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A stable decomposition for certain spaces

by

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§1:

In this paper we define a space $C(Y, X)$, depending on spaces Y and X, and prove a stable decomposition for it. We want such a result for our work on function and configuration spaces [6]. For example, $H_*(C(Y, X))$ gives the E_2 term of various spectral sequences including:

a) One due independently to D. W. Anderson [2] and P. Trauber [18] for calculating $H_*(X^Y)$ if Y is a compact manifold (where homology is taken with field coefficients).

b) One due to Gelfand and Fuks [7, 8] for calculating the continuous Lie algebra cohomology of C^∞ vector fields with compact supports on the smooth manifold Y. X is a wedge of spheres depending only on the dimension of Y.

For another example, $C(M, S^0)$ has the same homotopy type as D. McDuff's approximations for the space of sections, $\Gamma(M)$, [13].

Another reason for excising this paper from [6] is because of the independent interest of these decompositions. They specialize to some classical ones: Milnor [14], D. S. Kahn [9], M. G. Barratt and P. J. Eccles [3], V. Snaith [16], and C. L. Reedy [15], produced stable decompositions of $\Omega^n\Sigma^n X$ for $n = 1$, $n = \infty$, and $1 \leq n \leq \infty$ respectively. These results can be recovered from our more general result. Moreover, we use models

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for $\Omega^n\Sigma^n X$ so that these stable decompositions are particularly easy with which to work.

As an example, the first author has commutative diagrams exhibiting the relation between the maps in these stable decompositions, the action of the little cubes operad on the spaces involved, compositions, and smash products. Such explicit data together with methods in [5] ought to give new calculations in homotopy theory.

An outline of the rest of the paper follows: section 2 contains the basic definitions, constructions, and concludes with a statement of the main theorem; section 3 contains a description of our basic map; section 4 contains a proof of the main theorem; section 5 contains a superficial comparison of our stable decomposition with Snaith's; and section 6 is a technical section proving some earlier assertions.

§2:

Throughout all spaces are k-spaces, that is, compactly generated and Hausdorff. Let $\mathcal{U}$ denote the category whose objects are all k-spaces with an infinite number of points and whose morphisms are 1 - 1 maps. A space X is said to be NDR-based if the pair (X, *) is an NDR pair. Let $\mathcal{T}$ denote the category of all NDR-based path connected spaces and based maps. All products, subspaces, function spaces, etc. are formed in the category of k-spaces. [17] is a good reference.

Definition: $F(\ ,j) : \mathcal{U} \to \mathcal{U}$, the configuration space functor, assigns to each space Y of $\mathcal{U}$ the following subspace of Y^j: $(y_1, \ldots, y_j) \in Y^j$ is in $F(Y, j)$ if and only if $y_r = y_s$ implies $r = s$. Let Σ_j be the symmetric group on j letters. Σ_j acts on Y^j by permuting the coordinates. This action on the j-fold product restricts to a free Σ_j action on $F(Y, j)$. Let $B(\ , j) : \mathcal{U} \to \mathcal{U}$ denote the functor with $B(Y, j) = F(Y, j)/\Sigma_j$.

Definition: For Y in $\mathcal{U}$ and X in $\mathcal{T}$ define

$$C(Y, X) = \coprod_{j \geq 0} F(Y, j) \times_{\Sigma_j} X^j / \approx .$$

By convention, $F(Y, 0) \times X^0 = *$, the base point in X. $\approx$ denotes the equivalence relation generated by

$$((y_1, \ldots, y_j), (x_1, \ldots, x_j)) \approx ((y_1, \ldots, \hat{y}_i, \ldots, y_j), (x_1, \ldots, \hat{x}_i, \ldots, x_j))$$

if $x_i = *$ ($\hat{a}$ means delete a). $\coprod$ denotes disjoint union. C(Y, X) is topologized below.

Remark: The above is a generalization of construction 2.4 of J. P. May [12].

