Geometric realizations in general categories

by

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Introduction. In [6], we defined the geometric realization of simplicial $\mathcal{C}$-objects, for a quite general category $\mathcal{C}$, and there we initiated the study of this construction. In [7], the sequel to [6], we will need further properties of this general geometric realization functor. It is the purpose of this paper to develop these properties. It should be mentioned at the outset that the properties we discuss are all well-known in the classical case; it is not always a trivial matter, however, to pass to an abstract ambient category, and it is occasionally instructive to be forced to consider the general case. If nothing else, this work shows that simplicial theory is essentially applied category theory, the only geometric input being some properties of $\underline{\Delta}$ and the classical triangulation of the product of simplices.

For the classical case we refer once and for all to [2],[3],[5]. We will systematically use the categorical notions of ends and coends, discussed in [4].

§1 - Simplicial objects

In order to have sufficient flexibility to discuss products, function space objects, as well as geometric realizations, we will consider, not just a single ambient category, but the data below:

$$T : \mathcal{C} \times \mathcal{U} \to \mathcal{V}, \quad H : \mathcal{U}^* \times \mathcal{V} \to \mathcal{C}$$

such that there is a natural isomorphism

$$(*) \qquad \mathcal{V}(T(A,B),C) \approx \mathcal{T}(A,H(B,C))$$

(1.1) Examples. (i) $\mathcal{T} = \mathcal{U} = \mathcal{V}$ and $\mathcal{T}$ is closed; a suitable version of Top is such a $\mathcal{T}$; Top will always refer to such a version. (ii) $\mathcal{T} = \mathcal{V}$, $\mathcal{U}$ = Sets; then $T : \mathcal{T} \times \text{Sets} \to \mathcal{T}$ is defined by $T(X,A) = \coprod_{a \in A} X_a$, with $X_a \approx X$, and $H : \text{Sets}^* \times \mathcal{T} \to \mathcal{T}$ is defined by $H(A,X) = \prod_{a \in A} X_a$, with $X_a \approx X$. This makes sense, of course, only if $\mathcal{T}$ has products and coproducts. The condition (*) is easily verified. We will use P,F for T,H in examples of this kind. (iii) $\mathcal{T}$ = Spectra, $\mathcal{U}$ = Top, $\mathcal{V}$ = Spectra, with T the small smash product and H the small function spectrum functors.

In what follows, Δ will denote an index category, small, but otherwise arbitrary. Since $\underline{\Delta}$, the category of finite ordered sets and order-preserving maps, is the prototype for Δ, we will think of functors $X : \Delta^* \to \mathcal{T}$, $Y : \Delta \to \mathcal{U}$, for example, as simplicial $\mathcal{T}$-objects and cosimplicial $\mathcal{U}$-objects, respectively, and we will use notation familiar from ordinary simplicial theory. Rather than X,Y, as above, we will often write X_*, Y^* and the values of X,Y at an object n of Δ will be denoted by X_n, Y^n more often than by X(n),Y(n). Categories of simplicial and co-simplicial $\mathcal{T}$-objects will be denoted by $[\Delta^*, \mathcal{T}]$, $[\Delta, \mathcal{T}]$.

(1.2) Definition. If $A_* \in [\Delta^*, \mathcal{T}]$, $B_* \in [\Delta^*, \mathcal{U}]$, then we define $T_*(A_*, B_*) \in [\Delta^*, \mathcal{V}]$ by $T_n(A_*, B_*) = T(A_n, B_n)$. Clearly, this is contravariant in n and defines a simplicial $\mathcal{V}$-object. If T is as in example (1.1)(ii), we will write $P_*(A_*,B_*)$ for $T_*(A_*,B_*)$. The asterisk will sometimes be omitted.

Given any object n of Δ, there is a standard simplicial set $\Delta[n]$ defined by $\Delta[n](m) = \Delta(m,n)$. If A is any object of a category $\mathcal{C}$, then $P_*(A,\Delta[n])$ is defined and is a simplicial $\mathcal{C}$-object. We will call such a simplicial $\mathcal{C}$-object a <u>generalized n-simplex</u> in $\mathcal{C}$; category-theorists call these <u>generalized representable functors</u>, [12].

(1.3) <u>Theorem</u>. Any simplicial $\mathcal{C}$-object is the colimit of generalized simplices in $\mathcal{C}$.

