SOME PERSONAL MEMORIES OF THE EARLY YEARS OF TOPOLOGY

by

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Some of the participants of this colloquium* have kindly pointed out that their research, which led to new and beautiful results have been motivated to some extend by my papers. I have been asked to speak about these papers. This lecture, improvised, will be a "review" but I shall not speak as an "historian" who tries to be "objective" and "complete" but as a "reporter" who himself played a role and is therefore incomplete and subjective. Whenever I tell of my own contributions, my purpose is not mainly to discuss the results, but rather to point out the influences under which these works have been created – influences, which came partly from persons, partly from the mathematical climate of the times.

0. My first contact with topology took place in Summer 1917, in a lecture on "Set Theory" by Erhard Schmidt at the University of Breslau: he proved the theorem of the "Invariance of the Dimension" which can be formulated as follows. In euclidean space R^n there exists no topological image of an N-dimensional cube with $N>n$. The theorem had been proved in 1911 by L.E.J. Brouwer using the notion of "degree of a map"-- the difference between the numbers of positive and of negative coverings of a point under a map -- and Schmidt gave a lecture on this proof.

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* Brussels 1964

Ever since this time I have been fascinated by the notion of the degree of a map. It has decisively influenced important parts of my publications. And when I ask myself now the reasons for this influence I see two main answers: first the penetration and spirit of lectures by Erhard Schmidt and secondly the awakening of my perceptions during a 14 day vacation from military service.

1. In 1920, I followed Erhard Schmidt to the University of Berlin. Among his many stimulating ideas, I mention here only the ones which concern topology directly:

1.) as in Breslau in 1917 some of the works of Brouwer

2.) Schmidt's own proof of the Jordan curve theorem (Preuss. Akad. d. Wiss. 1923), a proof which cannot be generalized, but is very elegant

3.) and finally; Schmidt asked his assistant G. Feigl to give a series of lectures on the great works of Poincaré on "Analysis Situs".

Schmidt encouraged me to work in topology. I read some papers of Brouwer -- this was bitter work -- and the article by Hadamard (in the book on Theory of Functions by J. Tannery). The result of my studies was a group of publications: "On the Integral Curvature of closed hypersurfaces", "Classes of mappings of n-dimensional manifolds", and "Vectorfields on n-dimensional manifolds". The results of these works are contained in the lecture "Mappings of closed manifolds on spheres in n-dimensions", given at the annual meeting of the German Mathematical Society in Danzig, (September 1925) (printed in the "Selecta" for my 70th birthday (Springer 1964) pp.1-4). I would like to make the following remark about the work on vector fields.

The main theorem states: The sum of the indices of the singularities of a vector field in a closed oriented manifold, is an invariant of the manifold, the Euler characteristic. This theorem is already found without the concept of characteristic, in the note of Hadamard mentioned above -- but without proof. There was really at that time (1910) no proof, because of a misunderstanding between Brouwer and Hadamard. The first proof was given by Lefschetz (1923). My own proof (1925) relies on a ponderous induction: I warn people who are interested. The easiest proof uses my generalization of the Euler-Poincaré formula (Göttingen Nachrichten 1928).

The three works mentioned above on Integral Curvature, Classes of mappings, and Vectorfields were a part of my thesis and of my "Habilitation". But before I obtained my "Habilitation" as Privatdozent in Berlin, I went to Göttingen for one year, in the Autumn of 1925.

2. My most important experience in Göttingen was my meeting with Paul Alexandroff. From this meeting grew a deep friendship: not only topology and not only mathematics were discussed: this was a very happy and a very merry time, too. Our good times were not restricted to Göttingen, but over our common travels.

Alexandroff was, when I met him, already an important man in general topology; but he was just about to introduce the notion of "Nerve" which would end the separation between general and algebraic topology.

Let $U = (F_1, F_2, \ldots, F_n)$ be a covering of a compact metric space R by closed sets F_i: to each set F_i one associates a point p_i of an n-dimensional euclidean space such that $p_1, \ldots, p_n$ are in general position. One constructs the simplex $(p_{i_0}, p_{i_1}, \ldots, p_{i_r})$ if and only if $F_{i_0} \cap F_{i_1} \cap \ldots \cap F_{i_r} \neq \phi$. The constructed complex N(R) is a nerve of U. The complexes N(R), constructed as nerves of finer coverings of R, give an "abstract approximation" of R. The algebraic notions (dimension, Betti numbers) which are well defined on the complexes N(R), can be extended to R by this approximation. The classical example is the covering theorem of Lebesgue: The space R has dimension n if and only if it can be abstractly approximated arbitrarily well through n-dimensional complexes, but not through complexes of smaller dimension.

