The Real Connective K-Theory of Brown-Gitler Spectra.

by

Bruce Wallace McQuistan

A ~~dissertation~~ thesis submitted in partial fulfillment of the requirements for the degree of

Masters of Science

University of Washington 1985

Approved by : ______________________

(Chairperson of the Supervisory Committee)

Program Authorized to offer Degree ______ Mathematics ______

Date : ______________________

Heading

University of Washington

Abstract

The Real Connective K-Theory of Brown-Gitler Spectra

by Bruce Wallace McQuistan

Chairperson of the Supervisory Committee: Douglas Ravenel
Dept of Mathem

~~I) Introduction / Abstract~~ Let bo denote the spectrum representing real connective K-theo:

Start 1st paragraph

It's the objective of this paper to compute $bo_*(\Omega^2 S^3)$ using the classical Adams spectral sequence:

$$\mathrm{Ext}_A(H^*(bo \wedge \Omega^2 S^3), \mathbb{Z}/2) = E_2 \Rightarrow \pi_*^S(bo \wedge \Omega^2 S^3).$$

where $\pi_*^S(bo \wedge \Omega^2 S^3) = bo_*(\Omega^2 S^3)$ by definition, and A is the Steenrod algebra. ~~Here, bo is the spectrum representing real connective K-Theory.~~

Since we are computing stable homotopy, we may regard $\Omega^2 S^3$ as a stable complex and as such, according to Snaith and Peterson et al, $\Omega^2 S^3$ decomposes into a wedge of spectra $B([\frac{n}{2}])$ for $n \geq 0$, where $[\ ]$ is the greatest integer function, called Brown-Gitler spectra. This gives us a firm grip on the cohomology of $\Omega^2 S^3$, since the cohomology of the $B(k)$'s are known to be certain cyclic A-modules. Better yet the $B(k)$'s can be modified so as to fit nicely into some computational frameworks set up by Adams and Mahowald.

The information so garnered on the modified $B(k)$'s can be lifted back to the $B(k)$'s using some exact sequences and a bit of clever algebra suggested by Ravenel. This will complete the computation

Table of Contents

I) A stable decomposition of $\Omega^2 S^3$: the Brown-Gitler Spectra

~~[illegible]~~ If X is a non degenerately based, path connected, compactly generated and Hausdorff space then $\Omega^2\Sigma^2 X$ has the weak homotopy type of ~~[illegible]~~ points in X labelled by points in $\mathbb{R}^2$. To be more precise, let $C(\mathbb{R}^2, X)$ be equivalence classes of pairs (S, f) where S is a finite subset of $\mathbb{R}^2$ and $f: S \longrightarrow X$ is a function with the equivalence relation being given by $(S,f) \sim (S - \{s_0\}, f|_{S-\{s_0\}})$ if $f(s_0) = x_0$, the basepoint in X. We can filter $C(\mathbb{R}^2, X)$ by putting $F_k C(\mathbb{R}^2, X)$ equal to the subspace of points represented by (S, f) where the cardinality of S is at most k.

According to Smith $\Omega^2\Sigma^2 X \simeq \bigvee_{k\geq 1} F_k/F_{k-1}$ as a stable complex where $F_k = F_k C(\mathbb{R}^2, X)$. The most legible account of this result is outlined by F. Cohen [6]. This immediately reduces the problem of computing $h_*(\Omega^2 S^3)$ to computing $h_*(F_k/F_{k-1})$ where $X = S^1$. I now state a result of Brown and Gitler [4].

Definition 1 There exist spectra $B(n)$ such that

$$H^*(B(n), \mathbb{Z}/2) \cong A /\!/ A\{\chi(Sq^i) \mid i > n\} = M(n)$$

Such spectra are called the Brown-Gitler spectra. I'll call $M(n)$ the n^{th} Brown-Gitler module.

not defined by coh.

<u>Theorem 1</u> (Mahowald [8]) : $H^*(F_n/F_{n-1}) \cong M([\frac{n}{2}])$ as left A-modules with an appropriate dimension shift.

It has been shown by Peterson et al [5] that in fact F_n/F_{n-1} is a realization of the spectra $B([\frac{n}{2}])$ ~~$B(n)$~~. It's also worth noting the somewhat uncanny fact that the $B(n)$'s arise as Thom spectra over the n^{th} filtration of $\Omega^2 S^3$ according to Mahowald [8].

Note that $A\{\chi(Sq^i) \mid i > n\} \subset A\{Sq^1, \chi(Sq^i) \mid i > n\}$

Set $M_1(n) = A/\!/A\,\chi\{Sq^1, \chi(Sq^i) \mid i > n\}$ and let

$\pi : M(n) \longrightarrow M_1(n)$ be the projection. Much is known

about the left A_1-modules $M_1(n)$ and we exploit these

results to the fullest. It's interesting to observe ~~note~~ that this

map π is realized geometrically since $M_1(k)$ arises as the cohomology of a Thom spectrum over $F_k(W)$ where as a space $\Omega^2 S^3 \simeq S^1 \times W$ and $F_k(W)$ is the filtration induced on W from the filtration on $\Omega^2 S^3$ ~~under the inclusion~~ $F_k(W)$ ~~$\to F_k(\Omega^2 S^3)$~~ by the inclusion $i : W \hookrightarrow \Omega^2 S^3$.

II) The Adams Spectral Sequence

In this section ~~I'd~~ I would like to give a brief outline of the Adams Spectral Sequence. ~~I'll~~ I will assume a basic knowledge of the Steenrod algebra for the prime 2, denoted A. Also $H^*(\ )$ denotes mod 2 homology.

Let X be an n-connected complex of finite type, $n>1$, and consider ~~$H^*(X, \mathbb{Z}_2)$~~ ~~$H^*(X, \mathbb{Z}_2)$~~ ~~as~~ $H^*(X)$ as a module over the Steenrod algebra. For each generator $\alpha_i \in H^*(X)$, let $g_i : X \to K(\pi, n)$ be a map inducing a surjection in that dimension. Taking the product of all such g_i's we obtain a map

$$g : X \longrightarrow \prod_{j>0} K(H^j(X), j) = K_0$$

which induces a surjection in ~~H~~ mod 2 cohomology. Let X_1 be the homotopy theoretic fibre of this map. Apply the same construction to ~~the other~~ $H^*(X_1)$ as we did to $H^*(X)$ to obtain another product of Eilenberg-MacLane spaces which ~~I'll~~ I will denote K_1. Reiterating this process we construct an Adams Resolution for X:

$$\begin{tikzcd}
X \arrow[r] & K_0 \\
X_1 \arrow[u] \arrow[r] & K_1 \\
X_2 \arrow[u] \arrow[r] & K_2 \\
\vdots \arrow[u] &
\end{tikzcd}$$

Note that by the usual Barratt-Puppe construction we obtain maps $\Omega K_i \longrightarrow X_{i+1}$ or $K_i \longrightarrow \Sigma X_{i+1}$ if we recall that in the stable category fibrations and cofibrations are the same. Therefore we get short exact sequences in the stable range

$$H^*(\Sigma X_{i+1}) \rightarrowtail H^*(K_i) \twoheadrightarrow H^*(X_i).$$

where $H^*(K_i)$ is a free A-module

We can splice all these short exact sequences together to get an A-free resolution of $H^*(X)$

$$\cdots \cancel{\to H^*(K_1) \to H^*(K_0) \to H^*(X)}$$

$$\to H^*(\Sigma^{-2}K_2) \to H^*(\Sigma^{-1}K_1) \longrightarrow H^*(K_0) \twoheadrightarrow H^*(X) \qquad \star$$

The genius of this construction is that now the force of homological algebra ~~can~~ can be brought to bear on the following elegant observations.

(i) the fibrations $X_{i+1} \to X_i \to K_i$ yield long exact sequences of homotopy groups.

(ii) $\pi_*(K_i) \cong Hom_A(H^*(K_i), \mathbb{Z}/2)$.

(or more explicitly $[S^{n+t-s}, {}^nK_s] \cong Hom_A^{t-s}(H^*({}^nK_s), \mathbb{Z}/2)$

where ${}^nK_s$ is the s^{th} stage in the Adams resolution for $\Sigma^n X$.

and $Hom^{t-s}(\;, \;)$ denotes maps of degree $t-s$.

Thus it can be seen that the differential d_1 in the exact ~~com~~ couple of homotopy groups coming from the Adams resolution

$$d_1 : [S^{n+t-s}, \Omega K_s] \longrightarrow [S^{n+t-s}, K_{s+1}]$$

can be construed as a map induced from ☆ :

$$\mathrm{Hom}_A^{t-s}(H^*(^nK_s), \mathbb{Z}/2) \longrightarrow \mathrm{Hom}_A^{t-s-1}(H^*(^nK_{s+1}), \mathbb{Z}/2).$$

In short, without proving the convergence properties of the spectral sequence this outline is enough to indicate that the ~~the~~ sequence

$$K_0 \longrightarrow \Sigma^{-1}K_1 \longrightarrow \Sigma^{-2}K_2 \longrightarrow \cdots$$

gives a ~~c~~ cochain complex of homotopy groups whose cohomology is $\mathrm{Ext}_A(H^*(X), \mathbb{Z}/2)$. For details see [11] [1]

Theorem 2 (Adams [1]) There's a spectral sequence converging to the 2-component of $\pi_{n+k}(X)$ for $k < n-1$ with

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X), \mathbb{Z}/2) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

The groups $E_\infty^{s,t}$ form the associated graded group to a filtration of the 2-component of $\pi_{n+k}(X)$.