<u>Proof</u>. Let $X_* \in [\Delta^*, \mathcal{C}]$ and $\Theta : m \to n$ in Δ. Consider the diagram

$$(**) \qquad \begin{array}{ccc} P(X_n,\Delta[m]) & \xrightarrow{P(\Theta^*,1)} & P(X_m,\Delta[m]) \\ \big\downarrow {\scriptstyle P(1,\Delta[\Theta])} & & \\ P(X_n,\Delta[n]) & & \end{array}$$

Recall that every k-simplex of $\Delta[n]$, i.e., every element of $\Delta[n](k)$ has the form $\alpha^*\sigma_n$, where $\alpha : k \to n$ and σ_n is the <u>basic</u> n-simplex of $\Delta[n]$, $\sigma_n = 1 : n \to n$. Thus $P(X_k,\Delta[k]) \approx \coprod_\alpha P(X_k,\alpha^*\sigma_k)$ with codomain $\alpha = k$. Define $\pi_k : P(X_k,\sigma_k) \approx X_k \to X_k$ to be the identity and $\pi_k : P(X_k,\alpha^*\sigma_k) \to X_\ell$, where $\alpha : \ell \to k$, to be $P(X_k,\alpha^*\sigma_k) \approx X_k \xrightarrow{\alpha^*} X_\ell$. Clearly π_k is a simplicial map $P(X_k,\Delta[k]) \to X_*$.

It is easy to see that $\pi_m \cdot P(\theta^*,1) = \pi_n \cdot P(1,\Delta(\theta))$ and that X_*, $\{\pi_k\}$ satisfy the universal property for a coend of the diagrams (**). We will write $X_* \cong \mathrm{coend}_k P(X_k,\Delta[k])$. The standard description of coends in terms of equalizers and coproducts shows that

$$\coprod_{\theta:m\to n} P(X_n,\Delta[m]) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_k P(X_k,\Delta[k]) \xrightarrow{(\pi_k)} X_*$$

is an equalizer. We call this the canonical presentation of X_*.

(1.4) Remark. This result, for $\mathcal{C}$ = Sets, is classical ([3], p.21, or [1]. p.87); in full generality, using the language of functors rather than simplicial objects, it is also well-known, ([12], p.97).

(1.5) Corollary. If $F,G : [\Delta^*, \mathcal{C}] \to \mathcal{D}$ commute with colimits and agree on generalized simplices in $\mathcal{C}$, then $F = G$.

(1.6) Proposition. If $T : \mathcal{C} \times \mathcal{U} \to \mathcal{V}$ commutes with colimits in either variable, then

$$T(P(A,S),B) \cong T(A,P(B,S)) \cong P(T(A,B),S)$$

for all $A \in \mathcal{C}$, $B \in \mathcal{U}$, $S \in$ Sets.

The proof is trivial, all three objects of $\mathcal{V}$ being isomorphic to the coproduct of copies of $T(A,B)$, indexed by the elements of S. Similarly,

(1.7) Proposition. If $H : \mathcal{U}^* \times \mathcal{V} \to \mathcal{C}$ commutes with limits in either variable, then

$$H(P(A,S),B) \approx H(A,F(S,B)) \approx F(S,H(A,B))$$

(1.8) Definition. If $A_* \in [\Delta^*,\mathcal{U}]$, $B_* \in [\Delta^*,\mathcal{V}]$, then we define $H_*(A_*,B_*) \in [\Delta^*,\mathcal{T}]$ by

$$H_n(A_*,B_*) = \text{end}_k\ H(P(A_k,\Delta[n]_k),\ B_k)\ .$$

This agrees with the classical definition of function complexes when $\mathcal{T} = \mathcal{U} = \mathcal{V} =$ Sets. Such a construction is also used by D.W. Anderson, [1], in the case $\mathcal{U} =$ Sets, $\mathcal{T} = \mathcal{V}$.

(1.9) Theorem. The adjointness relation (*) is inherited by T_*,H_*;

$$[\Delta^*,\mathcal{V}](T_*(A_*,B_*),\ C_*) \approx [\Delta^*,\mathcal{T}](A_*,H_*(B_*,C_*)),$$

if T commutes with colimits in the second variable.

Proof. Note that (*) implies that T commutes with colimits in the first variable, since $T(-,B)$ has a right adjoint $H(B,-)$. So,