The notion of "nerve" has many other applications: it was the first successful attempt to introduce algebraic concepts in general topology - and displeased supporters of the "purity of method". The spirit and enthusiasm of Alexadroff were required to bring acceptance to his conviction that in topology algebraic and set-theoretical methods are connected in a natural way. I myself was influenced in a deep and lasting way, by this insight in the role of algebra in topological problems.

3. The center of the algebraic influences in Göttingen was naturally Emmy Noether. She knew Alexandroff for many years and their main fields of research, the abstract algebraic and abstract topological structures, were closely connected. But what we, Alexandroff and I, learned from her was direct and concrete, namely the foundation of homology theory in simplicial complexes. Let X^r be the r-dimensional chain group, ∂ the boundary homomorphism $X^{r+1} \rightarrow X^r$; then $\partial\partial = 0$ as one can easily verify in the case of a single simplex; therefore the image ∂X^{r+1} is contained in the kernel Z^r of $\partial : X^r \rightarrow X^{r+1}$. The factor groups $H^r = Z^r / \partial X^{r+1}$ is the r-th homology groups.

This basis-free foundation of homology, ie. without the use of Poincaré's incidence matrices, was entirely new (I don't know whether the notion of homology groups had explicitly appeared before). I used it myself for the first time in my note "A generalization of the Euler-Poincaré formula" (Gottinger Nachrichten 1928).

4. During the academic year 1927-28. Alexandroff and I held a Rockefeller Fellowship in Princeton. At that time Princeton was anidyllic small university town. The famous "Institute" and even Fine Hall did not exist. Alexandroff and I were the only regular foreign guests in the "French Restaurant" (On Sundays, we got prohibited wine served in a coffee cup.) At the university, there were lectures by O. Veblen, S. Lefschetz, and J. W. Alexander and we had interesting discussions with each of them. For us the most important was Lefschetz because he was Alexandroff's ally in the fight for application of algebraic

methods in set-theoretical topology and because my work on fixpoints followed his pioneering ideas. Moreover, Lefschetz and I discussed the following question.

The Lefschetz product method in the study of mappings f: X→Y, where X and Y are n-dimensional manifolds, consists in the consideration of the graph $\Delta f = \{(x, fx) \mid x \varepsilon X\}$; one assumes here that Δf is an n-dimensional cycle but f is not necessarily one-valued. My question was: "Which properties characterize graphs Δf associated to one-valued maps?" The Lefschetz product method does not give a direct answer; but one can give, within the frame work of this theory, necessary conditions for the graph Δf, which are quite satisfactory. A crucial role is played by the "inverse homomorphism" ϕ induced by f, of the intersection ring R(Y) into R(X). This allows to prove: "If the degree of f is different of zero, then f maps R(X) onto R(Y)" — (with rational coefficients). Other consequences are obtained in my paper "On the Algebra of Mappings of Manifolds" [Crelles Journal 163 (1930)]. I applied the inverse homomorphism in other problems (X and Y might not have the same dimension).

5. 1935 was for several reasons an important year in the development of topology. In September, the "First International Conference on Topology" took place in Moscow. The lectures given independently at this confer[illegible] by J.W. Alexander, I. Gordon, and A.N. Kolmogoroff can be considered as the beginning of Cohomology - Theory, but I should say that Lefschetz's "pseudo-cycles" anticipated this idea.

The fact which most impressed me, and probably many other topologist, was not the existence of the cohomology groups - which are the character groups of the homology groups - but the multiplicative structure, i.e., the cohomology rings, defined for arbitrary complexes and more general spaces. The cohomology-ring generalizes the intersection-ring of manifolds: we thought that such a product structure was made possible only by the local euclidean structure.

6. My own contribution at the Moscow conference was a report on the thesis of E. Stiefel "Fields of Directions and Parallelisability of n-dimensional Manifolds" which had not yet been published [Comm. Math. Helvet. 8(1935-36)]. (I mention here that I had been Professor at the Swiss Institute of Technology in Zurich since 1931.) The problem which I had proposed to E. Stiefel was the generalization of the following: one knows (see 1.) that, in a compact orientable, differentiable manifold M^n, there exists a continuous field of directions if and only if the Euler characteristic of M^n is zero. For which M^n does there exist m fields of directions, continuous and linear independent, at every point; $(m = 1,2,\ldots,n-1,n)$?