Define a filtration and topology on $C(Y, X)$ as follows. $\coprod_{j=0}^{r} F(Y, j) \times X^j$ maps to $C(Y, X)$ and $F_r C(Y, X)$ denotes its image. $\coprod_{j=0}^{r} F(Y, j) \times X^j$ is given the topology of the disjoint union and $F_r C(Y, X)$ is given the quotient topology. $C(Y, X)$ is given the topology of the union of the $F_r C(Y, X)$. It is easy to check that $C(Y, X)$ and $F_r C(Y, X)$ are in $\mathcal{T}$. (See Lemma 4.4.)

Definition: Let Y be in $\mathcal{U}$ and X be in $\mathcal{T}$. Define

$$D_j(Y, X) = F(Y, j) \times_{\Sigma_j} X^{[j]}/F(Y, j) \times_{\Sigma_j} *$$

where $X^{[j]}$ denotes the j-fold smash product.

Facts: If X is in $\mathcal{T}$, X^j and $X^{[j]}$ are in $\mathcal{T}$. $F(Y, j) \times X^{[j]}$ is a k-space which maps onto $D_j(Y, X)$. Give $D_j(Y, X)$ the quotient topology. Then $D_j(Y, X)$ is in $\mathcal{T}$. Lemma 8.2 of Steenrod [17] is used in the verifications.

From these facts it is clear that $D_j(,)$ defines a functor $D_j : \mathcal{U} \times \mathcal{T} \longrightarrow \mathcal{T}$.

Recall the definition of the functor $Q : \mathcal{T} \longrightarrow \mathcal{T}$ where $QX = \varinjlim \Omega^n \Sigma^n X$ and where $\Sigma^n X$ denotes the n-fold reduced suspension of X. In our proofs we require models for QX. Such models have been given by a number of people; we insert the following definition of "model" to obviate technical points which arise from different choices of contractible spaces on which the symmetric group acts freely and also to insure maximum functoriality of our splittings.

<u>Definition</u>: A functor $q : \mathcal{T} \longrightarrow \mathcal{T}$ is a model for Q if and only if

(A) There exist natural transformations $M \longrightarrow Q$ and $M \longrightarrow q$ such that for each space X in $\mathcal{T}$ the maps $MX \longrightarrow QX$ and $MX \longrightarrow qX$ are weak equivalences.

(B) There exists a natural transformation of $1 \longrightarrow M$ so that $X \longrightarrow MX \longrightarrow QX$ is the usual inclusion. The induced map $X \longrightarrow qX$ is called the natural inclusion.

Let $S(\ )$ denote the singular chain functor and $|S(\ )|$ its geometric realization. There is a natural transformation $|S(\ )| \longrightarrow 1$ so that, for any space X, $|S(X)| \longrightarrow X$ is a weak equivalence. (See for example Prop. 4.11 and Thm. 6.7 of [11].)

By applying $|S(\ )|$ to the diagram in (A) of the definition of model it is easy to produce a natural homotopy class of maps $|S(qX)| \longrightarrow QX$ if X is in $\mathcal{T}$. One chooses a homotopy inverse for the homotopy equivalence $|S(MX)| \longrightarrow |S(qX)|$.

<u>Definition</u>: If $q : \mathcal{T} \longrightarrow \mathcal{T}$ is a model for Q, and if X is in $\mathcal{T}$, a map $f : Y \longrightarrow qX$ is a stable weak equivalence if and only if the map $|S(Y)| \xrightarrow{|S(f)|} |S(qX)| \longrightarrow QX$ has the property that the induced map $Q|S(Y)| \longrightarrow QX$ is a weak equivalence.

For example, if X is in $\mathcal{T}$, the natural inclusion $X \longrightarrow qX$ is a stable weak equivalence.

<u>Remark</u>: A map $f : Y \longrightarrow qX$ is a stable map in the sense that f

corresponds to a unique map $\{f\} : \{S(Y)\} \longrightarrow \{S(X)\}$ in Adams' stable category [1] where $\{S(A)\}$ denotes the suspension spectrum of the CW complex $|S(A)|$. f is a stable weak equivalence if and only if $\{f\}$ is an equivalence.

We define one more functor to insure maximum generality. Let $E_j = R^1 \times Co(B(Y, j))$ where $Co(\ )$ denotes the unreduced cone functor. Define W_Y as $\prod_{j=1}^{\infty} E_j$ (the weak product). W_Y gives a functor from $\mathcal{U}$ to $\mathcal{U}$.