$$\begin{aligned}
[\Delta^*,\mathcal{T}](A_*,H_*(B_*,C_*)) &= \text{end}_k \mathcal{T}(A_k,H_k(B_*,C_*)) \\
&= \text{end}_k \mathcal{T}(A_k,\text{end}_\ell H(P(B_\ell,\Delta[k]_\ell),C_\ell)) \\
&\approx \text{end}_k \text{end}_\ell\ \mathcal{T}(A_k,H(P(B_\ell,\Delta[k]_\ell),C_\ell)) \\
&\approx \text{end}_k \text{end}_\ell\ \mathcal{V}(T(A_k,P(B_\ell,\Delta[k]_\ell)),C_\ell) \\
&\approx \text{end}_k \text{end}_\ell\ \mathcal{V}(T(P(A_k,\Delta[k]_\ell),B_\ell),C_\ell), \text{ by (1.6)} \\
&\approx \text{end}_\ell \text{end}_k\ \mathcal{V}(T(P(A_k,\Delta[k]_\ell),B_\ell),C_\ell), \text{ by the Fubini theorem for ends} \\
&\approx \text{end}_\ell\ \mathcal{V}(\text{coend}_k\ T(P(A_k,\Delta[k]_\ell),B_\ell),C_\ell)
\end{aligned}$$

but $\text{coend}_k T(PA_k, \Delta[k]_\ell), B_\ell) \approx T(\text{coend}_k P(A_k, \Delta[k]_\ell), B_\ell)$

$$= T(\text{coend}_k P(A_k, \Delta[k])_\ell, B_\ell)$$

$$\approx T((A_*)_\ell, B_\ell) \text{ by } (1.3)$$

$$= T(A_\ell, B_\ell).$$

Hence, $[\Delta^*, \mathcal{C}](A_*, H_*(B_*, C_*)) \approx \text{end}_\ell \ \mathcal{V}(T(A_\ell, B_\ell), C_\ell)$

$$= [\Delta^*, \mathcal{V}](T_*(A_*, B_*), C_*)$$

(1.10) Corollary. If B_* is constant $(B_k \approx B$, all $k)$, then $H_n(B_*, C_*) \approx H(B, C_n)$.

Proof. $T_*(A_*, B_*)$ is given by $T_n(A_*, B_*) = T(A_n, B)$. Hence,

$$[\Delta^*, \mathcal{C}](A_*, H_*(B_*, C_*)) \approx [\Delta^*, \mathcal{V}](T_*(A_*, B_*), C_*)$$

$$\approx \text{end}_k \ \mathcal{V}(T_k(A_*, B_*), C_k)$$

$$= \text{end}_k \ \mathcal{V}(T(A_k, B), C_k)$$

$$\approx \text{end}_k \ \mathcal{C}(A_k, H(B, C_k))$$

$$= [\Delta^*, \mathcal{C}](A_*, H(B, C_*))$$

§2 - Geometric Realization.

(2.1) Definition. A geometric realization datum for $[\Delta^*, \mathcal{C}]$ is (T, ϕ) where $\phi : \Delta \to \mathcal{C}'$ and $T : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$. Given such a datum, we define the corresponding geometric realization $RA_* = R_{T,\phi}A_*$, where $A_* \in [\Delta^*, \mathcal{C}]$, as the coend of

$$\Delta^* \times \Delta \xrightarrow{A\times\phi} \mathcal{C} \times \mathcal{C}' \xrightarrow{T} \mathcal{C}''.$$

Thus, $R : [\Delta^*,\mathcal{C}] \to \mathcal{C}''$. If $H : \mathcal{C}'^* \times \mathcal{C}'' \to \mathcal{C}$ satisfies $\mathcal{C}''(T(A,B),C) \approx \mathcal{C}(A,H(B,C))$, then the singular functor $S = S_{T,\phi} = S_{H,\phi} : \mathcal{C}'' \to [\Delta^*,\mathcal{C}]$ is defined by $(SA)_n = H(\phi(n),A)$. This is clearly contravariant in n, so that $SA \in [\Delta^*,\mathcal{C}]$. The classical adjointness relation between R,S generalizes:

(2.2) Theorem. R is left-adjoint to S.

Proof.
$$\begin{aligned}\mathcal{C}''(RA_*,B) &\approx \mathcal{C}''(\text{coend } (T(A\times\phi)),B)\\ &\approx \text{end } \mathcal{C}''(T(A\times\phi),B)\\ &\approx \text{end } \mathcal{C}(A,H(\phi,B)) = [\Delta^*,\mathcal{C}](A_*,SB).\end{aligned}$$

The definition of realization datum may seem unnecessarily general; the examples (1.1)(i), (ii), (iii) show that, in practice, $\mathcal{C}''$ will usually coincide with $\mathcal{C}$ or with $\mathcal{C}'$; the generality of our definition enables us to consider both cases simultaneously.

We now study the behavior of R with respect to products: suppose $T : \mathcal{C} \times \mathcal{U} \to \mathcal{V}$, $A_* \in [\Delta^*,\mathcal{C}]$, $B_* \in [\Delta^*,\mathcal{U}]$, so that $T_*(A_*,B_*) \in [\Delta^*,\mathcal{V}]$, what relationship holds between RA_*,RB_* and $RT_*(A_*,B_*)$?