The case $m = 1$, is settled by the characteristic; on the other hand, if n-1 independent fields can be constructed, then because of the orientation of M^n one can add one more field. The interesting cases are $1<m<n$ and the smallest values of m

and are $m = 3$, $m = 2$. Since the best thing to do is always to discuss the simplest case of a general problem, I gave E. Stiefel the following advice: "Find in which M^3 there are 2 (and therefore 3) independent fields and in which M^3 2 independent fields are impossible".

This was well meant but in fact bad advice. As hard as Stiefel tried to construct with sophisticated methods (Heegard diagram, knot-spaces, etc.) 3-dimensional manifolds, they always turned out to be parallelisable. At last he told me the general answer: he had constructed the theory of "Characteristic homology classes". For each m, there exists a characteristic class F^{m-1} and $F^{m-1} = 0$ is a necessary and sufficient condition for the existence of m fields with singularities contained in an $(m-2)$-dimensional complex. Furthermore $F^o = F^1 = \dots = F^{m-1} = 0$ is a necessary condition for the existence of m fields without singularities (see the paper of Stiefel mentioned above, for further properties of the classes F^i and especially for various coefficients). Stiefel proved that $F^1 = 0$ for all M^3: each M^3 is therefore parallelisable. The surprising fact is that $n = 3$ (and $n = 1$) are the only dimensions where each M^n is parallelisable.

After my lecture in Moscow, H. Whitney pointed out that an important part of these results were contained on this paper "Sphere spaces" [Proc. Nat. Acad. Sci. 21 (1935)]: he was right, but Stiefel and I did not know this paper. Anyway it is entirely justified that the "characteristic" classes are now called "Stiefel-Whitney classes". I find that in Whitney's work everything is more general than in Stiefel's whereas

Stiefel's interest is directed on special problems which do not appear in Whitney.

7. I want to discuss now such a special problem. Stiefel studied in a latter publication [Comm. Math. Helv. 13 (1940-41)] fields of directions in the real projective spaces P^n and proved: If there exist n independent vector fields on P^n, then $n+1 = 2^k$. A corollary is: The degree of an hypercomplex system over the real numbers, where the associativity is not assumed but without divisors of zero, is equal to 2^k.

This theorem which follows from Stiefel's work on vector fields on P^n, can be proved easily by using the following result: If there is a map $F: P^n \times P^n \to P^n$ such that, (using homology with coefficients in Z_2) F(Point × Line)~F(Line × Point)~Line (*) then n+1 is a power of 2.

This theorem can be proved by using the Inverse Homomorphism (see section 4 in this paper and Comm. Math. Helv. 13). We will return later to this subject. In the results we mentioned above, one can replace 2^k by 2,4 or 8 (M.Kervaire Proc. Nat.Acad. Sc. USA 44 (1958) R. Bott and J. Milnor (Bull. Amer. Math. Soc. 64)).

8. In the same year, 1935, the second International Congress on Topology took place in Geneva. Elie Cartan was the president of this congress and gave a talk "La Topologie des Groupes de Lie" (Actualités scientifiques et industrielles 358, Paris 1936).

A part of it was a report on the following theorem, which gives the homology (with rational or real coefficients) of all Lie groups contained in the four Killing - Cartan classes.

Theorem. The intersection ring is isomorphic to the intersection ring of the cartesian product

$$S^{m_1} \times S^{m_2} \times S^{m_3} \times \dots \quad .$$

The Poincaré polynomial is therefore

$$(1+t^{m_1})(1+t^{m_2})\dots \quad ;$$

The exponents m_i are odd and are given explicitly: for example, in the class of the unimodular unitary groups: 3, 7, ... $2\ell+1$.

The theorem had been proved, with entirely different methods by L. Pontrjagin, R. Brauer, and C. Ehresmann [C.R. Paris 200, 201 (1935) and 208 (1939)]. But the homology rings of the five exceptional groups were not included in this pattern. And Cartan ended his talk with the following words: " ... even if one succeeds in the computation of the homology rings of the exceptional Lie groups, a general answer explaining the particular structure of the Poincaré polynomials of all simple compact, Lie groups, remains to be found".

In 1939, I answered this question of Cartan; I used the discussion of mappings $F: P^n \times P^n \to P^n$ (see 7). F can be considered as a multiplication in P^n and the relations (*) can be interpreted as the existence of an element one for this multiplication. I tried therefore to use the inverse homomorphism to study the homology ring of a Lie group. This attempt was successful; the axiom of associativity played no role in the proof and I obtained:

Theorem. Let M be a compact manifold and let a continuous multiplication (not necessarily associative, but with homotopy unit) be defined on M. The homology ring of M is isomorphic to the ring of a product of spheres S^{m_i} (m_i odd).