The Adams spectral sequence can also be set up ~~for~~ to calculate the stable homotopy of a connective spectrum X, $\pi_*^S(X)$. ~~With~~ The provision that $H^*(X)$ is ~~locally~~ of finite type guarantees that the spectral sequence will converge.

Let's recall some facts about spectra and the stable homotopy category as covered in Adams [1]. Let $H\mathbb{Z}/2$ denote the mod 2 Eilenberg - MacLane spectrum.

(i) $H^*(X) = [X, H\mathbb{Z}/2]$

(ii) $H^*(H\mathbb{Z}/2) = A$

(iii) If K is a wedge of suspensions of $H\mathbb{Z}/2$ then $\pi_*^S(K) = \mathrm{Hom}_A(H^*(K), \mathbb{Z}/2)$.

(iv) A map $f: X \longrightarrow K$ is equivalent to a locally finite collection of elements in $H^*(X)$ in appropriate dimensions

~~(v) If a locally finite collection of elements in $H^*(X)$ generates it as an A mod~~

(v) $H\mathbb{Z}/2 \wedge X$ is a wedge of suspensions of $H\mathbb{Z}/2$ with one wedge summand for each $\mathbb{Z}/2$ generator of $H^*(X)$.

(vi) the ~~map~~ composition $X \simeq S^0 \wedge X \xrightarrow{u \wedge 1} H\mathbb{Z}/2 \wedge X$ induces the A-module structure $A \otimes H^*(X) \to H^*(X)$. In particular this map is onto.

(vii) If a locally finite collection collection of elements in $H^*(X)$ generate it as an ~~e~~ A-module, then the corresponding map $f : X \to K$ induces a surjection in cohomology

~~Us~~ Using the above we ~~have~~ can construct an Adams resolution for the spectrum $X = X_0$ by ~~setting~~ setting $K_0 = X \wedge H\mathbb{Z}/2$ and taking X_1 to be the fiber of the map $f_0 : X_0 \to K_0$ where $f_0^* : H^*(K_0) \to H^*(X_0)$ is onto. We inductively set

$$K_{i+1} = X_{i+1} \wedge H\mathbb{Z}/2 \quad \text{where } X_{i+1} \text{ is the fibre of}$$

$$f_i : X_i \to K_i$$ ~~inducing where~~ and choose

$$f_{i+1} : X_{i+1} \to K_{i+1}$$ to be a map inducing a surjection in cohomology.

Since the f_i^* is surjective we have short exact sequences

$$H^*(\Sigma X_{i+1}) \rightarrowtail H^*(K_i) \twoheadrightarrow H^*(X_i).$$

These can be spliced together to form an A-free resolution of $H^*(X)$. Without proving the convergence properties of this spectral sequence I claim that ~~them~~ Theorem 2 holds for spectra X provided $H^*(X)$ has finite type. For a detailed account of these convergence properties see Ravenel [11]

III $\pi_*^S(bo)$: The Stable Homotopy of Real Connective K-Theory.

This will be the first application of the Adams Spectral Sequence. According to the previous chapter there is a spectral sequence with E_2-term

$$E_2 = \mathrm{Ext}_A(H^*(bo), \mathbb{Z}/2) \Rightarrow E_\infty$$

where E_∞ is ~~an~~ graded group associated to $\pi_*^S(bo)$

First, we should talk about the spectrum representing real connective K-Theory, bo. It is constructed as follows:

Let BO be the ~~real~~ spectrum representing real ~~connective~~ K-Theory. Define a spectrum bo by $bo(8k) = BO[8k, 8k+1, \ldots] \wedge M_2$ where $BO[8k, 8k+1, \ldots]$ denotes the k-1 connective cover of BO in its Postnikov cover and M_2 denotes the Moore space for $\mathbb{Z}$ localized at 2. The structure maps $\Sigma^8 bo(8k) \longrightarrow bo(8(k+1))$ are adjoints to the identifications

$$BO[8k, 8k+1, \ldots] \longrightarrow \Omega^8 BO[8(k+1), 8(k+1)+1, \ldots]$$

coming from Bott periodicity.

~~By a calculation~~

According to Stong [12] $H^*(bo, \mathbb{Z}/2) = A/\!/A_1$ where $A_1 = \langle Sq^1, Sq^2 \rangle$.

We want to compute $\pi_*^S(bo)$ using the Adams spectral sequence, so the first objective in this is to calculate the E_2 term of, $Ext_A^{s,t}(H^*(bo), \mathbb{Z}/2)$. We employ a sneaky standard trick called a change of rings theorem.

Let $\mathbb{Z}/2 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ be an A_1-free resolution of $\mathbb{Z}/2$. We can tensor this resolution ~~with~~ on the left with A over A_1 to obtain a free A-resolution of $A \otimes_{A_1} \mathbb{Z}/2$

$$A \otimes_{A_1} \mathbb{Z}/2 \leftarrow A \otimes_{A_1} F_0 \leftarrow A \otimes_{A_1} F_1 \leftarrow \cdots$$

In this way we obtain an A-free resolution of $H^*(bo) = A \otimes_{A_1} \mathbb{Z}/2$. In order to calculate ~~Ext we would~~ $Ext_A(H^*(bo), \mathbb{Z}/2)$ we would apply

$Hom_A(\text{—}, \mathbb{Z}/2)$ to this resolution and then compute

the cohomology of the resulting cochain complex.

But note that $\quad Hom_A(A \otimes_{A_1} F_i, \mathbb{Z}/2) \cong Hom_{A_1}(F_i, \mathbb{Z}/2)$

and hence $\quad Ext_A^{s,t}(H^*(bo), \mathbb{Z}/2) \cong Ext_{A_1}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2)$.

This isomorphism greatly reduces our work.

One way of computing $\mathrm{Ext}_{A_1}(\mathbb{Z}/2,\mathbb{Z}/2)$ is to use ~~the~~ a spectral sequence associated to the extension of algebras

$$E(Q_1) \longrightarrow A_1 \longrightarrow E(\overline{Sq^1},\overline{Sq^2})$$ ~~where~~ ~~Q~~

(see Ravenel [11]) where

$Q_1 = Sq^1Sq^2 + Sq^2Sq^1$ is the Milnor exterior generator. Since $E(Q_1)$ is a 2-sided ideal in A_1, this extension is called normal and hence the Cartan-Eilenberg ~~spectral sequence~~ change of rings spectral sequence is applicable. This is a spectral sequence with

$$\mathrm{Ext}^{s_1}_{E(\overline{Sq^1},\overline{Sq^2})}(\mathbb{Z}/2, \mathrm{Ext}^{s_2,t}_{E(Q_1)}(\mathbb{Z}/2,\mathbb{Z}/2)) = E_2^{s_1,s_2,t}$$

~~and~~ with $d_r : E_r^{s_1,s_2,t} \longrightarrow E_r^{s_1+r,s_2-r+1,t}$ ~~and~~ a derivation and

$$E_2 \Rightarrow E_\infty = \mathrm{Ext}^{s_1+s_2,t}_{A_1}(\mathbb{Z}/2,\mathbb{Z}/2).$$

It's clear that $\mathrm{Ext}^{s_2,t}_{E(Q_1)}(\mathbb{Z}/2,\mathbb{Z}/2) = P[h_2]$ is a polynomial algebra on h_2 and that this is a trivial module over the exterior algebra $E(\overline{Sq^1},\overline{Sq^2})$ so we can write

$$\mathrm{Ext}_{E(\overline{Sq^1},\overline{Sq^2})}(\mathbb{Z}/2, \mathrm{Ext}_{E(Q_1)}(\mathbb{Z}/2,\mathbb{Z}/2)) \cong$$

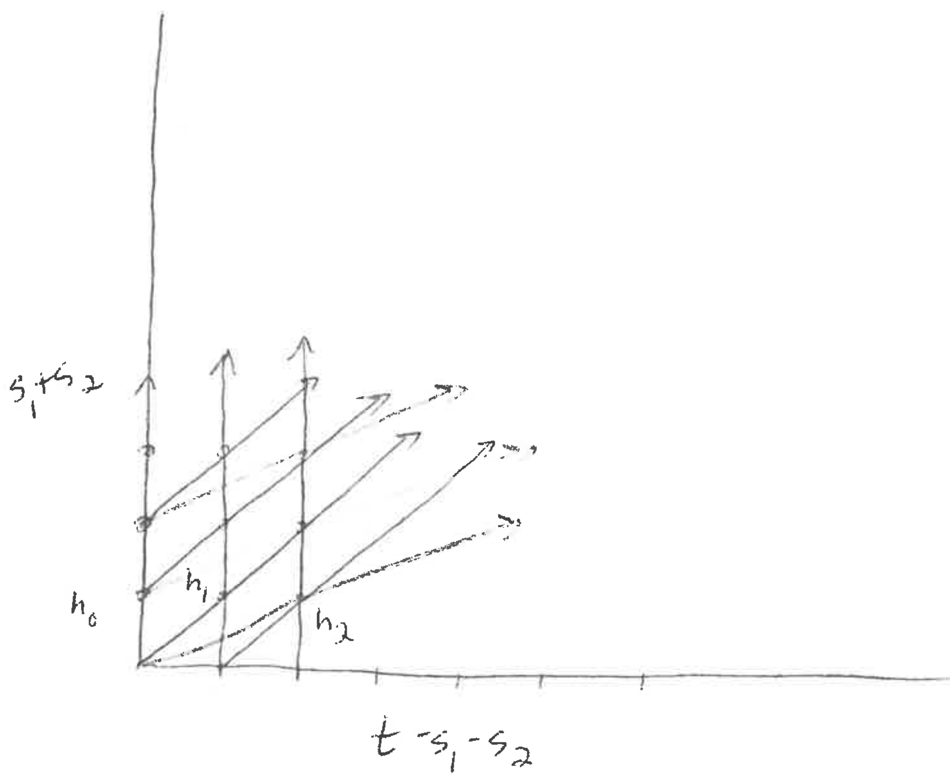
$$\mathrm{Ext}_{E(\overline{Sq^1},\overline{Sq^2})}(\mathbb{Z}/2, P[h_2]) \cong$$