For this question to make sense, we must have geometric realization data:

$$\phi_1 : \Delta \to \mathcal{C}',\ T_1 : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'' \quad \text{defining } R_1$$

$$\phi_2 : \Delta \to \mathcal{U}',\ T_2 : \mathcal{U} \times \mathcal{U}' \to \mathcal{U}'' \quad \text{defining } R_2$$

$$\phi_3 : \Delta \to \mathcal{V}',\ T_3 : \mathcal{V} \times \mathcal{V}' \to \mathcal{V}'' \quad \text{defining } R_3$$

(2.3) <u>Theorem</u>. $R_3T_*(A_*,B_*) \approx T''(R_1A_*,R_2B_*)$, provided:

(i) There exist $T' : \mathcal{C}' \times \mathcal{U}' \to \mathcal{V}'$, $T'' : \mathcal{C}'' \times \mathcal{U}'' \to \mathcal{V}''$ such that

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C}'\times\mathcal{U}\times\mathcal{U}' & \xrightarrow{1\times\tau\times1} & \mathcal{C}\times\mathcal{U}\times\mathcal{C}'\times\mathcal{U}' \\ \downarrow{\scriptstyle T_1\times T_2} & & \downarrow{\scriptstyle T\times T'} \\ \mathcal{C}''\times\mathcal{U}'' & & \mathcal{V}\times\mathcal{V}' \\ & {\scriptstyle T''}\searrow \quad \mathcal{V}'' \quad \swarrow{\scriptstyle T_3} & \end{array}$$

commutes up to a natural isomorphism (τ will always denote the obvious order-interchange isomorphism).

(ii) T_i commutes with colimits in the first variable, $i = 1,2,3$.

(iii) T and T'' commute with colimits in either variable.

(iv) $T'(\phi_1(m),\phi_2(n)) \approx R_{P,\phi_3}(\Delta[m] \times \Delta[n])$, where $P : \text{Sets} \times \mathcal{V}' \to \mathcal{V}'$.

Before proving (2.3), we consider some preliminary results. As we have already done in (iv) above, we will denote $\text{Sets} \times \mathcal{C} \xrightarrow{\tau} \mathcal{C} \times \text{Sets} \xrightarrow{P} \mathcal{C}$ by P also; this should cause no difficulty, since $P(S,C) \approx P(C,S)$.

(2.4) <u>Proposition</u>. The geometric realization functor $R = R_{T,\phi}$ commutes with colimits if T commutes with colimits in the first variable.

<u>Proof</u>. $R \operatorname{colim} F = \operatorname{coend}_k T(\operatorname{colim} F_k, \phi(k))$

$\approx \operatorname{coend}_k \operatorname{colim} T(F_k,\phi(k))$

$\approx \operatorname{colim} \operatorname{coend}_k T(F_k,\phi(k))$, since colimits commute

$\approx \operatorname{colim} RF$

(2.5) Lemma. If $S_1, S_2 \in$ Sets, $C \in \mathcal{C}$, then $P(P(C,S_1),S_2) \approx P(C,S_1 \times S_2)$.

The proof is a trivial computation.

(2.6) Lemma. If $\phi : \Delta \to \mathcal{C}'$, $T : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$, $C \in \mathcal{C}$, $S_* \in [\Delta^*, \text{Sets}]$ and T commutes with colimits in either variable, then

$$R_{T,\phi}P(C,S_*) \approx T(C,R_{P,\phi}S_*) \ .$$

Proof.
$$\begin{aligned} R_{T,\phi}P(C,S_*) &= \text{coend}_k\ T(P(C,S_k),\phi(k)) \\ &\approx \text{coend}_k\ T(C,P(S_k,\phi(k)) \quad \text{by (1.6)} \\ &\approx T(C,\text{coend}_k\ P(S_k,\phi(k))) \\ &\approx T(C,R_{P,\phi}S_*) \end{aligned}$$

(2.7) Corollary. $R_{T,\phi}P(C,\Delta[n]) \approx T(C,\phi(n))$.

Proof. By (2.6), it suffices to check that $R_{P,\phi}\Delta[n] \approx \phi(n)$. Now, $R_{P,\phi}\Delta[n] = \text{coend}_k\ P(\Delta[n]_k,\phi(k))$, and $P(\Delta[n]_k,\phi(k)) \approx \coprod_{\alpha:k\to n} P(\alpha^*\sigma_n,\phi(k)) \approx \coprod_{\alpha} \phi(k)_\alpha$. Define $\zeta_k : P(\Delta[n]_k,\phi(k)) \to \phi(n)$ by $\zeta_k = (\phi(\alpha))$, where $\phi(\alpha) : \phi(k) \to \phi(n)$. It is easily verified that $\phi(n)$ and $\{\zeta_k\}$ satisfy the universal property for coends.