Theorem 1.1: For X in $\mathcal{J}$ and Y in $\mathcal{U}$, there exists a natural transformation of functors

$$\alpha(Y, X) : C(Y, X) \longrightarrow C(W_Y, \bigvee_{j=1}^{\infty} D_j(Y, X)) \quad .$$

$C(W_Y,\ )$ is a model for Q, and $\alpha(Y, X)$ is a stable weak equivalence.

($\bigvee_{j=1}^{\infty}$ denotes the wedge product in the category $\mathcal{J}$.)

Remark: The point of introducing the construction W_Y here is to insure that the stable splitting given above is as functorial as possible. The reader who is willing to accept functoriality on faith need only read section 3 and section 4 while thinking of $C(W_Y, X)$ in terms of either the Barratt-Eccles approximation [3] or J. P. May's approximation [12] for QX.

In addition, these splittings generalize further where $F(Y, j)$ is replaced in our construction by any other "sufficiently natural" Σ_j space with appropriate "equivariant degeneracies." We do not elaborate on this point. The crucial point is that one must be able to write down the stable

Hopf invariants described in Barratt-Eccles [3] in sufficient generality. Perhaps the main observation of this paper is that configuration spaces provide a very convenient setting in which to describe these stable Hopf invariants.

C. Reedy [15] has given stable decompositions of some spaces which support operad actions; we do not require operad actions in our splittings.

§3:

The purpose of this section is to describe functorial maps

$$h_r^j : F_rC(Y, X) \longrightarrow F_\ell C(B(Y, j), D_j(Y, X)) \quad \text{where } \ell = \begin{cases} \binom{r}{j} & \text{if } r \geq j \\ 0 & \text{if } r < j \end{cases} .$$

We begin by giving a map

$$\psi_k : F(Y, k) \times X^k \longrightarrow C(B(Y, j), D_j(Y, X)) .$$

If $k < j$, ψ_k is just the constant map to the base point. If $k \geq j$, let $m = \binom{k}{j}$. There are m different projection maps $\pi_I : Y^k \longrightarrow Y^j$ defined as follows. Choose a subset, I, of $\{1, \ldots, k\}$ of cardinality j. This subset has a natural ordering on it so let $\{i_1, \ldots, i_j\}$ be the set where $i_1 < i_2 < \ldots < i_j$. $\pi_I(y_1, \ldots, y_k) = (y_{i_1}, \ldots, y_{i_j})$.

The product of all the π_I's in some order induces a map $Y^k \longrightarrow (Y^j)^m \longrightarrow (Y^j/\Sigma_j)^m$. Clearly $F(Y, k)$ lands in $F(F(Y, j), m)$ and $F(B(Y, j), m)$. Since $F(F(Y, j), m) \subseteq (F(Y, j))^m$, we also have a map $F(Y, k) \longrightarrow F(Y, j)^m$. Using the π_I in the same order gives a map $X^k \longrightarrow (X^j)^m$. Thus we get a map

$$F(Y, k) \times X^k \longrightarrow F(B(Y, j), m) \times (F(Y, j) \times X^j)^m .$$

Passage to quotients gives a map

$$F(Y, k) \times X^k \longrightarrow F(B(Y, j), m) \times_{\Sigma_m} (F(Y, j) \times X^j)^m$$

which is independent of the order used to multiply the π_I. Since $F(Y, j) \times X^j$ maps onto $D_j(Y, X)$, we obtain a map

$$F(Y,\ k) \times X^k \longrightarrow F(B(Y,\ j),\ m) \times_{\Sigma_m} D_j(Y,\ X)^m \quad .$$

Again, passage to quotients and a check of the definition of C gives an induced natural map

$$\psi_k : F(Y,\ k) \times X^k \longrightarrow C(B(Y,\ j),\ D_j(Y,\ X))$$

which, in fact, lands in $F_m C$.