$$\mathrm{Ext}_{E(\overline{Sq^1},\overline{Sq^2})}(\mathbb{Z}/2,\mathbb{Z}/2) \otimes P[h_2] \cong P[h_0,h_1,h_2]$$

a polynomial algebra on 3 generators with

$h_0 \in E_2^{1,0,1}$, $h_1 \in E_2^{1,0,2}$, $h_2 \in E_2^{0,1,3}$.

This yields the following picture of the E_2 term



<u>Lemma 3</u> $d_2(h_0) = 0$
$d_2(h_1) = 0$
$d_2(h_2) = h_1 \cdot h_0$, $d_2(h_2^2) = 0$.

<u>prf</u>: $d_2(h_0) = 0$ is clear, as is $d_2(h_1) = 0$

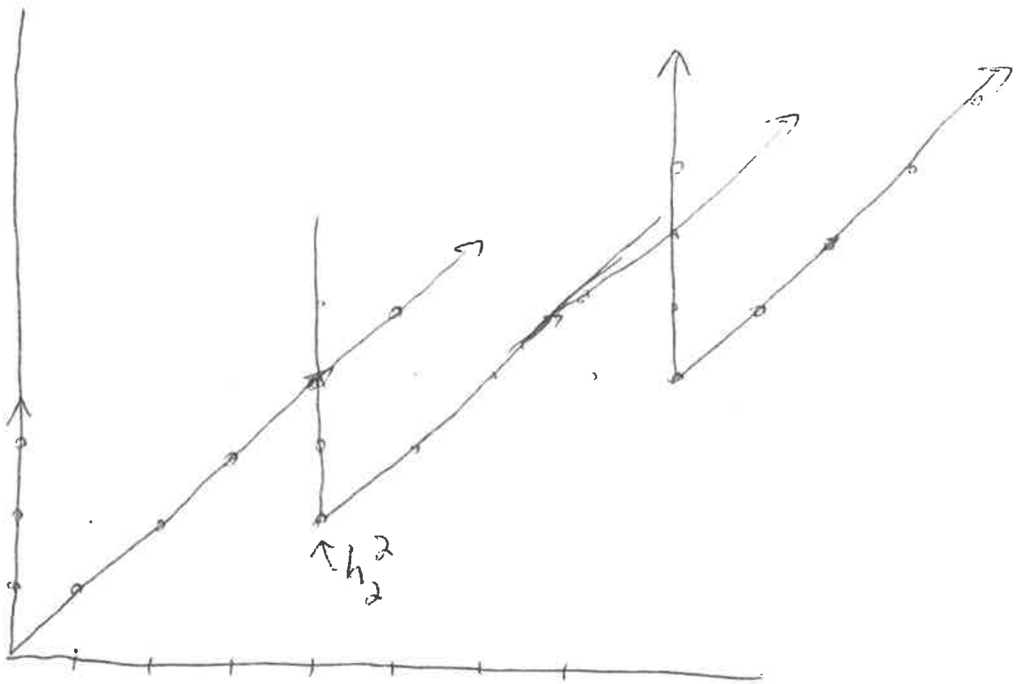
It can be shown that $d_2(h_2) = h_1 \cdot h_0$ ~~on the~~ by inspecting the cobar complex for $\mathbb{Z}/2$ over A_1. This is done by setting up the reduced bar resolution as follows: let $\overline{A}_1 = \ker(\varepsilon : A_1 \to \mathbb{Z}/2)$ be the augmentation ideal. Then the following sequence is exact

$$\mathbb{Z}/2 \xleftarrow{\varepsilon} A_1 \xleftarrow{1\otimes i} A_1 \otimes \overline{A}_1 \xleftarrow{1\otimes i \otimes i} A_1 \otimes \overline{A}_1 \otimes \overline{A}_1 \leftarrow \cdots$$

where $i : \overline{A}_1 \rightarrowtail A_1$ is the inclusion. Applying $\mathrm{Hom}_{A_1}(-, \mathbb{Z}/2)$ to this free resolution is a way of obtaining a ~~free~~ cochain complex from which $\mathrm{Ext}_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2)$ can be calculated. Note that if $\overline{A}^n$ denotes the n-fold product of $\overline{A}$ with itself then $\mathrm{Hom}_{A_1}(A_1 \otimes \overline{A}^n, \mathbb{Z}/2) \cong$

~~Hom_{A_1}~~ $\mathrm{Hom}_{\mathbb{Z}/2}(\overline{A}^n, \mathbb{Z}/2) \cong \overline{(A_1^*)}^n$ where $\overline{A}_1^*$ is

the dual of A_1. The resulting complex is called the cobar complex, where the differential is an alternating sum of the diagonal $\psi\ A_1^* \to A_1^* \otimes A_1^*$ ~~re~~ restricted to $\overline{A_1^*}$. For details see MacLane [8]. Since we know that $h_0 h_1$ is represented by $\xi_1 | \xi_1^2$ in the cobar complex and $\psi(\xi_2) = \xi_1 | \xi_1^2$, the element $h_0 h_1$ must be hit by something in the Cartan-Eilenberg spectral sequence. ~~The end~~ After inspecting the gradings, we see that the only possibility is that $d_2(h_2) = h_0 h_1$ $\boxtimes$ This yields the following picture of the E_3-term



Lemma 4 $d_3(h_2^2) = h_1^3$.

prf: Again we employ the cobar complex to do specific calculations. After some simple, but tedious inspection of the cobar complex, we find that

$$\partial(\xi_1\xi_2 | \xi_1^2) = (\xi_1^2|\xi_1^2|\xi_1^2) + (\xi_1|\xi_2|\xi_1^2) + (\xi_2|\xi_1|\xi_1^2) + (\xi_1|\xi_1^3|\xi_1$$

$$\partial(\xi_1 | \xi_1^2\xi_2) = (\xi_1|\xi_2|\xi_1^2) + (\xi_1|\xi_1^2|\xi_2) + (\xi_1|\xi_1^3|\xi_1^2)$$

$$\partial(\xi_2|\xi_2) = (\xi_1|\xi_1^2|\xi_2) + (\xi_2|\xi_1|\xi_1^2)$$ and hence

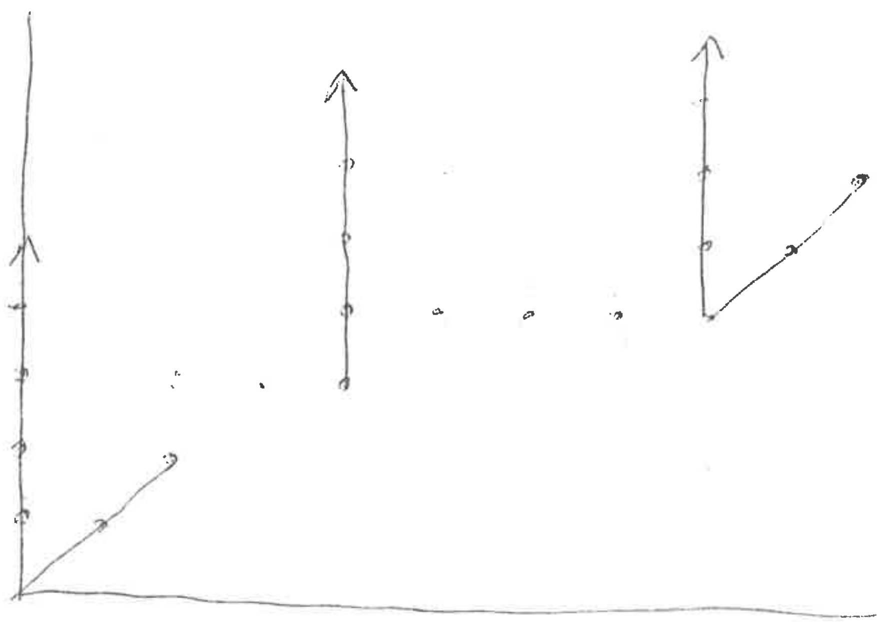
that $\partial\left[(\xi_1\xi_2|\xi_1^2) + (\xi_1|\xi_1^2\xi_2) + (\xi_2|\xi_2)\right] = \xi_1^2|\xi_1^2|\xi_1^2$

which represents h_1^3 in the cobar complex.