Proof of (2.3). By (i),(ii), and (2.4), we see that both $R_3T_*(A_*,B_*)$ and $T''(R_1A_*,R_2B_*)$, considered as functors on A_* or B_*, commute with colimits. Hence, by (1.3), it suffices to verify the desired isomorphism for generalized simplices.

Let, then, $A_* = P(A,\Delta[m])$, $B_* = P(B,\Delta[n])$ with $A \in \mathcal{T}$, $B \in \mathcal{U}$. Then $T_*(A_*,B_*) = T_*(P(A,\Delta[m]),\ P(B,\Delta[n]))$ and

$$
\begin{aligned}
T_p(A_*,B_*) &= T(P(A,\Delta[m]_p),P(B,\Delta[n]_p)) \\
&\approx P(T(A,P(B,\Delta[n]_p)),\Delta[m]_p) \quad \text{by (1.6)} \\
&\approx P(P(T(A,B),\Delta[n]_p),\Delta[m]_p) \quad \text{by (1.6)} \\
&\approx P(T(A,B),\Delta[m]_p \times \Delta[n]_p) \quad \text{by (2.5)}
\end{aligned}
$$

so that $T_*(A_*,B_*) \approx P(T(A,B),\ \Delta[m] \times \Delta[n])$. Hence,

$$
\begin{aligned}
R_3T_*(A_*,B_*) &\approx R_3P(T(A,B),\Delta[m] \times \Delta[n]) \\
&\approx T_3(T(A,B),\ R_{p,\phi_3}\Delta[m] \times \Delta[n]) \quad \text{by (2.6)} \\
&\approx T_3(T(A,B),\ T'(\phi_1(m),\phi_2(n))) \quad \text{by (iv)} \\
&\approx T''(T_1(A,\phi_1(m)),T_2(B,\phi_2(n))) \quad \text{by (iii)} \\
&\approx T''(R_1A_*,R_2B_*) \quad \text{by (2.7)} \ .
\end{aligned}
$$

In [8], 2.1, Mielke uses a similar argument.

(2.8) <u>Remark</u>. Of conditions (i)-(iv), the first three are usually trivially verified; (iv) is where the geometry enters. In most cases $\phi_1 = \phi_2 = \phi_3$, $\mathcal{T}' = \mathcal{U}' = \mathcal{V}'$, and T' is the product. Condition (iv) then states that the product of 2 models has the classical triangulation ([9], §2; [3], p.52; [5], p.104). In practice, most of the pairings T_i, T, T', T'''' coincide, and (i) follows from various commutativity and associativity relations.

(2.9) Proposition. (a) $R_3 T_*(A_*,B) \approx T''(R_1 A_*, T_2(B,\phi_2(0))$, provided (i),(ii),(iii), and (iv)' hold, where (iv)' : $\phi_3(m) \approx T'(\phi_1(m),\phi_2(0))$. (b) $R_3 T_*(A,B_*) \approx T''(T_1(A,\phi_1(0)),R_2 B_*)$, provided (i),(ii),(iii), and (iv)" hold, where (iv)" : $\phi_3(m) \approx T'(\phi_1(0),\phi_2(m))$.

Proof. Essentially the same as that of (2.3). Note that (iv)', (iv)" have a much more elementary character than (iv); they will usually be satisfied trivially.

§3. Bisimplicial objects.

Since the index category Δ is arbitrary, we may replace it by $\Delta \times \Delta$, in which case we have bisimplicial theory. We will investigate under which conditions the standard results on bisimplicial sets generalize.

Let (T,ϕ) be a geometric realization datum for $[\Delta^*,\mathcal{C}]$, and $\overline{T} : \mathcal{C}' \times \mathcal{C}' \to \mathcal{C}'$. Then let $\overline{\phi} = \overline{T} \cdot (\phi \times \phi) : \Delta \times \Delta \to \mathcal{C}'$, and $(T,\overline{\phi})$ is a geometric realization datum for $[\Delta^* \times \Delta^*, \mathcal{C}]$.

(3.1) Theorem. Assume T commutes with colimits in the first variable, and

$$(+) \qquad \begin{array}{ccc} \mathcal{C} \times \mathcal{C}' \times \mathcal{C}' & \xrightarrow{1 \times \overline{T}} & \mathcal{C} \times \mathcal{C}' \\ \downarrow{\scriptstyle T \times 1} & & \downarrow{\scriptstyle T} \\ \mathcal{C} \times \mathcal{C}' & \xrightarrow{T} & \mathcal{C} \end{array}$$

is commutative; then, for any bisimplicial $\mathcal{C}$-object A_{**}, we have $R_{T,\overline{\phi}} A_{**} \approx R^v_{T,\phi} R^h_{T,\phi} A_{**}$.

