122% × 150%

Half exact functors and cohomology

by Albrecht Dold

The theory of half exact functors generalizes and simplifies the theory of general cohomology. Further generalizations are possible and useful (see section 8) although it is not clear just how far they should go.

1. Half exact functors of spaces

Let $\underline{W}$ be the category whose objects are finite CW-complexes and whose morphisms are homotopy classes of maps. Let $\underline{A}$ be an abelian category. A (contravariant) functor $t: \underline{W} \to \underline{A}$ is called <u>half exact</u> if the sequence

(1.1) $\qquad t(X/A) \to tX \to tA$

is exact for every $X \in \underline{W}$ and subcomplex $A \subset X$.

The objects $t(S^i)$, where S^i denotes the i-sphere $i = 0,1,2,\ldots$ are called the <u>coefficients</u> of t .

If we apply 1.1 to X = A = a point we find t(a point) = 0 . If we apply it to $X = A \vee B$ (wedge) we find $t(A \vee B) = tA \oplus tB$; more generally

(1.2) $$t\left(\bigvee_{j=1}^{r} A_j\right) = \prod_{j=1}^{r} t(A_j)$$

I.e., t takes sums into products. There is, of course, no difference between (finite) sums and products in $\underline{A}$; we chose to write the product sign in 1.2 because this is the adequate form if one wants to deal with infinite complexes. Indeed, everything that follows holds for half exact functors on infinite CW-complexes (of finite dimension) provided the "infinite analogon" of 1.2 is satisfied.

If $X \in \underline{W}$ is arbitrary and $A \subset X$ contains exactly one point in every component of X then X/A is connected and 1.1, 1.2 imply

(1.3) $tX = t(X/A) \oplus tA$, a natural splitting.

Moreover, $A = \bigvee_j S^0$, $tA = \prod_j t(S^0)$. We shall therefore look at connected CW-complexes only, having a single 0-cell, and shall consider it as obvious from 1.3 how to deal with the general case.

Another simplifying remark concerns base points. One could consider half exact functors which are defined on homotopy classes with base points. However, it turns out that these are the same as those defined on free homotopy classes, the reason being that we assume an additive domaine category. For set valued functors (compare [2]) the situation would be quite different.

Examples (1.4) Let H be a homotopy-associative, homotopy-commutative H-space (e.g. $H = \Omega^2 Y$). Then $tX = [X,H]/\pi_0 H$ (homotopy classes modulo those of constant maps) is half exact. If $\underline{A}$ is the category of abelian groups then every half exact t with countable coefficients is of this form ([2]).

(1.5) Let L be a fixed space (or a spectrum) and put $tX = \{X,L\} = \lim_i [\Sigma^i X, \Sigma^i L]$ = group of stable homotopy classes.

(1.6) Put tX = group of stable vector bundles (real or complex) over X , i.e., $t = \tilde{K}_R$ resp. $\tilde{K}_C$.

(1.7) If h is a cohomology theory on pairs (X,A) with values in $\underline{A}$ (satisfying the Eilenberg-Steenrod axioms except possibly the dimension axiom) then $tX = \mathrm{Coker}\,[h^q(\text{point}) \to h^q X] = h^q X$ = reduced q-th cohomology of X is half exact. Examples can be found in [1], [4] et al.

(1.8) If τ: $\underline{A} \to \underline{A}'$ is an exact functor or σ: $\underline{W} \to \underline{W}$ a functor which takes cofibrations into cofibrations then composition τt resp. $t\sigma$ with every half exact t is again half exact. Examples for σ are: the suspension functor Σ, the $\#$-product with a fixed space, the passage to the n-dual in the sense of Spanier-Whitehead (the last works in the stable category only).

2. Comparing half exact functors

2.1 Proposition. *Let $\rho: t \to t'$ be a natural transformation of half exact functors.* If $\rho(S^i): t(S^i) \cong t'(S^i)$ *is an equivalence for all* $i \leq n$ *then* $\rho: tX \cong t'X$ *for all* X *of dimension* $\leq n$.

Proof. The Puppe sequence [5]

$$A \xrightarrow{f} B \to Cf \to \Sigma A \to \Sigma B \to \dots$$

is defined for every continuous map f, and every term is (up to homotopy equivalence) the cofibre of the preceeding map. Applying any half exact t therefore gives an exact sequence

(2.2) $\quad tA \leftarrow tB \leftarrow tCf \leftarrow t\Sigma A \leftarrow t\Sigma B \leftarrow .$

Every CW-complex is obtained by successively attaching wedges of cells; let $\nu(X)$ denote the number of such operations which are required to construct X. For instance, $\nu(X) \leq 1$ if and only if X is a wedge of spheres ; further $\nu(\Sigma X) \leq \nu(X)$. The proposition is proved by induction on ν. We can then assume $X = Cf$ where $f: A \to B$ and $\nu(A) = 1 = \nu(\Sigma A)$, $\nu(B) < \nu(X)$ hence $\nu(\Sigma B) < \nu(X)$. We apply ρ to the exact sequence 2.2 and get $tX \cong tX'$ by the five lemma, qed.