For fixed j, the maps ψ_k "patch" together to give a map

$$F_r C(Y,\ X) = \coprod_{k=0}^{r} F(Y,\ k) \times_{\Sigma_r} X^r/\approx \ \longrightarrow C(B(Y,\ j),\ D_j(Y,\ X)) \quad .$$

This is our map h_r^j after we note that the image is contained in $F_\ell C(B(Y,\ j),\ D_j(Y,\ X))$, $\ell = \binom{r}{j}$.

h_r^j is clearly a natural transformation from $F_r C$ to the composite functor $F_\ell C(B(\ ,\ j),\ D_j(\ ,\ ))$. If $r \leq s$, then $\ell = \binom{r}{j} \leq k = \binom{r}{s}$ and

$$\begin{array}{ccc} F_r C(Y,\ X) & \xrightarrow{h_r^j} & F_\ell C(B(Y,\ j),\ D_j(Y,\ X)) \\ \cap\!| & & \cap\!| \\ F_s C(Y,\ X) & \xrightarrow{h_s^j} & F_k C(B(Y,\ j),\ D_j(Y,\ X)) \end{array}$$

commutes.

§4:

By taking appropriate "sums" of the maps h_r^j which were defined in section 3, we obtain the map $\alpha(Y, X)$ of Theorem 1.1. The stable splitting of $C(Y, X)$ then follows directly from the definition of $\alpha(Y, X)$.

To sum the h_r^j, we give $C(Y, X)$ an "addition" for appropriate Y. In particular, let $Z = \coprod_{i=1}^{r} A_i$ and consider the map

$$\prod_{i=1}^{r} F(A_i, k_i) \xrightarrow{\text{inclusion}} \prod_{i=1}^{r} A_i^{k_i} \xrightarrow{\text{inclusion}} \prod_{i=1}^{r} Z^{k_i} \xrightarrow{\text{inclusion}} Z^k ,$$

$k = k_1 + \ldots + k_r$. This composite evidently factors through the natural inclusion $\prod_{i=1}^{r} F(A_i, k_i) \longrightarrow F(Z, k)$. Hence juxtaposition of coordinates gives a map

$$\prod_{i=1}^{r} C(A_i, B) \longrightarrow C(\coprod_{i=1}^{r} A_i, B)$$

which clearly restricts to

$$\mu : \prod_{i=1}^{r} F_{\ell_i} C(A_i, B) \longrightarrow F_{\ell} C(\coprod_{i=1}^{r} A_i, B)$$

where $\ell = \ell_1 + \ldots + \ell_r$.

Remark 4.1: Any fixed open embedding $e : \mathbb{R}^n \coprod \mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be used to give $C(Y \times \mathbb{R}^n, X)$ the structure of H-space using the map defined immediately above. More is true: $C(Y \times \mathbb{R}^n, X)$ is a $\mathcal{C}_n$-space in the sense of J. P. May [12] and hence $C(Y \times \mathbb{R}^n, X)$ is weakly homotopy equivalent to an n-fold loop space if it is connected.

Next, we use the natural inclusion $\iota : D_j(X, Y) \longrightarrow \bigvee_{i=1}^{r} D_i(X, Y)$, $1 \leq j \leq r$, to give a composite map, H_r^j, defined by commutativity of the diagram

$$\begin{array}{ccc} F_r C(Y, X) & \xrightarrow{H_r^j} & F_{\ell_j} C(B(Y, j), \bigvee_{i=1}^{r} D_i(Y, X)) \\ \downarrow h_r^j & \nearrow F_{\ell_j} C(1, \iota) & \\ F_{\ell_j} C(B(Y, j), D_j(Y, X)) & & \end{array}$$

where $\ell_j = \binom{r}{j}$.

Let $\ell = \ell_1 + \ldots + \ell_r = 2^r - 1$ and define $\beta_r(X, Y)$ as the composite

$$F_r C(Y,X) \xrightarrow{\Delta^r} (F_r C(Y,X))^r \xrightarrow{\Pi H_r^j} \prod_{j=1}^{r} F_{\ell_j} C(B(Y,j), \bigvee_{i=1}^{r} D_i(X,Y)) \xrightarrow{\mu} F_\ell(\coprod_j B(Y,j), \bigvee_i D_i(X,Y))$$

where Δ^r is the r-fold diagonal.