Proof. $R_{T,\overline{\phi}}A_{**} = \mathrm{coend}_{k\ell}\ T(A_{k\ell}, \overline{T}(\phi(k),\phi(\ell)))$

$= \mathrm{coend}_{\ell}\mathrm{coend}_{k}\ T(T(A_{k\ell},\phi(k)),\phi(\ell))$

$= \mathrm{coend}_{\ell}T(\mathrm{coend}_{k}\ T(A_{k\ell},\phi(k)),\phi(\ell))$

$= \mathrm{coend}_{\ell}\ T(R^{h}_{T,\phi}A_{*\ell},\phi(\ell))$

$= R^{v}_{T,\phi}R^{h}_{T,\phi}A_{**}$.

(3.2) Remark. If $\overline{T}$ is commutative, or more generally, if

$$\begin{array}{ccccc}
\mathcal{C} \times \mathcal{C}' \times \mathcal{C}' & & \xrightarrow{1 \times \overline{T}} & & \mathcal{C} \times \mathcal{C}' \\
\downarrow{\scriptstyle 1 \times \tau} & & & & \downarrow{\scriptstyle T} \\
\mathcal{C} \times \mathcal{C}' \times \mathcal{C}' & \xrightarrow{T \times 1} & \mathcal{C} \times \mathcal{C}' & \xrightarrow{T} & \mathcal{C}
\end{array}$$

is commutative, then $R_{T,\overline{\phi}}A_{**} \approx R^{h}_{T,\phi}R^{v}_{T,\phi}A_{**}$ also.

The diagonal functor $\Delta \to \Delta \times \Delta$ $(n \mapsto (n,n))$ induces the diagonal functor $D : [\Delta^* \times \Delta^*, \mathcal{C}] \to [\Delta^*, \mathcal{C}]$. Given an object (m,n) of $\Delta \times \Delta$, the standard simplex $(\Delta \times \Delta)[(m,n)]$ in $[\Delta^* \times \Delta^*, \mathrm{Sets}]$ will be denoted by $\Delta[m] \divideontimes \Delta[n]$. Clearly, $D(\Delta[m] \divideontimes \Delta[n]) \approx \Delta[m] \times \Delta[n]$.

(3.3) Theorem. Assume T commutes with colimits in either variable and (+) is commutative. Assume, further, that $R_{p,\phi}\Delta[m] \times \Delta[n] \approx \overline{T}(\phi(m),\phi(n))$. Then, for any bisimplicial $\mathcal{C}$-object A_{**}, $R_{T,\overline{\phi}}A_{**} \approx R_{T,\phi}DA_{**}$.

Proof. By (2.4), $R_{T,\overline{\phi}} : [\Delta^* \times \Delta^*, \mathcal{C}] \to \mathcal{C}''$ and $R_{T,\phi} : [\Delta^*, \mathcal{C}] \to \mathcal{C}''$

both commute with colimits. Clearly, so does D. Hence, by (1.3), it suffices to verify the desired isomorphism on generalized (bi)simplices, $P(A,\Delta[m] \times\!\!\!\times \Delta[n])$:

$$R_{T,\overline{\phi}}P(A,\Delta[m] \times\!\!\!\times \Delta[n] \approx T(A,\overline{\phi}(m,n)) \quad \text{by (2.7)}$$

$$= T(A,\overline{T}(\phi(m),\phi(n)))$$

while $R_{T,\phi}DP(A,\Delta[m] \times\!\!\!\times \Delta[n]) = R_{T,\phi}P(A,\Delta[m]\times \Delta[n])$

$$\approx T(A,R_{P,\phi}\Delta[m]\times \Delta[n]) \quad \text{by (2.6).}$$

Note that the required property of $\Delta[m]\times \Delta[n]$ is a special case of (iv) in (2.3); see remark (2.8); thus (3.3) involves the geometry of models much more than does (3.1).

(3.4) Corollary. Let (T,ϕ) be the classical geometric realization data for $[\Delta^*,\mathcal{C}]$, $\mathcal{C}$ = Top or $[\Delta^*,\text{Sets}]$, and $\overline{T}$ the product in $\mathcal{C}$. Then $R_{T,\overline{\phi}}A_{**} \approx R_{T,\phi}DA_{**}$.