Application. A contravariant functor $t\colon \underline{W} \to \underline{A}$ is called stable if there is a natural factorization

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\ t\ } & \mathrm{Hom}(tY,tX) \\ {\scriptstyle s}\big\downarrow & \nearrow_{T} & \\ \{X,Y\} & & \end{array}$$

where s is the passage to stable homotopy classes (s.1.5); the examples 1.5, 1.6, 1.7 are stable. If t is half exact then T is a homomorphism of abelian groups (see 3.3), and is therefore adjoint to a morphism

(2.3) $\quad e(X,Y,t)\colon \{X,Y\}\otimes tY \to tX.$

In particular we have a natural transformation

(2.4) $\quad E(X,t) = \left\{ e(X,S^i,t) \right\} \; : \; \bigoplus_i \{X,S^i\}\otimes tS^i \to tX ,$

the "Hurewicz-map".

2.5 Proposition. If $\otimes tS^i$

(a) is an exact functor (i.e. tS^i is flat),

(b) kills finite groups

then the morphism 2.4 is an isomorphism.

Proof. (a) implies by 1.8 that $u(X) = \bigoplus_i \{X,S^i\}\otimes tS^i$ is half exact. It therefore suffices to show that $E(S^j,t)$ is isomorphic for all j. If $j \neq i$ then $\{S^j,S^i\}$ is finite, hence $\{S^j,S^i\}\otimes tS^i = 0$ by (b), hence $u(S^j) = \{S^j,S^j\}\otimes tS^j \cong tS^j$, qed.

More generally we have

2.6* Proposition. Assume t is as in 2.5 and t' is an arbitrary half exact (additive suffices) stable functor. If $\rho^i\colon tS^i \to t'S^i$ is a

(* A more elaborate argument shows that 2.6 holds also for non-stable t,t'.)

sequence of morphisms then there is a unique natural transformation $\rho: t \to t'$ such that $\rho(S^i) = \rho^i$.

Proof. ρ is the unique filler of the following diagram

$$\begin{array}{ccc}
\bigoplus_j \{X, S^j\} \otimes tS^j & \xrightarrow{\bigoplus id \otimes \rho^j} & \bigoplus_j \{X, S^j\} \otimes t'S^j \\
\cong \downarrow E & & \downarrow E' \\
tX & \xrightarrow{\quad \rho \quad} & t'X
\end{array}$$

, qed.

2.7 Example If $tX = \hat{K}_C X \otimes Q$ then $tS^j = Q$ for even $j > 0$, and is zero otherwise $= H^{even}(S^j, Q)$ where $H^{even} = \bigoplus_\mu H^{2\mu}$. There is a unique $\rho: t \to H^{even}(-, Q)$, the Chern-character, which extends the identity on spheres, and it is an equivalence by 2.1.

3. Lemmas on homotopy groups

3.1 Lemma. Assume a diagram

$$\begin{array}{ccccccc}
S^n & \xrightarrow{f} & B & \to & Cf = B \cup e^{n-1} & \to & Cf/B = S^{n+1} = \Sigma S^n \\
\downarrow g' & & \downarrow g|B & & \downarrow g & & \downarrow \bar{g} \simeq \Sigma g' \\
\bigvee_j S^n & \xrightarrow{f'} & B' & \to & Cf' & \longrightarrow & Cf'/B' = \bigvee_j S^{n+1} = \Sigma(\bigvee_j S^n)
\end{array} \qquad (3.2)$$

$, n > 0$

is given where $g: Cf \to Cf'$ is such $g(B) \subset B'$, and $\bar{g}$ is the induced map on quotients; the 2nd and 3rd square are then commutative. g' has been so chosen that $\Sigma g' \simeq \bar{g}$.

Conclusion: The two elements $[f'g']$, $[(g|B)f] \in \pi_n B'$ differ only by a sum of elements of the form $x - \gamma x$ where $x \in \pi_n B'$, $\gamma \in \pi_1 B'$.

i.e., the first square in 3.2 may not be commutative (it isn't, in general) but the defect is not too bad. The proof relies on the fact that the kernel of the Hurewicz map $\pi_{n+1}(Cf', B') \to H_{n+1}(Cf', B')$ is generated by elements of the form $x - \gamma x$ with $x \in \pi_{n+1}(Cf', B')$ and $\gamma \in \pi_1 B'$.

3.3 *Lemma. If* $t \colon \underline{W} \to \underline{A}$ *is half exact then the map*

(3.4) $\quad \pi_n X \to \operatorname{Hom}(tX, tS^n), \quad [f] \rightsquigarrow tf$

is a homomorphism of abelian groups. More generally, if Y *is an H'-space then*

(3.4') $\quad [Y, X] \to \operatorname{Hom}(tX, tY), \quad [f] \rightsquigarrow tf$

is a homomorphism.

This holds because addition on both sides of 3.4' is based on the diagram

$$Y \to Y \vee Y \to X \vee X \to X.$$

3.5 *Corollary. If* $X = S^n$, *and* $f \colon S^n \to S^n$ *has degree* r *then* $tf \colon tS^n \to tS^n$ *is multiplication by* r.

3.6 *Corollary. Under the assumptions of lemma 3.1 the diagram*

$$\begin{array}{ccc} tS^n & \longleftarrow & tB \\ \uparrow & & \uparrow \\ t(\bigvee_j S^n) & \longleftarrow & tB' \end{array}$$

is commutative, i.e., $t(g')t(f') = t(f)t(g'B)$.