Since ~~this~~ ~~E~~ the E_3-term is the last opportunity for h_1^3 to be hit, ~~#~~ it must be the case that

$$d_3(h_2^2) = h_1^3. \quad \boxtimes$$

Hence we have the following picture of the E_4-term



<u>Lemma 5</u>: $E_4 = E_\infty$

prf: This is easy to see from the above picture as none of the d_r for $r \geq 4$ have anything to hit. $\boxtimes$

The motivating philosophy behind this calculation is that the Cartan-Eilenberg spectral sequence is a good bookkeeping device, but a bit too vague for explicit calculations. Thus we employ the C.E.S.S to tell us where to inspect the cobar complex for relations. Using the cobar complex itself is unreasonable since its too dense with information for feasible calculation.

This completes the calculation of $\mathrm{Ext}_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2)$ and hence of $\pi_*^S(bo)$. This information will be fed ~~in to~~ into the Adams Spectral Sequence for $bo_*(B(n))$.

$bo_*(B(n))$ Applying the Adams Spectral sequence

In this section we will compute the real connective K-theory of the Brown-Gitler spectra using the ~~/~~ classical Adams spectral sequence

Recall our definitions of the n^{th} Brown ~~Git~~ Gitler module $M(n)$, ~~and~~ the modified Brown-Gitler module $M_1(n)$ and the map ~~$\pi: M(n) \to M_1(n)$~~ $\pi: M(n) \to M_1(n)$. Initially we will concentrated on some of the properties of $M_1(n)$.

__Theorem__ 6 (A B P S [3]) : $H_*(bo) = \mathbb{Z}/2[x_1^4, x_3^2, x_7, \ldots]$ with left A_1-action given by

$$Sq^1(x_{(i)}) = x_{(i-1)}^2 \quad \text{and} \quad Sq^2(x_{(i)}) = 0 \quad \text{where}$$

$(i) = 2^i - 1$. $\boxtimes$

__Theorem__ 7 (Mahowald [10]) : Let $g: \bigoplus_{k\geq 0} \Sigma^{4k} M_1(2k) \longrightarrow A/\!/A_1$ be given by

$$g|_{4k} : \Sigma^{4k} M_1(2k) \longrightarrow M_1(2k)\cdot \chi(Sq^{4k}) \subset A/\!/A_1$$

Then g is an isomorphism of left A_1-modules $\boxtimes$

In $H_*(bo)$, let $x_{(i)}$ have weight 2^{i-1}. Then this weight is multiplicative and is preserved by the left A_1-action.
Let $N_{4k} \subset H_*(bo)$ consist of the vector space generated by all monomials of weight $4k$. By the above observation, N_{4k} is an A_1-submodule too.

Recall that Q_0 and Q_1 act on $H_*(bo)$ as derivations, hence we can regard N_{4k} as a differential group with respect to these actions. It's been proved by Adams [1] that the homology of N_{4k} with respect to this action of Q_0 and Q_1 contains a great deal of information. Before I launch into spelling this out explicitly, lets record some information about N_{4k}.

Lemma 8 (ABP [3]) $H_*(N_{4k}, Q_0) = \mathbb{Z}/2$ in weight $4k$
and $H_*(N_{4k}, Q_1) = \mathbb{Z}/2$ " " " $\boxtimes$

next page ⟶

see attached pages

~~Corollary~~: ~~$H_*(M_1(2n), Q_0)$~~ $= \mathbb{Z}/2$ in dimension 0
$H_*(M_1(2n), Q_1) = \mathbb{Z}/2$ " " $2[2n - \alpha(n)]$.
where $\alpha(n)$ is the number of nonzero coefficients in the dyadic expansion of n.

prf: Suppose that $n = \sum_{j=2}^{s} i_j 2^{j-2}$ where $i_j = 0$ or 1

Then $4n = \sum_{j=2}^{s} i_j 2^j = \sum_{j=2}^{s} 2i_j(2^{j-1})$. Recall that

x_{2^i-1} has weight 2^{i-1}. Since we've made the identification $\sum^{4n} M_1(2n) = N_{4n}$, we can say

$H_*(M_1(2n), Q_1) = \mathbb{Z}/2$ in dimension

$\sum_{j=2}^{s} 2i_j(2^j - 1) - \sum_{j=2}^{s} 2i_j(2^{j-1}) =$ WHY???

something is missing here.

~~First proof could be simplified~~

Corollary 9 : $H_*(M_1(2n), Q_0) \cong \mathbb{Z}/2$ in degree 0 and
$H_*(M_1(2n), Q_1) \cong \mathbb{Z}/2$ in degree $2(2n - \alpha(n))$.

proof : The fact that $H_*(M_1(2n), Q_0) \cong \mathbb{Z}/2$ in degree zero follows from that fact that $\Sigma^{4n} M_1(2n) \cong N_{4n}^*$ as left A_1-modules, $H_*(N_{4n}, Q_0)$ is one dimensional and $Q_0(1) = 0$ in $M_1(2n)$.

The other fact can be observed by considering N_{4n}. Suppose that $m \in N_{4n}$ is some monomial so that $[m]$ generates $H_*(N_{4n}, Q_1)$. This can be arranged by a suitable choice of basis for N_{4n}. Let $m = \prod_{j=1}^{n} x_{(i)_j}^{\alpha_j}$ where

$x_{(i)_j}$ is a polynomial generator of weight $2^{i_j - 1}$ and of degree

$2^{i_j} - 1$ and $x_{(i)_j} \neq x_{(i)_k}$ for $j \neq k$ in this representation.
Note that $x_{(i)}^2$ doesn't occur in this representation.
It's clear that α_j must be even for $1 \le j \le n$ since otherwise $Q_1(m) \neq 0$ by the Cartan formula. Also if $\alpha_K \geq 4$ for some K then $m = x_{(i)_K}^4 \cdot x_{(i)_K}^{\alpha_K - 4} \cdot \prod_{\substack{j=1 \\ j \neq K}}^{n} x_{(i)_j}^{\alpha_j} =$

$$Q_1\left[x_{(i+2)_K} \cdot x_{(i)_K}^{\alpha_K - 4} \cdot \prod_{\substack{j=1 \\ j \neq K}}^{n} x_{(i)_j}^{\alpha_j} \right].$$

$*$ Contradiction to our choice of m. Thus $\alpha_j = 2$ for $1 \le j \le n$.
We can now find the weight of m ~~which is~~ in terms of the weight of its factors which is

$$\sum_{p=2}^{s} 2b_p \cdot (2^{p-1}) \quad \text{where}$$

$$b_p = \begin{cases} 1 & \text{if } p = i_j \text{ for some } 1 \le j \le n \\ 0 & \text{otherwise} \end{cases}$$

But we know that the weight of m is $4n$. Thus $4n = \sum_{p=2}^{s} 2b_p(2^{p-1})$. It's useful to observe that this representation of $4n$ is twice the unique dyadic expansion of $2n$ and that the monomial $\hat{m}$ is of degree $\sum_{p=2}^{s} 2b_p(2^p - 1)$.

We have the identification $\Sigma^{4n} M_1(2n) \cong N_{4n}^{*}$

so the monomial $m' \in M_1(2n)$ corresponding to $m \in N_{4n}$

is of degree

$$\sum_{p=2}^{s} 2b_p(2^p - 1) - \sum_{p=2}^{s} 2b_p(2^{p-1})$$

$$= \sum_{p=2}^{s} 2b_p(2^{p-1} - 1)$$

$$= 2 \sum_{p=2}^{s} b_p(2^{p-1} - 1)$$

$$= 2\,[2n - \alpha(n)].$$ This completes the proof. $\boxtimes$

~~It would be easier to compute $H_*(H_*(W); Q_1)$ directly and proceed from there.~~

$\cancel{= \sum_{j=?}^{s} 2 i_j (2^{j-1} - 1) = 2 \sum_j i_j (2^{j-1} - 1)}$

$\cancel{= 2 [2n - \alpha(n)]}$ ~~The $H_*(M_1(2n), Q_0)$~~ case

~~obvious.~~

We now set up some machinery of Adams through which to push this information.

<u>Definition 2</u>: A module M over A_1 is called invertible if $\text{rank}(H_*(M, Q_0)) = \text{rank}(H_*(M, Q_1)) = 1$ where the rank is as a $\mathbb{Z}/2$ vector space

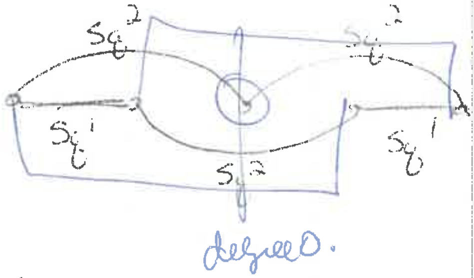
<u>Definition 3</u>: Two modules M and N over A_1 are said to be stably equivalent if there exist A_1 free modules F and G so that $M \oplus F \cong N \oplus G$. This implies that $\text{Ext}_{A_1}^{s,t}(M, \mathbb{Z}/2) \cong \text{Ext}_{A_1}^{s,t}(N, \mathbb{Z}/2)$ for $s \geq 1$.