Proof. The condition $R_{P,\phi}\Delta[m]\times \Delta[n] \approx \overline{T}(\phi(m),\phi(n))$ is proved for $\mathcal{C}$ = Top in the standard references (see (2.8)); For $\mathcal{C} = [\Delta^*,\text{Sets}], \phi(n) = \Delta[n]$ and $R_{P,\phi}\Delta[m]\times \Delta[n] = \text{coend}_k \Delta[m]_k\times \Delta[n]_k\times \Delta[k] \approx \Delta[m]\times \Delta[n]$, by (1.3).

§4 - Cosimplicial objects

We now dualize some of the preceding results to obtain corresponding results for cosimplicial objects. We take the basic situation of §1 and replace $\mathcal{C}$ by $\mathcal{C}^*$, $\mathcal{V}$ by $\mathcal{V}^*$, but do not dualize $\mathcal{U}$. Thus, we have $T : \mathcal{C}^* \times \mathcal{U} \to \mathcal{V}^*$, $H : \mathcal{U}^* \times \mathcal{V}^* \to \mathcal{C}^*$, or $T : \mathcal{C} \times \mathcal{U}^* \to \mathcal{V}$, $H : \mathcal{U} \times \mathcal{V} \to \mathcal{C}$. Let $\hat{\mathcal{C}} = \mathcal{V}$, $\hat{\mathcal{V}} = \mathcal{C}$, $\hat{H}(A,B) = T(B,A)$, $\hat{T}(A,B) = H(B,A)$, and (*) becomes

$$\hat{\mathcal{V}}(\hat{T}(A,B),C) \approx \hat{\mathcal{C}}(A,\hat{H}(B,C))$$

so that all the results depending on (*) immediately dualize. We drop the "hats" from our notation and obtain:

If $Y^* : \Delta \to \mathcal{V}$ is a <u>cosimplicial</u> $\mathcal{V}$-object, its <u>geometric realization</u> $R^{\bullet}Y^* = R^{H,\phi}Y^* =$ end of $\Delta^* \times \Delta \xrightarrow{\phi \times Y} \mathcal{U}^* \times \mathcal{V} \xrightarrow{H} \mathcal{C}$. Thus $R^{\bullet} : [\Delta,\mathcal{V}] \to \mathcal{C}$. Note $\phi : \Delta \to \mathcal{U}$ as in §1. The <u>co-singular</u> functor $S^{\bullet} = S^{H,\phi} = S^{T,\phi} : \mathcal{C} \to [\Delta,\mathcal{V}]$ is given by $(S^{\bullet}A)(n) = T(A,\phi(n))$, and $S^{\bullet}$ is left-adjoint to $R^{\bullet}$.

Let $\mathcal{C}$ be any category and $F : Sets^* \times \mathcal{C} \to \mathcal{C}$ the functor described in (1.1)(ii); then $F(\Delta[n],A)$, for any object A of $\mathcal{C}$, is a cosimplicial $\mathcal{C}$-object; these are the <u>generalized n-cosimplices</u> in $\mathcal{C}$, and (1.3) states that each cosimplical $\mathcal{C}$-object is the limit of generalized cosimplices; more precisely, every $A^* \in [\Delta,\mathcal{C}]$ has the <u>canonical presentation</u> $A^* \approx \mathrm{end}_n F(\Delta[n],A^n)$.

Given $A^* \in [\Delta,\mathcal{C}]$, $B_* \in [\Delta^*,\mathcal{U}]$, $C^* \in [\Delta,\mathcal{V}]$, define $T^*(A^*,B_*) \in [\Delta,\mathcal{V}]$ and $H^*(B_*,C^*) \in [\Delta,\mathcal{C}]$ by $T^n(A^*,B_*) = \mathrm{coend}_k T(A^k,P(\Delta[n]_k,B_k))$, $H^n(B_*,C^*) = H(B_n,C^n)$, and (1.9) states that

$$[\Delta,\mathcal{V}](T^*(A^*,B_*),C^*) \approx [\Delta,\mathcal{C}](A^*,H^*(B_*,C^*)).$$

We leave the dualization of further results to the reader.

§5. Simplicial homotopy

The discussion in §11 of [5] generalizes without difficulty; in this section, $\Delta = \underline{\Delta}$.

(5.1) Proposition. If there is a simplicial homotopy $h_* : X_* \to Y_*$ between simplicial maps f_*, g_*, then there is a simplicial map $H_* : P(X_*, \Delta[1]) \to Y_*$ such that $H_* \cdot P(1, i_0) = f_*$, $H_* \cdot P(1, i_1) = g_*$ (identifying $P(X_*, \Delta[0])$ and X_*), where $i_\alpha : \Delta[0] \to \Delta[1]$, $\alpha = 0,1$, are the obvious maps.

