ring K_P is called the localization of K at the prime P. Similarly if A is a K-module, the K_P-module A_P is called the localization of A at the prime P.

Theorem 2.14: Let K be a ring.

i) If A is a K-module, then $A = 0$ if and only if $A_M = 0$ for every maximal ideal M in K.

ii) If $f: A \longrightarrow B$ is a morphism of K-modules, then f is an epimorphism if and only if $f_M: A_M \longrightarrow B_M$ is an epimorphism for every maximal ideal M in K, and f is a monomorphism if and only if $f_M: A_M \longrightarrow B_M$ is a monomorphism for every maximal ideal M in K.

Proof: If $A \neq 0$, then A has a non-zero submodule A', with 1-generator, so to prove part i) of the theorem it suffices to prove it under the additional assumption that A has a single generator in view of 2.8, iii). Thus we may assume $A = K/I$ where I is an ideal in K. Now if $(K/I)_M = 0$ for every maximal ideal M in K, then $I_M = K_M$ for every such ideal, and by 2.13, iv), $I \not\subset M$ for any maximal ideal M. Consequently $I = K$, and $K/I = 0$. This proves part i).

Suppose now $f: A \longrightarrow B$. We have an exact sequence $0 \longrightarrow \text{Ker } f \longrightarrow A \xrightarrow{f} B \longrightarrow \text{Coker } f \longrightarrow 0$, and for every maximal ideal M in K , an exact sequence $0 \longrightarrow (\text{Ker } f)_M \longrightarrow A_M \xrightarrow{f_M} B_M \longrightarrow (\text{Coker } f)_M \longrightarrow 0$. Consequently part i) implies part ii), and the theorem is proved.

Notice that in the preceding it was not necessary to study localizations at every prime in K , but only at the maximal ideals.

Exercises

1. Show that if I is an ideal in K, and $\sqrt{I}$ is a maximal ideal in K, then K/I is a local ring.

2. Let A be a K-module, and let I be the annihilator of A. Show that if $\sqrt{I}$ is a maximal ideal in K, then $A = A_{\sqrt{I}}$ and $A_M = 0$ for M a maximal ideal in K, $M \neq \sqrt{I}$.

3. Suppose that if K is a local ring with maximal ideal M, and A is a finitely generated K-module such that $MA = A$; then $A = 0$.

4. Give an example to show that there is a local ring K with maximal ideal M, and a K-module A such that $MA = A$ and $A \neq 0$.

5. Let K be a local ring with maximal ideal M, and let A and B be K-modules such that $A \subset B$, and $A + MB = B$. Show that $A = B$ if B is finitely generated. (The preceding is known as Nakayama's lemma.)

6. List all monoids with 3 elements.

7. Show that if X is a set, there exists a monoid $M(X)$ and a function $i: X \longrightarrow M(X)$ such that if N is a monoid, and $f: X \longrightarrow N$ is a function, then there is a unique morphism of monoids $\tilde{f}: M(X) \longrightarrow N$ such that $\tilde{f}i = f$. The monoid $M(X)$ is called the free (commutative) monoid generated by X .

8. Show that if M is a monoid, there exists a group $G(M)$ and a morphism of monoids $i: M \longrightarrow G(M)$ such that if H is a group and $f: M \longrightarrow H$ is a morphism of monoids, there is a unique morphism of groups $\tilde{f}: G(M) \longrightarrow H$ such that $\tilde{f}i = f$.

9. Determine necessary and sufficient conditions that the morphism $i: M \longrightarrow G(M)$ of the preceding exercise be injective, i.e. that $i(m_1) = i(m_2)$ implies $m_1 = m_2$. Show that if M is finite, a necessary and sufficient condition for M to be a group is that i be injective, and that in this case i is an isomorphism.

10. Suppose that N is a submonoid of M . Show that there exists a monoid denoted by $M/_N$ and a morphism of monoids $\pi: M \longrightarrow M/_N$ such that if $f: M \longrightarrow M'$ is a morphism of monoids such that $f(n) = 1$ for $n \in N$, then there is a unique morphism of monoids $\tilde{f}: M/_N \longrightarrow M'$ such that $\tilde{f}\pi = f$. Let $\overline{N} = \{m \mid m \in M,$ and $\pi(m) = 1\}$. Show that $\overline{N}$ is a submonoid of M containing N but that it may happen that $\overline{N} \neq N$.

11. If M is a monoid, let $K(M)$ be the K-algebra such that as a K-module it is the free K-module generated by M, and such that the natural map $i: M \longrightarrow K(M)$ is a morphism of monoids. Show that if Λ is a commutative K-algebra, and $f: M \longrightarrow \Lambda$ is a morphism of monoids, there is a unique morphism of $\tilde{f}: K(M) \longrightarrow \Lambda$ such that $\tilde{f}i = f$. The algebra $K(M)$ is called the monoid algebra of M over K. If M is a group it is called the group algebra of M over K.

12. Show that if M is a submonoid of K, and $\pi: K \longrightarrow \Lambda$ is the ring of fractions of K relative to M, then there is a commutative diagram

$$\begin{array}{ccc} K(M) & \xrightarrow{\pi'} & K(G(M)) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ K & \xrightarrow{\pi} & \Lambda \end{array}$$

of morphisms of K-algebras where π_1 and π_2 are epimorphisms, and π' is induced by the morphism $M \longrightarrow G(M)$. Further show that $\pi': K(M) \longrightarrow K(G(M))$ is the ring of fractions of $K(M)$ relative to the submonoid M of $K(M)$.

13. A monoid M is finitely generated if there exists a finite set X, and a function $f: X \longrightarrow M$ such that the corresponding morphism of monoids $\tilde{f}: M(X) \longrightarrow M$ is onto. Show that the algebra $K(M)$ is Noetherian if and only if the ring K is Noetherian and the monoid M is finitely generated.

14. Show that if X is a set, then $K(M(X))$ is just the polynomial algebra $K[X]$.

Suggested Reading

Bourbaki, N.
Algèbre, Livre II, Chapitre IV, Polynomes et fractions rationnelles, Paris, 1950.

Chevalley, C.
Fundamental concepts of Algebra, New York, 1956. Particularly read chapter 1.

Krull, W.
Idealtheorie, Berlin, 1935.