On the Cohomology Algebra of Fibre Bundles whose Fibre is Totally Non-homologous to Zero

by

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Throughout this paper we will assume that our coefficient ring is Z_2, though much can be carried out with Z_p coefficients for any prime p. Let $p: E \to B$ be a fibre bundle with fibre F and group G. Let $i: F \to E$ be the inclusion map. If the fibre is totally non-homologous to zero, then $H^*(E) \approx H^*(B) \otimes H^*(F)$ as an $H^*(B)$-module. An $H^*(B)$-basis for $H^*(E)$ can be obtained by choosing inverse images under i^* of a Z_2-basis for $H^*(F)$. This isomorphism, however, does not describe the algebra structure of $H^*(E)$ nor its structure as a module over the Steenrod algebra A. We will attack these problems.[3]

First of all we define some algebraic notions. Let R be a graded commutative left algebra over the Hopf algebra[4] A. Let M be a module over R and over A. We will call M an <u>A-R-module</u> if $\alpha(r \cdot m) = \Sigma \alpha_i'(r) \cdot \alpha_i''(m)$ where $\psi(\alpha) = \Sigma \alpha_i' \otimes \alpha_i'' \in A \otimes A$ (ψ denotes the diagonal map in A). For example, for any pair (X, Y), $H^*(X,Y)$ is an $A-H^*(X)$-module and the cohomology of the total space $H^*(E)$ is an $A-H^*(B)$-module.

We would like to reduce the study of A-R-modules to the study of modules over a single algebra. To this end, we define the <u>semi-tensor product</u> of algebras, $R \odot A$, as follows. As a Z_2-module, $R \odot A = R \otimes A$. The multiplication is defined as follows:

$$(r \otimes \alpha)(s \otimes \beta) = \Sigma\, r \cdot \alpha_i'(s) \otimes \alpha_i''\beta, \quad r, s \in R, \quad \alpha, \beta \in A.$$

If M is an A-R-module, define $(r \otimes \alpha)(m) = r \cdot (\alpha(m))$.

THEOREM 1. With these two definitions, $R \odot A$ becomes an associative, graded algebra with unit and M becomes a left $R \odot A$ module. There is a natural 1-1 correspondence between $A - R$ modules and $R \odot A$-modules.

Note also that A and R are imbedded isomorphically in $R \odot A$ as subalgebras.

An $R \odot A$-module is called unstable if it is unstable as an A-module in the sense of Steenrod and Epstein, p. 27. An $R \odot A$-algebra is an $R \odot A$-module which is an algebra over the Hopf Algebra A (in the sense of Steenrod) and an algebra over the commutative ring R (in the usual sense). An $R \odot A$-algebra is unstable if it is an unstable $R \odot A$-module and $S_q^i(x) = x^2$ if $\dim x = i$. A base point for an $R \odot A$-module M is an $R \odot A$-homomorphism $R \to M$. We now define the free $R \odot A$-algebra generated by an $R \odot A$-module M with base point; this is a slight generalization of the definition by Steenrod and Epstein of $U(X)$ on p. 29 of their book. The free $R \odot A$-algebra generated by M is an $R \odot A$-algebra with base point, $U_R(M)$, and an $R \odot A$-homomorphism $\phi: M \to U_R(M)$ (all homomorphisms preserve the base point if there is one) such that, if $f: M \to \underline{\underline{A}}$ is an $R \odot A$-homomorphism of M into an $R \odot A$-algebra with base point, then there is a unique $R \odot A$-algebra-homomorphism $\bar{f}: U_R(M) \to \underline{\underline{A}}$ such that $f = \bar{f}\phi$. The uniqueness of $U_R(M)$ is proved easily and the proof of existence is similar to that for the case $R = Z_2$ given by Steenrod and Epstein on p. 28.

We now apply these algebraic notions to study the problems mentioned above. Though our theorems are somewhat more general, for ease of exposition we restrict ourselves to the case $F = V_{n,r}$ and $G = O(n)$. Let $\xi = (E, p, B, V_{n,r}, O(n))$ be such a fibre bundle.

Let $p_T: E_T \to B$ be the associated bundle with fibre $T(V_{n,r})$, the cone on $V_{n,r}$. (E_T is the mapping cylinder of p). P_T^* is an isomorphism, so the exact sequence of the pair (E_T, E) may be replaced by the following exact sequence:

$$\cdots \longrightarrow H^*(E_T, E) \xrightarrow{L} H^*(B) \xrightarrow{p^*} H^*(E) \xrightarrow{\delta} H^*(E_T, E) \longrightarrow \cdots,$$

which is an exact sequence of $H^*(B) \odot A$-modules and maps. The universal example for such bundles is the fibration $p: BO(n-r) \to BO(n)$ induced by the inclusion $O(n-r) \subset O(n)$. Here p^* is an epimorphism so the above exact sequence reduces to

$$0 \longrightarrow H^*(BO(n-r)_T, BO(n-r)) \xrightarrow{L} H^*(BO(n)) \longrightarrow H^*(BO(n-r)) \longrightarrow 0.$$

Define $U_i \varepsilon H^i(BO(n-r)_T, BO(n-r))$ by $L(U_i) = w_i$, the ith universal Stiefel-Whitney class, $n - r < i \leq n$. Given ξ, define $U_i(\xi) \varepsilon H^i(E_T, E)$ by $U_i(\xi) = g^*(U_i)$, where $g: (E_T, E) \to (BO(n-r)_T, BO(n-r))$ is induced by the classifying map. The well-known Wu formulae lead to the following formulae:

$$S_g^t(U_i(\xi)) = \sum_{S=0}^{t} \binom{i-t+s-1}{s} w_{t-s}(\xi) \cdot U_{i+s}(\xi).$$

These formula show that the $H^*(B)$-sub-module of $H^*(E_T, E)$ generated by $\{U_i(\xi) \mid n-r<i\leq n\}$ (which we denote by $M(\xi)$) is closed under the action of A and hence is an $H^*(B) \odot A$-submodule.