<u>Theorem 10 (Adams [1])</u> If M is an invertible A_1-module where $H_*(M, Q_0) \cong \mathbb{Z}/2$ in dimension $a + b = j$ and $H_*(M, Q_1) \cong \mathbb{Z}/2$ in dimension $a + 3b = k$ then

~~M is stably equivalent to $\Sigma^j \bar{A}^k - J^{\psi(M)}$ where $\bar{A}$ is the augmentation ideal, J is the "joker"~~

and $\psi(M)$ is some function of the $\mathbb{Z}/2$ rank of M

M is stably equivalent to $\Sigma^{i} \bar{A}_1^{k} J^{\varphi(M)}$ where the exponents denote an appropriately iterated tensor product, $\bar{A}_1$ is the augmentation ideal ~~~~, J is the "joker" pictured here:



and $\varphi(M)$ is some function of the $\mathbb{Z}/2$ rank of M. $\boxtimes$

We will denote such a stable equivalence by $M \underset{s}{\simeq} \Sigma^{J} \bar{A}_1^{K} J^{\varphi(M)}$.

It is proved by Adams in the BSO paper that $J^2 \underset{s}{\simeq} F$, a free module. I have left $\varphi(M)$ indeterminate here since Adams provides a convenient way of omitting it.

<u>Theorem</u> 11 (Adams [2]). The stable type of $A/\!/A_1$ as a left A_1-module is

$$\left(1 \oplus \Sigma^3 \bar{A}_1 J\right) \otimes \left(1 \oplus \left(\bigoplus_{i>0} \Sigma^{2^i+1} \bar{A}_1^{2^i-1}\right)\right)$$

$$= 1 \oplus \Sigma^3 \bar{A}_1 J \oplus \Sigma^5 \bar{A}_1^3 \oplus \Sigma^8 \bar{A}_1^4 J \oplus \Sigma^9 \bar{A}_1^7 \oplus \cdots \boxtimes$$

Corollary 12: $M_1(2n) \simeq_S \Sigma^{-c}\bar{A}_1^{\,c}\,J^{n}$ where $c = 2n - \alpha(n)$.

proof: We know that as left A_1 modules,

$$\bigoplus_{k \geq 1} \Sigma^{4k} M_1(2k) \cong A /\!/ A_1$$

and as stable modules

$$A /\!/ A_1 \simeq_S \left(1 \oplus \Sigma^{3}\bar{A}_1 J\right)\left(\bigoplus_{i \geq 1}\left(1 \oplus \Sigma^{2^i+1}\bar{A}_1^{\,2^i-1} J\right)\right).$$

From these facts and ~~the theorem of Adams [ ]~~ theorem **10** of Adams, the result follows. $\boxtimes$

~~Corollary~~ ~~$Ext_{A_1}^{s,t}(M_1(2n), \mathbb{Z}/2) \cong Ext_{A_1}^{s+c,t+c}($~~

Corollary 13: $Ext_{A_1}^{s,t}(M_1(2n), \mathbb{Z}/2) \cong Ext_{A_1}^{s+c,t+c}(J^n, \mathbb{Z}/2)$.

proof: This follows from the stable type of $M_1(2n)$ and the exact sequence $\bar{A}_1 \rightarrowtail A_1 \twoheadrightarrow \mathbb{Z}/2$ inducing the connecting map

$$\delta : Ext_{A_1}^{s,t}(\bar{A}_1, \mathbb{Z}/2) \xrightarrow{\cong} Ext_{A_1}^{s+1,t}(\mathbb{Z}/2, \mathbb{Z}/2). \boxtimes$$

Theorem 14. $M(2n+1) \cong M_1(2n) \otimes M(1)$

proof. Recall that, as a space, $\Omega^2 S^3 \simeq S^1 \times W$. This yields that $H_*(W) \cong \mathbb{Z}/2\,[x_1^2, x_3, x_7, \dots]$ by using the Eilenberg-Zilber theorem. The left action of A_1 on this is the same as that on $H_*(bo)$ so we can write

$$H_*(W) \cong H_*(bo) \otimes \langle 1, x_1^2, x_3, x_1^2 \cdot x_3 \rangle .$$

Filter $H_*(W)$ as follows: let $F_0 = H_*(bo)$, $F_1 = \cancel{H_*}\, F_0 \otimes \langle 1, x_1^2 \cdot x_3 \rangle$ and $F_2 = H_*(W)$. Then since the A_1 action must preserve this weight there's a split short exact sequence of left A_1-modules

$$F_1 \rightarrowtail F_2 \to F_2/F_1$$

where F_2/F_1 consists of all monomials of weight congruent to 2 mod 4 in $H_*(W)$.

It remains to be seen that such monomials correspond to $\chi(M(2k+1)^*)$, that is, to the dual of the oddly indexed Brown-Gitler modules. This is accomplished as follows.

Let $S^1 \vee W \hookrightarrow S^1 \times W \to \Sigma W$ be a cofibration inducing a long exact sequence

$$\begin{tikzcd} \tilde{H}_*(S^1) \oplus \tilde{H}_*(W) \arrow[r] & \tilde{H}_*(S^1 \times W) \arrow[r] & \tilde{H}_*(\Sigma W) \arrow[ll, bend left, "\delta"] \end{tikzcd}$$

in which δ is trivial, since it must preserve this weight. Thus there exist short exact sequences

$$\tilde{H}_*\left(F_n(W)/F_{n-1}(W)\right) \rightarrowtail \tilde{H}_*\left(F_n/F_{n-1}\right) \twoheadrightarrow \tilde{H}_*\left(\Sigma F_n(W)/\Sigma F_{n-1}(W)\right)$$

for $n > 1$.

Recall that $\tilde{H}_*(F_n/F_{n-1}) \cong \chi(M([\frac{n}{2}])^*)$ and that if n is even then (same line) $\tilde{H}_*(F_n/F_{n-1}) \cong \tilde{H}_*(F_n(W)/F_{n-1}(W))$.

Therefore $\tilde{H}_*(W) \cong \bigoplus_{k\geq 0} \chi(M(k)^*)$.

It's easy to see that $H_*(F_n/F_{n-1})$ consists of monomials of weight exactly n. Therefore

(i) $n \equiv 2 \pmod 4$ implies that $\chi(M([\frac{n}{2}])^*)$ consists of monomials of weight congruent to 2 mod 4.

(ii) $n \equiv 0 \pmod 4$ implies that $\chi(M([\frac{n}{2}])^*)$ consists of monomials of weight congruent to 0 mod 4.

Therefore, remark (i) tells us that

$$\chi(M(2k+1)^*) \subset H_*(bo) \otimes \langle x_1^2, x_3 \rangle \cong$$

$$\bigoplus_{n\geq 1} \Sigma^{4n} \chi(M_1(2n)) \otimes \chi(M(1)^*).$$ It follows that

$$M(2k+1) \cong M_1(2k) \otimes M(1). \quad \boxtimes$$

Lemma 15. There exist short exact sequences of left A_1-modules

(a) $\Sigma M_1(2n) \xrightarrow{\mu_1} M(2n+1) \xrightarrow{\pi_1} M_1(2n)$

(b) $\Sigma_1 M_1(2n-2) \xrightarrow{\mu_2} M(2n) \xrightarrow{\pi_2} M_1(2n)$

(c) $\Sigma^k M([\frac{k}{2}]) \xrightarrow{i} M(k) \xrightarrow{p} M(k-1)$ [Mahowald [9]]

prf: I have already described the maps π_1 and π_2. Let K_1 denote the kernel of π_1. Define $\varphi_1 : K_1 \longrightarrow \Sigma M_1(2n)$ by $\varphi_1(\chi(Sq^I)) = \Sigma(\pi_1(\chi Sq^J))$ where J is an admissible sequence and

$$\chi(Sq^I) = \chi(Sq^J)\cdot Sq^1.$$ This is well defined for if

$$\chi(Sq^I) = \chi(Sq^J)\, Sq^1 = \chi(Sq^K)\cdot Sq^1$$ for some admissible sequence K then $(\chi(Sq^J) - \chi(Sq^K))\, Sq^1 = 0 \in M(2n+1)$

so $\pi_1(Sq^J) = \pi_1(Sq^K) \in M_1(2n) = A_1/A_1\{\chi Sq^i, Sq^1 \mid i > 2n\}$

I claim that φ_1 is a morphism of left A_1-modules.

I also claim that φ_1 is injective, for if $\varphi_1(\chi(Sq^I)) = 0$ then $\chi(Sq^I) = \chi(Sq^J)\cdot Sq^1$ where $\chi(Sq^J) = \chi(Sq^K)\cdot Sq^1$

so $\chi(Sq^I) = \chi(Sq^K)\, Sq^1 \cdot Sq^1 = 0$. This proves the injective claim.