This holds because x and γx are *freely* homotopic, hence $t(x - \gamma x) = 0$.

3.7 *Corollary. Let* X *be a CW-complex,* X^n *its n-skeleton,* $\varphi \colon \bigvee_j S^n \to X^n$ *an attaching map for the (n+1)-cells* (hence $C\varphi = X^{n+1}$). *Then* $c(X) = t(\bigvee_j S^n)$ *is a functor on cellular maps, and*

$t\varphi$: $t(X^n) \to t(\bigvee_j S^n)$ _is a natural transformation. I.e., although the attaching map_ φ _is not unique,_ $t\varphi$ _is._

This follows from lemma 3.1 and Corollary 3.6.

4. Cochains $C(X,tS^n)$, cohomology $H^*(X,tS^n)$

A slight generalization of a result of Eilenberg and Watts shows:
If $K \in \underline{A}$ then there is a unique contravariant functor from finitely generated abelian groups to $\underline{A}$ which (a) is left exact, (b) takes Z into K. It is denoted by $\hom(-,K)$ (P. Freyd calls it the "symbolic hom-functor"). This extends to arbitrary abelian groups provided one requires $\hom(\bigoplus G_\lambda, K) \cong \prod_\lambda \hom(G_\lambda, K)$ whenever all $G_\lambda \cong Z$. It is in fact obvious how to construct $\hom(L,K)$ from a free resolution of L.

If now

$$C: \cdots \leftarrow C_{q-1} \leftarrow C_q \leftarrow C_{q+1} \leftarrow \cdots$$

is a chain complex of abelian groups then

$$\hom(C,K): \cdots \to \hom(C_{q-1},K) \to \hom(C_q,K) \to \hom(C_{q+1},K) \to \cdots$$

is a cochain complex in $\underline{A}$ whose homology is denoted by $H^*(C,K)$. Applying this to cellular chains C of a CW-pair (X,A) defines $C^q(X,A;K)$ and $H^q(X,A;K)$.

4.1 _Proposition. Let_ (X,A) _be a CW-pair,_ X^n _the n-skeleton of_ X, $\varphi = \varphi^n_A$: $\bigvee_j S^n \to X^n \cup A$ _an attaching map for the (n+1)-cells of X-A. There are natural (with respect to cellular maps) isomorphisms_

$$(4.2) \qquad C^n(X,A;tS^n) \cong t(X^n \cup A / X^{n-1} \cup A) ,$$

$$(4.3) \qquad C^{n+1}(X,A;tS^n) \cong t(\bigvee_j S^n)$$

which take the coboundary $\delta: C^n \to C^{n+1}$ into the composite

$$t(X^n\cup A/X^{n-1}\cup A) \to t(X^n\cup A) \xrightarrow{t(\varphi)} t(\bigvee_j S^n) .$$

Proof. 4.2 is clear because $X^n\cup A/X^{n-1}\cup A = \bigvee_k S^n$, with one sphere for each n-cell in X-A. Further, 4.3 follows from Corollary 3.7. The last statement is true because the incidence coefficients of C are the various degrees of the composite

$$\bigvee_j S^n \xrightarrow{\varphi} X^n\cup A \to X^n\cup A/X^{n-1}\cup A = \bigvee_k S^n$$

and because of 3.5.

5. The obstruction sequence

5.1 Proposition. There exists a natural (with respect to (X,A) and t) exact sequence

$$(5.2)\quad t(X^{n+1}\cup A) \to \mathrm{Im}[t(X^n\cup A) \to t(X^{n-1}\cup A)] \xrightarrow{\sigma} H^{n+1}(X,A;tS^n) .$$

In words: the map σ associates with every "element" x of $t(X^{n-1}\cup A)$ which admits an extension to $X^n\ A$ an obstruction $\sigma(x) \in H^{n+1}(X,A:tS^n)$, and $\sigma(x)=0$ if and only if x extends to $X^{n+1}\cup A$.

Proof. If $\varphi: \bigvee_j S^n \to X^n\cup A$ is an attaching map for the (n+1)-cells of X-A then we have an exact sequence

$$t(X^{n+1}\cup A) \to t(X^n\cup A) \to t(\bigvee_j S^n) = C^{n+1}(X,A;tS^n) .$$

Dividing by the image of $K = t(X^n\cup A/X^{n-1}\cup A)$ we get an exact sequence

$$t(X^{n+1}\cup A) \to t(X^n\cup A)/\mathrm{im}(K) \to C^{n+1}(X,A;tS^n)/\mathrm{im}(K)$$

hence by the exactness of $t(X^n\cup A/X^{n-1}\cup A) \to t(X^n\cup A) \to t(X^{n-1}\cup A)$ and by proposition 4.1, the exact sequence

$$t(X^{n+1}\cup A) \to \mathrm{Im}[t(X^n\cup A) \to t(X^{n-1}\cup A)] \xrightarrow{\sigma'} C^{n+1}/B^{n+1}$$

where B denotes coboundaries. It remains to show that σ' maps into cocycles, i.e., that $\delta\sigma' = 0$. This amounts to showing that the composite

$$t(X^n \vee A) \xrightarrow{t(\varphi)} C^{n+1}(X,A;tS^n) \xrightarrow{\delta} C^{n+2}$$

is zero.