The assumption of the existence of a homotopy unit may be weakened (Annals. of Math. 42(1941). The properties of the homology rings of such spaces have influenced Homological Algebra.

9. Let us turn back to 1935 to mention the most important result. W. Hurewicz introduced the concept of Homotopy-groups which deeply changed the structure of Topology. I repeat here what I have said very often: many mathematicians confessed with regard to this theory. "I could not make such a discovery — it would look too simple to me".

But I want to add that the exact definition of homotopy groups had been given -- without any application -- at the International Congress in Zurich (1932) (Verh. d. Internat. Math. Kongresses Zurich 1932, II. Bd p. 203).

10. The discovery of homotopy groups, the theory of fiber spaces introduced by Whitney and Stiefel, the progress in the study of

group-manifolds and their generalizations, led in the years following 1940 to many interesting works. I mention some of them here.

W. Gysin studied the homology of fibrations of manifolds by spheres and obtained many remarkable results. [Thesis C.M.H. 14 (1941)].

B. Eckmann discovered in his first papers [C.M.H. 14 (1941)] a large number of homotopy groups of fiber- and group-spaces. Moreover he proved that on spheres S^{4k+1} $(k = 1,2,...)$ two continuous vector fields must be linear dependent at some points. This result was a generalization of Stiefel's results and had been proved at the same time by G.W. Whitehead [Ann. of Math. 43(1942)]. The general problem of finding the maximal number of linearly independent vector on S^n has been solved by J.F. Adams [Ann. of Math. 75(1962)].

H. Samelson discussed in his thesis [Ann. of Math. 42(1941)] the connection between the "Pontrjagin product" (see 11) and my theorem (see 8) on product of spheres. He obtained interesting relations between the intersection ring of a Lie group G and the ring of its quotient space G/H.

More information of these papers can be found in the above mentioned "Selecta" (p.175 ff.)

I would like to quote another work, although its author did not belong to the "Zurich-circle". L.Pontrjagin proved

[(C.M.H. 13(1941)] that the group manifolds which have, according to 8, the homology ring of products of spheres, are not homeomorphic to this product. For example, the group A_2 of 3x3 unimodular unitary matrices (homology ring isomorphic to the ring of $\pi = S^3 \times S^5$) is not homeomorphic to π since $\pi_4(A_2) = 0$ and $\pi_4(S^3 \times S^5)$ is of order 2. In Pontrjagin's proof, the concept of homotopy group is not used; the fundamental groups of both manifolds are trivial.

11. The above mentioned "Pontrjagin product" in a group-manifold G is defined in the following way. Let U and V be cycles in G; let $u \varepsilon U$ and $v \varepsilon V$. The product $P = U.V$ is given by the set of all u.v: $\dim(P) = \dim(U)+\dim(V)$ and the product of two closed curves is a closed surface. At about the time I was interested in the Pontrjagin product, in connection with the work of H. Samelson, A. Preissmann met the following situation. In a closed Riemannian manifold with negative curvature (therefore no conjugate points on geodesics) let U and V be 2 closed geodesics which commute as elements of the fundamental group. Then using a kind of multiplication analogous to the Pontrjagin product, U and V will "span" a torus-like surface. I tried to bring the two multiplications under a common structure; I remarked at last that the group-manifolds and the special Riemannian spaces played no particular role. The underlying principle is quite general and describes the connection between the fundamental group and the second homology group of a complex (or of a more general space). This led to the isomorphism

$$H_2/S_2 = R\cap(F.F)/(R.F)$$

with the following notations: H_2 is the second homology group of the complex K, S_2 the subgroup of H_2 spanned by images of spheres; F is a free group, R a normal subgroup of F and $F/R = G$ the fundamental group of K. The symbol $(A.B)$ where A and B are subgroups of the same group, is the group generated by the commutators $aba^{-1}b^{-1}$, $a\varepsilon b$, $b\varepsilon B$. The right side of the formula is a "lower limit" for the homology groups H_2 which are compatible with the fundamental group G. (One sees easily that the right side depends only on G, not on the representation F/R). The right side can be interpreted as the group of "commutator relations modulo the trivial commutator relation". This, and much more, can be found in my paper "Fundamental groups and second Betti groups" (C.M.H. 14(1941-42) or Selecta p. 186 ff). I published three more papers, on this subject, the last in 1945.

I want to conclude my talk. The prehistoric era of today's topology ended with this period, the history began. A young generation appeared which worked on our old ideas and much more: this generation could solve many of our old problems with its new ideas and changed the image of Topology in an unexpected way.