Define $\psi_1 : \Sigma M_1(2n) \longrightarrow K_1$ by

$\psi_1(\Sigma\pi(\chi(Sq^J))) = \chi(Sq^J)\cdot Sq^1$. It's clear that

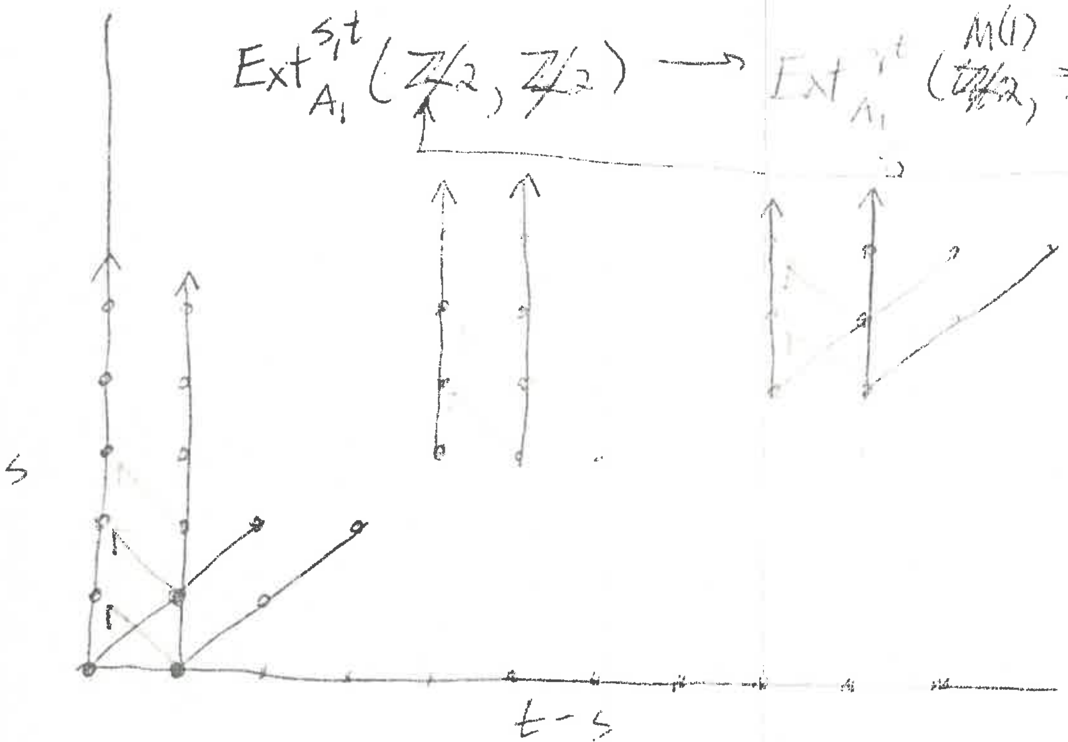
this is well defined. Suppose that $\chi(Sq^J)\cdot Sq^1 = 0$. Then $\chi(Sq^J) = \chi(Sq^K)\cdot Sq^1$ so $\pi_1(\chi(Sq^J)) = 0$. This proves the injectivity of ψ_1. Since both $\Sigma M_1(2n)$ and K_1 are finite the existence of these injective maps guarantees that they are isomorphisms. This proves the exactness of (a).

The proof of the exactness of sequence (b) is analogous to that of (a). Sequence (c) is due to Mahowald $\boxtimes$

We now have all the necessary information to start the calculation. First, consider the long exact sequence of Ext groups arising from the short exact sequence

$$\Sigma\mathbb{Z}/2 \xrightarrow{i} M(1) \xrightarrow{p} \mathbb{Z}/2.$$

$$\mathrm{Ext}^{s,t}_{A_1}(\mathbb{Z}/2,\mathbb{Z}/2) \longrightarrow \mathrm{Ext}^{s,t}_{A_1}(M(1),\mathbb{Z}/2) \longrightarrow \mathrm{Ext}^{s,t}_{A_1}(\Sigma\mathbb{Z}/2,\mathbb{Z}/2)$$



It can be shown that the differentials so indicated are nontrivial by inspecting the cobar complex for $\mathrm{Ext}^{s,t}_{A_1}(M(1),\mathbb{Z}/2)$. Alternatively, one might argue more simply that $\delta : \mathrm{Ext}^{0,1}_{A_1}(\Sigma\mathbb{Z}/2,\mathbb{Z}/2) \to \mathrm{Ext}^{1,1}_{A_1}(\mathbb{Z}/2,\mathbb{Z}/2)$

is nonzero since the previous map

$$i^* : \mathrm{Hom}_{A_1}(M(1),\mathbb{Z}/2) \longrightarrow \mathrm{Hom}_{A_1}(\Sigma\mathbb{Z}/2,\mathbb{Z}/2)$$

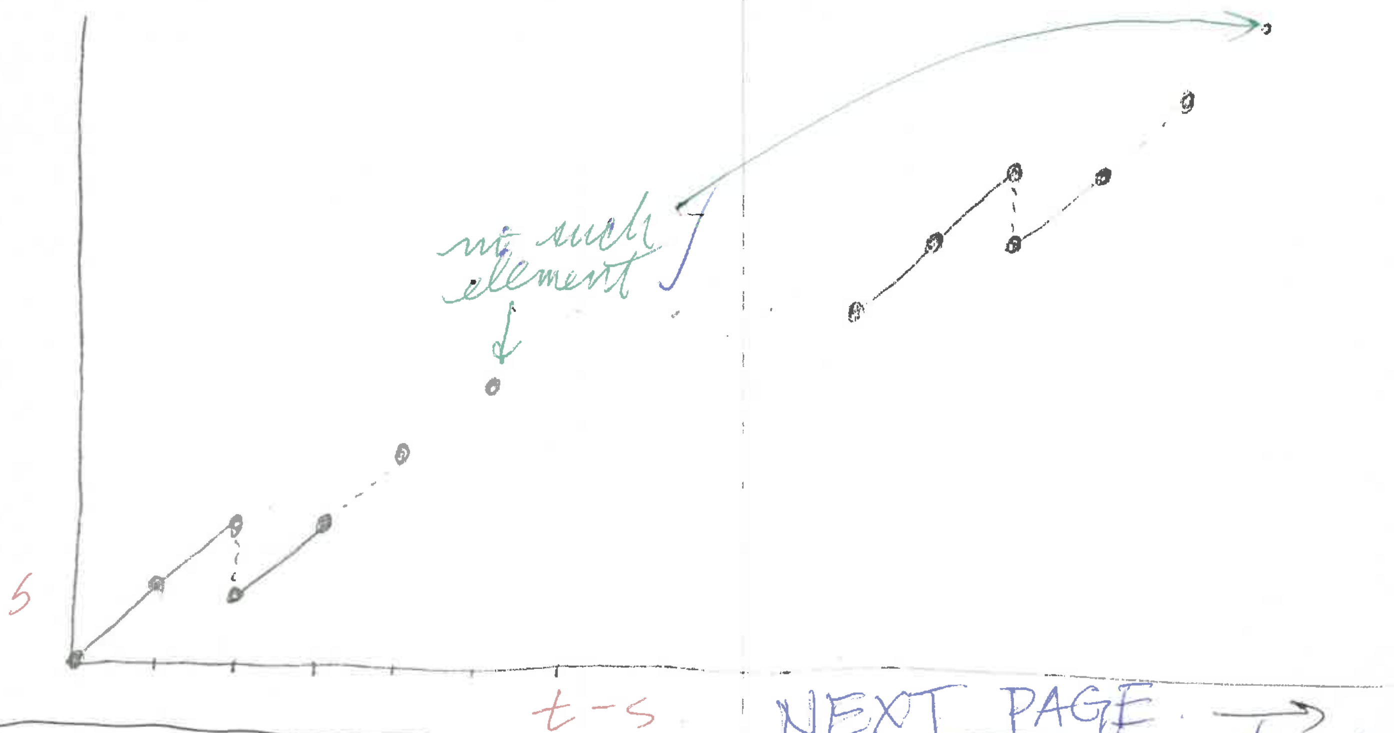
must be trivial and $\mathrm{Ext}^{1,1}_{A_1}(\mathbb{Z}/2,\mathbb{Z}/2)$ is nontrivial.

The other connecting map

$$\delta : \mathrm{Ext}_{A_1}^{3,8}(\Sigma\, \mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow \mathrm{Ext}_{A_1}^{4,8}(\mathbb{Z}/2, \mathbb{Z}/2)$$

~~[illegible] difficult to compute in this setting or in the other complex.~~
is zero by virtue of the periodicity of $\pi_*^S(bo)$

Thus we're left with the picture of the Adams E_2-term without the relations indicated by dotted lines.



NEXT PAGE →

~~see most recent installment.~~

We know see that these dotted lines do represent relations ~~on the~~ extension problems. induced by the $\pi_*^s(bo)$ multiplication. The method which we employ is due to W.S. Massey and the operations are called, appropriately, Massey Products. A detailed account is given in Ravenel [ ]

Insert them? between this page + next.

I want to prove that $v \cdot h_1 = 1 \cdot h_1^2$, ~~that~~ which is proving the existence of a vertical line in the picture above, where $v = \langle 1, h_0, h_1 \rangle$, is a Massey product.

By a method known as juggling (see Ravenel [ ]) we get that ~~[illegible]~~

M(1) is not a coalgebra so this Massey product is not defined.

In this picture of the Adams E_2-term*, we can show that the dotted vertical ~~lines~~ and diagonal lines do indeed represent relations induced by the $\pi_*^S(bo)$ multiplication. We can regard this module structure as the Yoneda product of exact sequences, but we prove that this product is nonzero by inspecting the cobar complex for $M(1)$ over A_1. Let us first label the element $v \in E_2^{1,3}$. ~~It's clear~~ We know that if $1 \in Ext_{A_1}(M(1), \mathbb{Z}/2)$ is represented by $\hat{1}$ in the cobar complex for $M(1)$ then $1 \cdot h_0 = 0$ by inspecting just the ~~E_2-term~~ additive structure of the E_2-term. By the same means we see that $h_1 h_0 = 0$ in $Ext_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2)$. Therefore, in the cobar complex for $M(1)$ over A_1 there exist $\overbrace{a \text{ and } b}^{\text{cochains}}$ so that $\partial(a) = \hat{1}|\xi_1$ and $\partial(b) = \xi_1|\xi_1^2$. It follows that $[a|\xi_1^2 + \hat{1}|b]$ is a cocycle. We can represent this as a Yoneda product $\langle 1, h_0, h_1 \rangle$ and by inspecting dimensions we see that we have constructed an explicit representative for v.