Let $Y = \bigvee_k e^{n-2} \xrightarrow{\ell} X$ be the characteristic map for the $(n+2)$-cells of $X-A$, where e^{n+2} is the standard $(n+2)$-cell; note that the n-skeleton Y^n consists of a single point. Naturality gives a commutative diagram

$$\begin{array}{ccccc} t(X^n \vee A) & \xrightarrow{t\varphi} & C^{n+1}(X,A) & \xrightarrow{\delta} & C^{n+2}(X,A) \\ \downarrow \rho & & \downarrow \rho & & \downarrow \rho \, \cong \\ 0 = t(Y^n) & \longrightarrow & C^{n+1}(Y) & \longrightarrow & C^{n+2}(Y) \end{array} ,$$

which proves the assertion.

5.3 <u>Corollary</u> <u>If</u> $H^{n+1}(X,A;tS^n) = 0$ <u>for all n then</u> $tX \to tA \to 0$ <u>is exact</u>

- because all obstructions vanish.

5.4 <u>Corollary</u>. <u>If</u> $H^n(Y,tS^n) = 0$ <u>for all n then</u> $tY = 0$.

<u>Proof</u>. Take $A = Y$, $X = CA$ = cone over A, remark that $H^{n+1}(X,A;tS^n)$ $H^n(Y,tS^n) = 0$, apply 5.3, and use $tX = tCA = 0$.

5.5 <u>Corollary</u>. <u>If</u> $f: A \to X$ <u>is a continuous map such that for all n</u> $f^*: H^i(X,tS^n) \to H^i(A,tS^n)$ <u>is epimorphic for</u> $i = n-1$, <u>isomorphic for</u> $i = n$, <u>and monomorphic for</u> $i = n+1$ <u>then</u> $tf: tX \cong tA$. In particular, this applies if ordinary integral homology is mapped isomorphically because the universal coefficient theorem then implies the assumption.

<u>Proof</u>. We can assume f is an inclusion. The assumptions then mean $H^n(X,A;tS^n) = 0$, $H^{n+1}(X,A;tS^n) = 0$ for all n. Therefore $tX \to tA$ is monomorphic by 5.4 and epimorphic by 5.3 .

5.6 <u>Proposition</u>. <u>If</u> $tS^i = 0$ <u>for</u> $i<n$ <u>then there exists a natural transformation</u> $\rho: tA \to H^n(A,tS^n)$ <u>such that</u> $\rho(S^n) = id$.

<u>Proof</u>. For every X Corollaries 5.5 and 5.3 give

$\mathrm{Im}[t(X^n \cup A) \to t(X^{n-1} \cup A)] = t(X^{n-1} \cup A) \cong tA$, so the obstruction map becomes $\sigma: tA \to H^{n+1}(X,A;tS^n)$. Putting $X = CA$ this is $tA \to H^{n+1}(CA,A;tS^n) \cong H^n(A,tS^n)$, as required.

5.7 Corollary (Hopf-Whitney). If $tS^i = 0$ for $i<n$ and $\dim(A) \leq n$ then

$$tA \cong H^n(A;tS^n) .$$

5.8 Corollary (Uniqueness of half exact functors satisfying a dimension axiom) If $tS^i = 0$ for $i \neq n$ then $tA = H^n(A;tS^n)$ for all A

Both Corollaries follow from 5.6 and 2.1 .

Another typical application of 5.1 gives a result of F. Peterson, as follows.

5.9 Proposition Assume $H^{2n+1}(X,A;Z)$ has no torsion dividing $(n-1)!$ Then a stable bundle $\eta \in K_{\mathbb{C}}A$ which extends to $X^{2n-1} \cup A$ has an extension to $X^{2n+1} \cup A$ if and only if $\delta^* c_n(\eta) = 0$ where c_n = n-th Chern class, and δ^* = coboundary homomorphism.

5.10 Corollary. If $H^{2i+1}(X,A;Z)$ has no torsion dividing $(i-1)!$, $i = 3,4,\ldots$, then a stable bundle $\eta \in \tilde{K}_{\mathbb{C}}A$ extends to X if and only if $\delta^* c(\eta) = 0$, where c = total Chern class.

5.11 Corollary. If $H^{2i}(A,Z)$ has no torsion dividing $(i-1)!$, $i = 3,4,$ then $\eta \in \tilde{K}_{\mathbb{C}}A$ is the trivial bundle if and only if $c(\eta) = 1$.

This follows from 5.10 by taking $X = CA$ = cone over A .

Proof of 5.9 It is clear that the condition is necessary: if η extends to X^{2n+1} A then $c_n(\eta)$ extends to X, hence $\delta^* c_n = 0$.