We now assume that $V_{n,r}$ is totally non-homologous to zero in ξ. Then the above long exact sequence reduces to

$$0 \longrightarrow H^*(B) \xrightarrow{p^*} H^*(E) \xrightarrow{\delta} H^*(E_T, E) \longrightarrow 0.$$

Define $N(\xi) = \delta^{-1}(M(\xi))$. Then we have the following exact sequence of unstable $H^*(B) \odot A$-modules:

$$0 \longrightarrow H^*(B) \xrightarrow{p^*} N(\xi) \xrightarrow{\delta} M(\xi) \longrightarrow 0.$$

We can now state our main theorem:

THEOREM 2. $H^*(E)$ *is isomorphic to* $U_{H^*(B)}(N(\xi))$ *as an* $H^*(B) \odot A$ *algebra; this isomorphism is natural and is induced by the inclusion map* $N(\xi) \to H^*(E)$.

Thus the study of the structure of $H^*(E)$ is reduced to the study of the module extension $0 \longrightarrow H^*(B) \xrightarrow{p^*} N(\xi) \xrightarrow{\delta} M(\xi) \longrightarrow 0$. It can be shown that $M(\xi)$ is a free $H^*(B)$-module on the basis $\{U_i(\xi) \mid n-r < i \leq n\}$; thus the extension splits over $H^*(B)$. One may apply relative homological algebra for the pair of rings $(H^*(B) \odot A, H^*(B))$ to this situation and obtain a group $\mathrm{Ext}^1_{H^*(B)\odot A, H^*(B)}(M(\xi), H^*(B))$ either by means of relatively projective resolutions or 1-cocycles. The values of the 1-cocycles give rise to secondary characteristic classes. This part of the problem needs further study.

In order to study secondary characteristic classes more thoroughly, we need to define and study the universal examples. Define a map $\pi: K \to BO(n)$ to be a *universal example* for the category of fibre bundles with fibre $V_{n,r}$ and group $O(n)$ whose fibre is totally non-homologous to zero if (1) every classifying map $g: B \to BO(n)$ for a bundle in this category can be factored through K, i.e. $g = \pi\bar{g}$, where $\bar{g}: B \to K$ and if (2) the $V_{n,r}$ bundle over K induced by π is in the category. A universal example is called *minimal* if π is a principal fibration with fibre $\mathcal{F}$ and if for every map $f: K \to K$ such that $\pi f = \pi$, then the induced map $f^*: \pi(K; \mathcal{F}) \to \pi(K; \mathcal{F})$ is an epimorphism (here the notation $\pi(X, Y)$ denotes the set of all homotopy classes of maps $X \to Y$).

The existence of universal examples is easily proved: construct a fibre space over $BO(n)$ with fibre a product of $K(Z_2, m)$ - spaces that "kills off" the universal Stiefel-Whitney classes w_i, $n - r < i \leq n$. The proof of the existence of a minimal universal example is a little more complicated.

THEOREM 3. Let $\pi: K \to BO(n)$ be a minimal universal example and let $\pi': K' \to BO(n)$ be any other universal example. Then there are maps $a: K \to K'$ and $a': K' \to K$ such that $\pi' a = \pi$, $\pi a' = \pi'$, and $a' a \simeq$ identity.

It follows that the minimal universal example is unique up to homotopy, and that any secondary characteristic classes that can be defined using an arbitrary universal example can also be defined using a minimal universal example. The cohomology $H^*(K)$ of a minimal universal example $\pi: K \to BO(n)$ can be computed explicitly by means of a general theorem, which we will now state without proof.

This theorem is concerned with principal fibre spaces $p: E \to B$ with fibre G a product of $K(\pi, m)$'s. Recall that $H^*(G) = U(Y)$, the free algebra over A generated by a certain A-module $Y \subset H^*(G)$. Let $i: G \to E$ denote the inclusion map.

THEOREM 4. Assume this principal fibre space $G \xrightarrow{i} E \xrightarrow{p} B$ satisfies the following four additional conditions:

(a) $H^*(B) = R \otimes Z_2[v_1, \ldots, v_r]$, the tensor product of a sub-algebra R and a polynomial algebra.

(b) Kernel p^* = the ideal generated by $v_1, \ldots, v_r$.

(c) In the spectral sequence of the fibre space,

$$E_\infty^{p,q} = E_\infty^{p,o} \otimes E_\infty^{o,q}.$$

(d) <u>Image</u> $i^* = U(Y')$, <u>where</u> $Y' = Y \cap$ <u>Image</u> i^*.
<u>Then there exists an</u> $R \odot A$ - <u>module</u> $N \subset H^*(E)$ <u>such that</u> $H^*(E) = U_R(N)$.

It is not difficult to see that a minimal universal example $\pi: K \to BO(n)$ satisfies all the hypotheses of this theorem. In certain cases they are also satisfied by 2-stage Postnikov systems. It would be interesting to find other conditions on a fibre space that imply $H^*(E) = U_R(N)$ for some $R \odot A$ - module N.

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[3]For previous work on this problem, see various papers by G. Hirsch; also, a recent paper in the Annals of Mathematics by F. Peterson and N. Stein.

[4]From here on, we assume that the reader has some familiarity with the material contained, for example, in Chapters I and II of the Annals of Mathematics Study No. 50 entitled "Cohomology Operations" by Steenrod and Epstein. This will be our standard reference in the rest of the paper.