The next objectives ~~is~~ are to show that $v \cdot h_0 = h_1^2$ and that $v \cdot h_1^2 \neq 0$. The first fact follows from the associativity

$$\langle 1, h_0, h_1 \rangle \cdot h_0 = 1 \langle h_0, h_1, h_0 \rangle$$

of Yoneda products and a little bit of computing in the cobar complex. for $M(1)$. In this complex, note that

$1 \cdot h_0 h_1 = 0$ since $\partial(\hat{1}|\xi_2) = \hat{1}|\xi_1|\xi_1^2$ and

$1 \cdot h_1 \cdot h_0 = 0$ since $\partial(\hat{1}|\chi(\xi_2)) = \hat{1}|\xi_1^2|\xi_1$

(~~* see back~~).

Therefore we can write $\langle h_0, h_1, h_0 \rangle = [\xi_2 | \xi_1 + \xi_1 | \chi(\xi_2)]$

$= [\xi_2 | \xi_1 + \xi_1 | \xi_2 + \xi_1 | \xi_1^3]$. But

$\cancel{\partial(\xi_2 | \xi_1)}$ $\partial(\xi_2 \cdot \xi_1) = \xi_2 | \xi_1 + \xi_1^2 | \xi_1^2 + \xi_1 | \xi_1^3 + \xi_1 | \xi_2$.

so $[\xi_2 | \xi_1 + \xi_1 | \chi(\xi_2)] = [\xi_1^2 | \xi_1^2] = h_1^2$.

Thus $\langle h_0, h_1, h_0 \rangle = h_1^2$ so $1 \cdot \langle h_0, h_1, h_0 \rangle = 1 \cdot h_1^2 \neq 0$.

Next we would like to see that $v \cdot h_1^2 \neq 0$ ~~by~~ thereby proving the ~~ext~~ existence of a diagonal line in the picture of the Adams Spectral Sequence from $E_2^{5,2}$ to $E_2^{7,3}$. Again by associativity we get that $\cancel{\langle 1, h_0, h_1 \rangle} =$

$$\langle 1, h_0, h_1 \rangle \cdot h_1^2 = 1 \cdot \langle h_0, h_1, h_1^2 \rangle$$

Thus it remains to be seen that $\langle h_0, h_1, h_1^2 \rangle \neq 0$. From previous definitions we see that we could define

~~$\langle h_0, h_1, h_1^2 \rangle = [a | \xi_1^2 + \xi_1 | b]$ for some a and b with $\partial(a) = \xi$~~

$\langle h_0, h_1, h_1^2 \rangle = [a | \xi_1^2 | \xi_1^2 + \xi_1 | b]$ for some cochains a and b in the cobar complex for $\mathbb{Z}/2$ over A_1, where $\partial(a) = \xi_1 | \xi_1^2$ and $\partial(b) = \xi_1^2 | \xi_1^2 | \xi_1^2$. Note that up to some cocycle we could choose $a = \xi_2$ and $b = \xi_2 | \xi_2 + \xi_1 | \xi_1^2 \xi_2 + \xi_1 \xi_2 | \xi_1^2$.

Lemma 16. ~~Claim~~ $[\xi_2|\xi_2|\xi_1] = [\xi_2|\xi_1^2|\xi_1^2 + \xi_1|\xi_2|\xi_2 + \xi_1|\xi_1|\xi_1^2\cdot\xi_2 + \xi_1|\xi_1\xi_2|\xi_1^2]$

prf: $\partial(\xi_2\cdot\xi_1|\xi_2) = \xi_1^2|\xi_1^2|\xi_2 + \xi_1|\xi_1^3|\xi_2 + \xi_2\cdot\xi_1|\xi_1|\xi_1^2$

and $\partial(\xi_1|\xi_1^3|\xi_2 + \xi_1|\xi_1|\xi_1^2\cdot\xi_2) =$

$\xi_1|\xi_1^3|\xi_1|\xi_1^2 + \xi_1|\xi_1|\xi_1^3|\xi_1^2 = 0.$

Thus $[\xi_2|\xi_1^2|\xi_1^2 + \xi_1|\xi_1|\xi_1^2\cdot\xi_2 + \xi_1|\xi_1\cdot\xi_2|\xi_1^2]$

$= [\xi_1^2|\xi_1^2|\xi_2 + \xi_1|\xi_1^3|\xi_2 + \xi_1|\xi_1\cdot\xi_2|\xi_1^2]$

$= 0.$ This proves the claim. $\boxtimes$

Therefore $\langle h_0, h_1, h_1^2\rangle \neq 0$ and hence the dotted diagonal line can be filled in.

~~But if this is the case, just note that since $|a|=3$, $a = \xi_1^3$ or ξ_2, neither of which are in the image of Sq. This completes the elimination of case (v) and proves~~ that $v \cdot h^2 \neq 0$.

Thus all the dotted lines in the picture for $\mathrm{Ext}_{A_1}(M(1), \mathbb{Z}/2)$ can be filled in to indicate multiplicative relations in $bo_*(M(1))$ as a $\pi_*^S(bo)$-module.

To assemble all these facts together for the final solution we recall that

$$M_1(2n) \simeq_s \Sigma^{-c}\,\overline{A}_1^{\,c}\, J^n$$

where $c = 2n - \alpha(n)$

and $M(2n+1) \cong M_1(2n) \otimes M(1)$ so

$$\mathrm{Ext}_{A_1}^{s,t}(M(2n+1), \mathbb{Z}/2) \cong \mathrm{Ext}_{A_1}^{s+c,t+c}(J^n \otimes M(1), \mathbb{Z}/2).$$

for $s > c$.

It's easy to see that the spectral sequence admits no nontrivial differentials from here on, hence collapses. Thus, we have computed $bo_*(B(2n+1))$.

We compute $bo_*(B(2n))$ by comparing the E_2 terms arising from the short exact sequences of A_1-modules

(A) $\Sigma M_1(2n-2) \rightarrowtail M(2n) \twoheadrightarrow M_1(2n)$

(B) $\Sigma^n M(n) \rightarrowtail M(2n) \twoheadrightarrow M(2n-1)$

It can be seen from the picture associated to A and B that the E_2 terms are completely complementary in that any differential or extension problem in one is solved in the other. To prove that we get the same picture from either short exact sequence we note that in ~~ko_*~~ $bo_*(B(1))$, the "periodic elements" in the lower left hand point of each lightening flash lie on the line $s = \frac{1}{2}(t-s)$. We compare to see that these elements are translated to the same spot in the E_2 term:

~~In~~ the case ~~$n \equiv 1(4)$~~ we inspect ~~the case associated to~~ the short exact sequence A. then $M(n-1) \underset{s}{\cong} \Sigma^{-c}\bar{A}$ where $c = (n-1) - \alpha\left(\frac{n-1}{2}\right)$. Therefore for $s > c$

$$\cancel{Ext_{A_1}^{s,t}\left(\Sigma^{-2n} M(n)\right) \cong Ext_{A_1}^{s+c,t+c}\left(\Sigma^{2n} M(1)\right)}$$

$$Ext_{A_1}^{s,t}\left(\Sigma^{2n} M(n), \mathbb{Z}/2\right) \cong Ext_{A_1}^{s+c,t+c}\left(\Sigma^{2n} M(1), \mathbb{Z}/2\right)$$

which looks like $Ext_{A_1}(M(1), \mathbb{Z}/2)$ ~~chopped off above the line $s = c$ and to the right of the line $t-s = 2n$~~