For the converse, put $t = \widetilde{K}_{\mathbb{C}}$, and let $\eta' \in t(X^{2n-1} \vee A)$ an extension of η. The exact sequence $t(X^{2n} \vee A) \to t(X^{2n-1} \vee A) \to t(\bigvee_i S^{2n-1}) = 0$ shows $\eta' \in \mathrm{Im}[t(X^{2n} \vee A) \to t(X^{2n-1} \vee A)]$, so we can use the obstruction sequence 5.2 . We map it into the corresponding sequence for $t' = H^{2n}(-,Z)$ via the Chern class $c_n : t \to t'$. As remarked in the proof of 5.6 we have

$$\mathrm{Im}[t'(X^{2n} \vee A) \to t'(X^{2n-1} \vee A)] = t'A = H^{2n}(A,Z) \ ;$$

so we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{Im}[t(X^{2n} \vee A) \to t(X^{2n-1} \vee A)] & \xrightarrow{\sigma} & H^{2n+1}(X,A;tS^{2n}) \\
\downarrow c_n & & \downarrow c_n = (n-1)! \\
H^{2n}(A,Z) & \xrightarrow{\delta^*} & H^{2n+1}(X,A;t'S^{2n})
\end{array} \quad ,$$

hence $\sigma(\eta') = 0 \Leftrightarrow (n-1)!\,\sigma(\eta') = 0 \Leftrightarrow \delta^* c_n(\eta) = 0$, qed.

6. Spectral sequences

For every p we have the Puppe sequence

$$\ldots \leftarrow \Sigma X^{p+1} \leftarrow \Sigma X^p \leftarrow X^{p+1}/X^p \leftarrow X^{p+1} \leftarrow X^p \xleftarrow{\rho} \bigvee_j S^p$$

where ρ is an attaching map for $(p+1)$-cells. Hence an exact sequence

$$\ldots \to t\,\Sigma X^{p+1} \to t\Sigma X^p \to t(X^{p+1}/X^p) \to tX^{p+1} \to tX^p \xrightarrow{t\rho} t(\bigvee_j S^p)$$

(in case $p = 0$ we can replace $t(\bigvee_j S^p)$ by 0 because $t\rho = 0$). This looks like an exact couple except that it stops on the right. There are several ways around this difficulty of which we describe one. Recall first that $t(\bigvee_j S^p) \cong C^{p+1}(X,tS^p)$, and $\delta(t\rho) = 0$ (see proof of 5.1), so we can replace the last term by cocycles $Z^{p+1}(X,tS^p)$. Then we can extend the sequence as follows to get an exact couple

$$
\begin{array}{cccccccc}
(6.1) & A^{p+1,-p-2} & & A^{p,-p-1} & & C^{p+1,-p-1} & A^{p+1,-p-1} & & A^{p,-p} \\
& \| & & \| & & \| & \| & & \| \\
\cdots & t\Sigma X^{p+1} & \longrightarrow & t\Sigma X^{p} & \longrightarrow & C^{p+1}(X,tS^{p+1}) \xrightarrow{i} & tX^{p+1} & \xrightarrow{f} & tX^{p} \xrightarrow{g^p}
\end{array}
$$

$$
\begin{array}{ccccccc}
Z^{p+1}(X,tS^{p}) & \xrightarrow{(1,0)} & \bigoplus_{\nu\le p}\mathrm{Coker}(g^{\nu}) & \xrightarrow{\text{proj.}} & \bigoplus_{\nu\le p-1}\mathrm{Coker}(g^{\nu}) \to & 0 \to 0 \to \cdots \\
\| & & \| & & \| & \| \\
C^{p+1,-p} & & A^{p+1,-p} & & A^{p,-p+1} & C^{p+1,-p+1}
\end{array}
$$

This leads to

6.2 <u>Proposition</u> <u>There exists a natural (with respect to X and t) spectral sequence</u>

$E(t,X) = \quad d_r : E_r^{pq} \to E_r^{p-r,q+r-1}$ such that

$E_2^{p,-q} = H^p(X,tS^q)$ <u>for</u> $p-q \le +1$, $E_2^{pq} = 0$ <u>for</u> $p-q>1$,

<u>and which converges to</u>

$E^{p,-q} = F_p t\Sigma^{q-p}X/F_{p-1}t\Sigma^{q-p}X$ <u>for</u> $p-q\le 0$

$E^{p,-p+1} = H^{p+1}(X,tS^p)/\mathrm{Im}(\sigma)$, $E^{p,-q}$ 0 <u>for</u> $p-q > 1$.

<u>Here</u> $F_p t\Sigma^m X = \mathrm{Ker}[t\Sigma^m X \to t\Sigma^m X^{p-1}]$, <u>and</u> σ <u>denotes the obstruction morphism of Proposition 5.1</u>

For the proof one has only to recall (4.2) that $t(\Sigma^m X^p/\Sigma^m X^{p-1}) = C^p(X,tS^{p+m}) \cong E_1^{p,-p-m}$ and that the composite $t(\Sigma^m X^p/\Sigma^m X^{p-1}) \to t(\Sigma^m X^p) \to t(\Sigma^{m-1}X^{p+1}/\Sigma^{m-1}X^p)$ agrees with the coboundary δ ; furthermore, an easy verification has to be made on the right edge terms $A^{p,-p+1}$.

Prop. 6.2 contains, of course, earlier results like 2.1 or 5.5. For example, under the assumptions of 5.5 we have a morphism of spectral sequences $E(f): E(X) \to E(A)$ such that $E_2^{i,-n}(f)$ is epimorphic for $i = n-1$, isomorphic for $i = n$, monomorphic for $i = n+1$. This implies the same with lower index 2 replaced by 3,4,... , in particular $E_\infty^{n,-n}(X) = E_\infty^{n,-n}(A)$, which immediately gives $tX = tA$.