(5.2) Corollary. If the product $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ commutes with colimits in either variable and $T(\phi(m), \phi(n)) \simeq R_{P,\phi}\Delta[m] \times \Delta[n]$, then the existence of a simplicial homotopy h_* induces that of a morphism $H = RH_* : T(RX_*, \phi(1)) \to RY_*$ such that $H \cdot T(1, \phi(i_0)) = Rf_*$, $H \cdot T(1, \phi(i_1)) = Rg_*$.

This follows immediately from (2.3) and (5.1). Thus, if homotopy in $\mathcal{C}$ is defined in terms of $\phi(1)$, geometric realization sends simplicial homotopies to homotopies, and preserves homotopy equivalences.

§6. Fiber-products

In this section, $\Delta = \underline{\Delta}$. In order to prove that geometric realizations behave well with respect to fiber-products, as stated in (6.1) below, we need $\mathcal{C}$ to resemble sets; this condition will take the form that colimits in $\mathcal{C}$ are universal; this means that if $A \to B$ in $\mathcal{C}$, $F : \mathcal{C} \longrightarrow \mathcal{C}/B$, then $\mathrm{colim}\,(A \times_B F) \simeq A \times_B \mathrm{colim}\, F$. For a more formal definition, see [11], p.78.

(6.1) <u>Theorem.</u> Let $T : \mathcal{T} \times \mathcal{T}' \to \mathcal{T}$, $\phi : \underline{\Delta} \to \mathcal{T}'$, $A \to B$ in $\mathcal{T}$, $C_* \in [\Delta^*, \mathcal{T}/B]$. Assume:

(i) T commutes with colimits in either variable.

(ii) colimits in $\mathcal{T}$ are universal.

(iii) $T[A,\phi(0)] \approx A$.

(iv) $T[A \underset{B}{\times} C, D] \approx A \underset{B}{\times} T[C,D]$ for D an object in the co-complete subcategory of $\mathcal{T}'$ generated by $\{\phi(n)\}$.

Then $R(A \underset{B}{\times} C_*) \approx A \underset{B}{\times} RC_*$.

<u>Proof.</u> First note that $A \underset{B}{\times} C_* \in [\Delta^*, \mathcal{T}]$ is defined; also that by (iii), we have for any D, as in (iv), a unique $D \to \phi(0)$, and $T[C,D] \to T[C,\phi(0)] \approx C$; hence $T[C,D] \to B$ if $C \to B$. Thus, (iv) makes sense.

We will prove (6.1) by induction on n, using the structure of R given by [6], 3.11. Thus, $R(A \underset{B}{\times} C_*) \approx \operatorname{colim}_n R_n(A \underset{B}{\times} C_*)$, where

$$\begin{array}{ccc} T[A \underset{B}{\times} C_n, \phi(n)] & \longrightarrow & R_n(A \underset{B}{\times} C_*) \\ \uparrow & & \uparrow \\ T[A \underset{B}{\times} C_n, \dot{\phi}(n)] \cup T[s(A \underset{B}{\times} C_*)_{n-1}, \phi(n)] & \rightarrow & R_{n-1}(A \underset{B}{\times} C_*) \end{array}$$

is a pushout. Using (ii), we see that $s(A \underset{B}{\times} C_*)_{n-1} \approx A \underset{B}{\times} sC_{n-1}$. By (ii),(iii),(iv), we see that the diagram above is isomorphic to

$$\begin{array}{ccc} A \underset{B}{\times} T[C_n, \phi(n)] & \longrightarrow & R_n(A \underset{B}{\times} C_*) \\ \uparrow & & \uparrow \\ A \underset{B}{\times} \{T[C_n, \dot{\phi}(n)] \cup T[sC_{n-1}, \phi(n)]\} & \longrightarrow & R_{n-1}(A \underset{B}{\times} C_*) \end{array}$$

For $n = 0$, $R_0C \approx T[C_0,\phi(0)]$, $R_0(A \times_B C_*) \approx T[A \times_B C_0,\phi(0)] \approx A \times_B T[C_0,\phi(0)] \approx A \times_B R_0C$. By induction, the universality of colimits, and the pushout diagram above, we see that $R_n(A \times_B C_*) \approx A \times_B R_nC_*$. Hence, using (ii) again, $R(A \times_B C_*) \approx \mathrm{colim}_n R_n(A \times_B C_*) \approx \mathrm{colim}_n A \times_B R_nC_* \approx A \times_B \mathrm{colim} R_nC_* \approx A \times_B RC_*$.

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