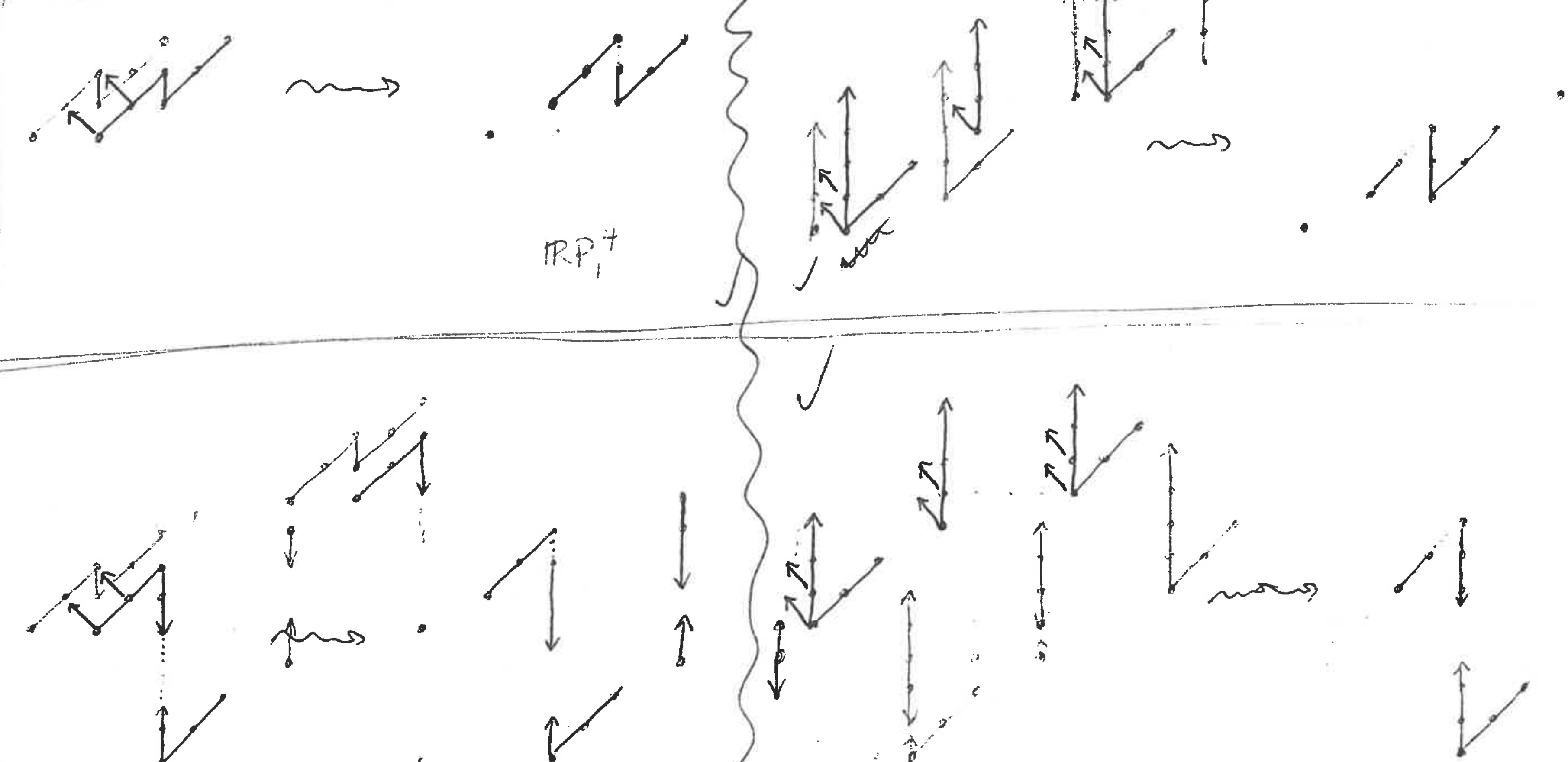
shoved to the right $2n$, ~~and~~ above the line $s = c$.

B $\Sigma^{2n} M(n) \rightarrowtail M(2n) \twoheadrightarrow M(2n-1)$

A $\Sigma M_1(2n-2) \rightarrowtail M(2n) \twoheadrightarrow M_1(2n)$

n odd

n even



Thus all multiplications in $\mathrm{Ext}_{A_1}(M(2n), \mathbb{Z}/2)$ are solved

In the spectral sequences associated to A it's clear that we get the associated diagrams. To prove this is the case for the ~~exact sequences~~ & spectral sequences associated to B we compute to see how the line $s = \frac{1}{2}(t-s)$ is translated in the 2 cases $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$.

If $n \equiv 1 \pmod 4$ then $M(n) \cong M_1(n-1) \otimes M(1)$ as left A_1-module and hence $\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{2n} M(n), \mathbb{Z}/2) \cong \mathrm{Ext}_{A_1}^{s+c,t+c}(\Sigma^{2n} M(1), \mathbb{Z}/2)$ where $c = (n-1) - \alpha(\frac{n-1}{2})$. Note that J is not a factor in the stable decomposition of $M(n)$ since $(n-1) \equiv 0 \pmod 4$. The picture of $\mathrm{Ext}_{A_1}^{s+c,t+c}(\Sigma^{2n} M(1), \mathbb{Z}/2)$ is just that of $\mathrm{Ext}_{A_1}^{s,t}(M(1), \mathbb{Z}/2)$ chopped off at $s = c$ and shoved to the right $2n$. Also $M(2n-1) \cong M_1(2n-2) \otimes M(1)$ as left $A(1)$-modules so

$\mathrm{Ext}_{A_1}^{s,t}(M(2n-1), \mathbb{Z}/2) \cong \mathrm{Ext}_{A_1}^{s+b,t+c}(M(1), \mathbb{Z}/2)$ where

$b = 2(n-1) - \alpha(n-1)$. Thus the picture of $\mathrm{Ext}_{A_1}^{s,t}(M(2n-1), \mathbb{Z}/2)$ is just that of $\mathrm{Ext}_{A_1}^{s,t}(M(1), \mathbb{Z}/2)$ chopped off at the line $s = b$. We can now check that the exact sequence B gives the picture we've indicated. In the case for $\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{2n} M(n), \mathbb{Z}/2)$ the line $s = \frac{1}{2}(t-s)$ is translated to

$s + c = \frac{1}{2}((t-s) - 2n)$ so

$s + (n-1) - \alpha(\frac{n-1}{2}) = \frac{1}{2}(t-s) - n$. In the other case

$s + b = \frac{1}{2}(t-s)$ so

$s + 2n - 2 - \alpha(n-1) = \frac{1}{2}(t-s)$. The fact that if $n \equiv 1 \pmod 4$ then $\alpha(n-1) = \alpha(\frac{n-1}{2})$ so we get

$s = \frac{1}{2}(t-s+2) - 2n + \alpha$ in the first case and
$s = \frac{1}{2}(t-s+4) - 2n + \alpha$ in the second case where
$\alpha = \alpha(n-1) = \alpha(\frac{n-1}{2})$. This proves the desired result.

In the case of $n \equiv 3\,(4)$ we see that
$M(n) \underset{s}{\cong} \Sigma^{-c} I^{c} J$ where $c = (n-1) - \alpha(\frac{n-1}{2})$. Therefore

$$\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{2n} M(n), \mathbb{Z}/2) \cong \mathrm{Ext}_{A_1}^{s+c,t+c}(\Sigma^{2n} M(1) \otimes J, \mathbb{Z}/2)$$

This is $\mathrm{Ext}_{A_1}^{s,t}(M(1), \mathbb{Z}/2)$ moved to the right $2n+4$ units and chopped off at $s = c-2$. Thus the line $s = \frac{1}{2}(t-s)$ has been translated to $s + c - 2 = \frac{1}{2}(t-s-2n-4)$
or $s = \frac{1}{2}(t-s) - 2n + 1 - \alpha$ where $\alpha = \alpha(\frac{n-1}{2}) = \alpha(n-1)$. The picture for $\mathrm{Ext}_{A_1}^{s,t}(M(2n-1), \mathbb{Z}/2)$ is the same as before so the line $s = \frac{1}{2}(t-s)$ in it is translated to $s = \frac{1}{2}((t-s)+4) - 2n + \alpha$. Thus we get the claimed ~~desired~~ result.

If n is even then the spectral sequence associated to the short exact sequence A is clear. To show that we get the picture indicated for B write $n = 2^k \cdot m$ where m is relatively prime to 2. ~~Then the first vertical tower in $M(2n)$ is in the same $t-s$ grading as $\Sigma^{2n}(M(2m))$ which is in the same~~ ~~Inductively it can be seen that this is in the same $t-s$ grading as $\Sigma^{r}(M(m))$ where~~
~~$r = 2n + n + \frac{n}{2} + \dots + \frac{n}{2^{k}} + m = m\left(\sum_{j=0}^{k+1} 2^{j}\right)$~~

Now observe that if we order the vertical towers ~~\$~~ appearing in the spectral sequence from left to right, then the first tower in $Ext_{A_1}^{s,t}(M(2n), \mathbb{Z}/2)$ appears in the same $t-s$ degree as the first tower appearing in $Ext_{A_1}^{s,t}$ $(\Sigma^{2n} M(n), \mathbb{Z}/2)$. This indicates that we should try to prove the claim by induction on the number of powers of 2 appearing in the prime factorization of n. Assume the result for $r = 2^l \cdot m$ for $l \le l < k$. Then for $n = 2^k \cdot m$ we have the following 2 exact sequences

$$(i)\quad \Sigma^{2n} M(n) \rightarrowtail M(2n) \longrightarrow M(2n-1)$$

and

$$(ii)\quad \Sigma^{n} M\left(\tfrac{n}{2}\right) \rightarrowtail M(n) \longrightarrow M(n-1).$$

The ~~vertical~~ first vertical tower in Ext for $M(2n)$ is in the same $t-s$ grading as that in Ext for $\Sigma^{2n} M(n)$. But note that $M(2n-1) \underset{s}{\cong} \Sigma^{-d} I^{d} J \otimes M(1)$ where $d = 2(n-1) - \alpha(n-1)$ and $M(n-1) \underset{s}{\cong} \Sigma^{-e} I^{e} J \otimes M(1)$ where ~~$e = (\frac{n-2}{2}) - \alpha(\frac{n-2}{2})$~~

$$e = n-2-\alpha\left(\tfrac{n-2}{2}\right).$$

Note that $\alpha(n-1) = \alpha\left(\tfrac{n-1}{2}\right)$. Thus d and e differ by n. This observation completes a verification of the claim since we've just proved that the picture associated to (i) is one associated to (ii) but shifted to the right $2n$.