The exact couples 6.1 for t and $t\Sigma$ obviously agree except for a shift of indices and except for values p,q in the neighborhood of $p-q=0$ (at the right end). Even there we have a map of $E(t\Sigma,X)$ into $E(t,X)$, and we get

6.3 Proposition (suspension isomorphism) There are natural isomorphisms

$$E_r^{p,-q}(t\Sigma,X) = E_r^{p,-q-1}(t,X) \qquad \text{for } p-q\leq 0 \text{ and all } r \;,$$

$$E_2^{p,-p+1}(t\Sigma,X) = E_2^{p,-p}(t,X)$$

which commute with the differentials d_r

In particular, if $h=\{h^m\}$ is a cohomology theory (1.7) then $h^m\Sigma = h^{m-1}$, and 6.3 allows to paste together the sequence of spectral sequences $E(h^m)$ to a single convergent spectral sequence $E(h,X)$ such that

$$E_2^{pq} = \tilde{H}^p(X,\tilde{h}^q(S^o)) \;,\; E^{pq} = F_p\tilde{h}^{p-q}X/F_{p+1}\tilde{h}^{p-q}X \qquad \text{(compare [1])} \;.$$

7. Pairings of half exact functors

Let $\underline{A}$, $\underline{A}'$, $\underline{A}''$ be abelian categories and $F\colon \underline{A}\times\underline{A}' \to \underline{A}''$ an additive right-exact covariant functor, i.e., $F(A,A')$ is covariant right-exact in each variable. We write $A\times A'$ instead of $F(A,A')$.

If t,t',t'' are half exact functors with values in $\underline{A},\underline{A}',\underline{A}''$ then a pairing is a natural transformation

$$\mu\colon t(X)\times t'(Y) \to t''(X*Y) \;.$$

7.1 Proposition Every pairing μ induces a pairing of spectral sequences

$$\mu_r\colon\; E_r^{pq}(t,X)\times E_r^{p'q'}(t',Y) \to E_r^{p+p',q+q'}(t'',X*Y) \;,\quad r\geq 1$$

with the following properties

(i) $\mu_2: H^p(X,tS^q)\times H^{p'}(Y,t'S^{q'}) \to H^{p+p'}(X*Y,t''S^{q+q'})$
is the ordianry exterior $\cup$-product associated with the coefficient pairing $tS^q \times t'S^{q'} \to t''(S^q * S^{q'}) = t''S^{q+q'}$.

(ii) d_r is a derivation, i.e. the following diagram is commutative

$$\begin{array}{ccc} E_r^{pq}\times E_r^{p'q'} & \xrightarrow{\mu_r} & E_r^{p+p',q+q'} \\ \downarrow{\scriptstyle (d_r\times id,(-1)^{p+q} id\times d_r)} & & \downarrow{\scriptstyle d_r} \\ E_r^{p+r,q-r+1}\times E_r^{p'q'} \oplus E_r^{pq}\times E_r^{p'+r,q'-r+1} & \xrightarrow{(\mu_r,\mu_r)} & E_r^{p+p'+r,q+q'-r+1} \end{array}$$

(iii) μ_{r+1} is induced by μ_r , $r\geq 1$ [!?]

(iv) μ_∞ is induced by μ .

Sketch of proof. It does not seem enough to look at the exact couple 6.1 ; one has to use terms like $t(X^{p+r}/X^p)$ for all possible values of p,r (not just $r = 1$ or ∞). I.e., one has to use a system $H(p+r,p)$ as in [2], Chap. [?] .

We first note the following equalities

$$X^{p+r}/X^p * Y^{p'+r}/Y^{p'} = X^{p+r}*Y^{p'+r}/X^{p+r}*Y^{p'}\cup X^p*Y^{p'+r} =$$
$$= X^{p+r}*Y^{p'+r}\cup(X*Y)^{p+p'+r+1}/(X*Y^p\cup X^p*Y)\cap \text{numerator} \quad ;$$

in the last expression there appear more cells in the numerator than in the second expression, but they are divided out.

By composition we now get a pairing

[?] inserted by reader.

[!?] inserted by reader: μ_r is induced by μ, and is "compatible" with μ_{r+1}, for every $r \geq 1$ (?)

$$t(X^{p+r}/X^p)\times t'(Y^{p'+r}/Y^{p'}) \xrightarrow{\mu} t''(X^{p+r}/X^p \# Y^{p'+r}/Y^p) =$$

$$t\left\{X^{p+r}\# Y^{p'+r}\cup(X\# Y)^{p+p'+r+1}/(X\# Y^{p'}\cup X^p\# Y)\cap \text{numerator}\right\} \longrightarrow$$

$$t''\left\{(X\# Y)^{p+p'+1+r}/(X\# Y)^{p+p'+1}\right\}$$

Since $\mathrm{Im}[t(X^{p+r}/X^p)\to t(X^{p+1}/X^p)] = Z_r^{p+1,-p-1}$ there results (replacing $p+1$ by p)

$$Z_r^{p,-p}\times Z_r^{p',-p'} \xrightarrow{\mu} {}''Z_r^{p+p',-p-p'} ;$$

similarly

$$B_r^{p,-p}\times {}'Z_r^{p',-p'} \xrightarrow{\mu} {}''B_r^{p+p',-p-p'} ,$$

hence by right exactness

$$\mu_r :\ E^{p,-p}\times {}'E_r^{p',-p'} \longrightarrow {}''E_r^{p+p',-p-p'}$$

This defines μ_r if $p+q = 0$, $p'+q' = 0$. The general case $p+q\leq 0$, $p'+q'\leq 0$ reduces to this via the suspension isomorphism 6.3 and the pairing

$$t\Sigma^m X\times t'\Sigma^n Y \to t''(\Sigma^m X \# \Sigma^n Y) = t'\Sigma^{m+n}(X\# Y) .$$

It remains to define μ_r if $p+q = 0$, $p'+q' = 1$, or vice versa.

Let $(\bigvee_i e^{p+1}, \bigvee_i S^p) \xrightarrow{\varphi} (X^{p+1}, X^p)$ resp. $(\bigvee_j e^{p'+1}, \bigvee_j S^{p'}) \xrightarrow{\varphi'} (Y^{p'+1}, Y^{p'})$ denote characteristic maps for the $(p+1)$- resp. $(p'+1)$-cells of X resp. Y. In $X\# Y$ we can then get characteristic maps by taking $\#$-products. Consider the maps

$$\partial(\bigvee_i e^{p+1} \# \bigvee_j e^{p'+1}) = (\bigvee_i e^{p+1} \# \bigvee_j S^{p'})\cup(\bigvee_i S^p \# \bigvee_j e^{p'+1}) \longrightarrow$$

$$(\bigvee_i e^{p+1}/S^p \# \bigvee_j S^{p'})\vee(\bigvee_i S^p \# \bigvee_j e^{p'+1}/S^{p'}) \xrightarrow{(\varphi\# id)\vee(id\#\varphi')}$$

$$(X^{p+1}/X^p \# \bigvee_j S^{p'})\vee(\bigvee_i S^p \# Y^{p'+1}/Y^{p'}).$$

Apply t'' :

$$t''(X^{p+1}/X^p \ast \bigvee_j S^{p'}) \oplus t''(\bigvee_i S^p \ast Y^{p'+1}/Y^{p'}) \to t''(\bigvee_{i,j}(e^{p+1} \ast e^{p'+1}) \subset$$

$$'E_1^{p+p'+2,\ p-p'-1}.$$

Finally compose with

$$t(X^{p+1}/X^p) \times t'(\bigvee_j S^{p'}) \xrightarrow{\mu} t''(X^{p+1}/X^p \ast \bigvee_j S^{p'})$$

to get

$$\mu_r : E_r^{p+1,-p-1} \times {}'E_r^{p'+1,-p'} \longrightarrow {}''E_r^{p+p'+2,-p-p'-1}.$$

Parts (iii) and (iv) of proposition 7.1 are now easily checked. Part (i) follows by looking at E_1-terms and the pairing μ_1. We have $E_1^{p,-p} = t(X^p/X^{p-1}) = t(\bigvee_j S^p) = \prod_j tS^p = C^p(X,tS^p)$, and by restricting μ_1 to single factors tS^p of $C^p(X,tS^p)$ we see that μ_1 is the ordinary exterior product of cellular cochains associated with the specified coefficient pairing. Since μ_2 is induced by μ_1 this proves (i).

Part (ii) requires a little more work. We consider the case $q = -p$, $q' = -p'$ only; the general case is similar, and it can be reduced to our case by the suspension isomorphism 6.3. The sign $(-1)^{p+q}$ of (ii) comes in under the form $t(w)$ where w is the mapping of $\Sigma^{p+q+p'+q'}(X \ast Y)$ into itself which brings the $(p+q+1)$-st suspension operator $\Sigma = S^1 \ast$ into the first position. As a homotopy class w is $(-1)^{p+q}id$, hence $t(w) = (-1)^{p+q}id$ by lemma 3.3.

Now $d_r : E_r^{p+1,-p-1} \to E_r^{p+1+r,-p-r}$ is induced (passage to quotients) by the morphism

$$t(X^{p+r}/X^p) \to tX^{p+r} \to t(\bigvee_i S^{p+r}) = C^{p+r+1}(X, tS^{p+r}),$$

where i ranges over the $(p+r+1)$-cells of X. It is therefore enough to show that the following diagram is commutative

$$\begin{array}{ccc}
t(X^{p+r}/X^p)\times t'(Y^{p'+r}/Y^{p'}) & \xrightarrow{\mu} & t''(X\#Y)^{p+p'+r+1}/(X\#Y)^{p+p'+1} \\
\downarrow & & \\
t(X^{p+r})\times t'(Y^{p'+1}/Y^{p'}) \oplus t(X^{p+1}/X^p)\times t'(Y^{p'+r}) & & \downarrow \\
\downarrow & & \\
t(\bigvee_i S^{p+r})\times t'(Y^{p'+1}/Y^{p'}) \oplus t(X^{p+1}/X^p)\times t'(\bigvee_{i'} S^{p'+r}) & \xrightarrow{\mu} & t''(\bigvee_{i''} S^{p+q'+r+1})
\end{array}$$

where the horizontal maps are induced by μ as explained above (see p. 15 for the first line and p. 16 for the third). It is enough to show that the diagram becomes commutative after projecting the lower right corner onto any one of its factors $tS^{p+p'+r+1}$. Now this sphere is the boundary of a cell $e^i \# e^j$ of $X\#Y$ where $i+j = p+p'+r+2$. The characteristic maps $e^i \to X$, $e^j \to Y$ induce a map of the above diagram into the corresponding one for e^i, e^j, and on the lower right we get precisely the projection onto $t\,(e^i \# e^j) = tS^{p+p'+r+1}$. By naturality it suffices therefore to prove commutativity in case $X = e^i$, $Y = e^j$, $i+j = p+p'+r+2$.

If $X = e^i$ then X^{p+r}/X^p is contractible (hence $t(X^{p+r}/X^p) = 0$) unless $i = p+1$ or $i = p+r+1$; i.e., we have only to study the two (symmetric) cases $(i,j) = (p+1, p'+r+1)$ or $(i,j) = (p+r+1, p'+1)$. Consider the first, then $X^{p+r} = e^i$, $X^{p+r}/X^p = S^i$, $Y^{p'+r} = S^{j-1}$, and the diagram reduces to

$$\begin{array}{ccc}
tS^i \times t'S^{j-1} & \longrightarrow & t''\partial(e^i \# e^j) = tS^{i+j-1} \\
\downarrow \mathrm{id} & & \\
tS^i \times t'S^{j-1} & & \downarrow \mathrm{id} \\
\downarrow \mathrm{id} & & \\
tS^i \times t'S^{j-1} & \longrightarrow & tS^{i+j-1}
\end{array}$$

qed.

8. Generalizations

One can generalize the preceding results to functors $t\colon \underline{W} \to \underline{A}$ where $\underline{A}$ is not necessarily abelian. For instance, one can take $\underline{A}$ = ENS, the category of sets, provided we replace half-exactness by the Meyer-Vietoris condition (e) of Brown [2]. Also for most results we then have either to restrict ourselves to simply connected CW-complexes X or we have to assume that t is <u>n-simple</u> for all n; this means: $t(\alpha) = t(\gamma\alpha)$ for all $\alpha \in \pi_n X$, $\gamma \in \pi_1 X$, and all X. Generalizing both abelian categories and ENS one can probably take for $\underline{A}$ any category such that the category of abelian group objects of $\underline{A}$ is an abelian category.

On the other hand $\underline{W}$ can be replaced by more general categories. As a guiding model consider the following: Let $u\colon \underline{W} \to \underline{A}$ be half exact, let $\pi\colon E \to B$ be a fibration, and for every $\alpha\colon X \to B$ let $\pi_\alpha\colon E_\alpha \to X$ be the induced fibration. For every $\alpha\colon X \to B$ and $Y \subset X$ put $t(X,Y,\alpha) = u(E_\alpha / E_{\alpha|Y})$. This is a functor on triples (X,Y,α) (i.e. on the category of pairs of spaces over B) to which the preceding results generalize. For instance, one has a spectral sequence such that $E_2^{p,-q} \cong \tilde{H}^p(B, \underline{u}(S^q \times F / {*} \times F))$ for $p-q \leq 1$,

$E_2^{p,-q} = 0$ for $p-q > 1$, and $E_\infty^{p,-q}$, for $p-q \leq 0$, is the graded object associated with a filtration of $u\Sigma^{q-p}(E/F)$. Here $F = \pi^{-1}(*)$ is the fibre, $*$ = base point and $\underline{u}$ indicates a local coefficient system (the group $\pi_1 B$ operates on $u(S^q \times F/* \times F) = u\Sigma^q F \oplus uS^q$). This spectral sequence has multiplicative properties as in 7.1. The proofs are very similar, and it is not hard to axiomatize the whole situation.

Other possibilities for $\underline{W}$ are the category of spectra, or the category of (free) chain complexes over a fixed ring. Dual to section 1, one could consider half-coexact functors t^*, i.e., functors which are exact on fibrations (instead of cofibrations). Guided by these examples one is lead to axiomatize $\underline{W}$, too, probably (following a suggestion of S. Eilenberg) by distinguishing in $\underline{W}$ a certain class of morphisms $X' \rightarrow X \rightarrow X''$ as "short exact sequences", which the functor would have to carry into exact sequences.

References

[1] ATIYAH-HIRZEBRUCH, Vector bundles and homogenuous spaces, Proc. Symp. Diff. Geom. Tucson 1960.

ATIYAH, M., Characters and cohomology of finite groups, Inst. Hautes Etud., Publ. Math., Paris 1961.

[2] BROWN, E., Jr., Cohomology theories, Annals of Math. 75(1962) 467-484.

[3] CARTAN-EILENBERG, Homological Algebra, Princeton Unov. Press 1956.

[4] DOLD, A., Relations between ordinary and extraordinary cohomology, Coll. on Algebr. Topology Aarhus 1962, p. 1-9.

[5] PUPPE, D., Homotopiemengen und ihre induzierten Abbildungen I, Math. Zeitschr. 69(1958) 299-344.

AMS summer institute for topology, Seattle 1963