We embed $\coprod_j B(Y, j)$ in W_Y : Recall that $W_Y = \prod_{j=1}^{\infty} E_j$. E_j has a natural base point given by (0, cone point) and so we obtain a natural embedding $E_j \longrightarrow W_Y$. Consider the embedding e_j given by the composite

$$B(Y, j) \xrightarrow{\cong} B(Y, j) \times 1 \longrightarrow Co(B(Y, j)) \xrightarrow{\cong} 0 \times Co(B(Y, j)) \hookrightarrow \mathbb{R}^1 \times Co(B(Y, j))$$
$$= E_j \longrightarrow W_Y \ .$$

Define $e = \coprod_{j=1}^{r} e_j : \coprod_{j=1}^{r} B(Y, j) \longrightarrow W_Y$ and notice that e is an embedding.

Define

$$\alpha_r(Y, X) : F_r C(Y, X) \longrightarrow C(W_Y, \bigvee_{i=1}^{r} D_i(Y, X))$$

as the composite

$$F_r C(Y, X) \xrightarrow{\beta_r(Y,X)} F_\ell C(\coprod_{j=1}^{r} B(Y, j), \bigvee_{i=1}^{r} D_i(Y, X)) \xrightarrow{F_\ell C(e,1)} F_\ell C(W_Y, \bigvee_{i=1}^{r} D_i(Y, X)) \ .$$

It is easy to check that the diagram

$$(4.2)\qquad \begin{array}{ccc} F_{r-1}C(Y,\ X) & \xrightarrow{\alpha_{r-1}} & C(W_Y,\ \bigvee_{i=1}^{r-1} D_i(Y,\ X)) \\ \downarrow 1 & & \downarrow 2 \\ F_rC(Y,\ X) & \xrightarrow{\alpha_r} & C(W_Y,\ \bigvee_{i=1}^{r} D_i(Y,\ X)) \end{array}$$

commutes, where 1 is just the inclusion and 2 is the functor $C(W_Y,\ )$ applied to the inclusion $\bigvee_{i=1}^{r-1} D_i(Y,\ X) \longrightarrow \bigvee_{i=1}^{r} D_i(Y,\ X)$.

Our goal is to use (4.2) and induction to prove

Theorem 4.3: For X in $\mathcal{J}$, $\alpha_r(Y,\ X)$ is a stable weak equivalence.

We require the following lemma which is proven by copying May's proof of Proposition 2.6 in [12] with $F(Y,\ j)$ replacing $\mathcal{C}(j)$ and $C(Y,\ X)$ replacing CX wherever they appear.

Lemma 4.4:

A) $F_{r-1}C(Y,\ X) \longrightarrow F_rC(Y,\ X) \longrightarrow D_r(Y,\ X)$ is a cofibration.

B) $C(Y,\ X)$ is in $\mathcal{J}$.

C) $F_1C(Y,\ X) = D_1(Y,\ X)$.

D) For fixed Y in $\mathcal{U}$, $C(Y,\ )$ is a limit preserving functor.

Remark: Lemma 4.4 except part (B) remains true if X is only assumed to be compactly generated and Hausdorff; in this case $C(Y,\ X)$ is still compactly generated and Hausdorff, but is not necessarily connected.

We require the following lemma.

Lemma 4.5: There exists a strictly commutative diagram

$$\begin{array}{ccc} F_rC(Y,\ X) & \xrightarrow{\alpha_r(Y,X)} & C(W_Y,\ \bigvee_{i=1}^{r} D_i(Y,\ X)) \\ \downarrow f & & \downarrow g \\ D_r(Y,\ X) & \xrightarrow{\iota_r} & C(W_Y,\ D_r(Y,\ X)) \end{array}$$

where (1) f is the quotient map of Lemma 4.4(A), (2) g is $C(W_Y, \ )$ applied to the pinch map $\bigvee_{i=1}^{r} D_i(Y, X) \longrightarrow D_r(Y, X)$, and (3) ι_r is homotopic to the standard inclusion of $D_r(Y, X)$ in $C(W_Y, D_r(Y, X))$.

<u>Proof</u>: We exhibit a homotopy

$$G_r : D_r(Y, X) \times [0, 1] \longrightarrow C(W_Y, D_r(Y, X))$$

such that $G_r(\ , 0)$ is the natural inclusion of $D_r(Y, X)$ in $C(W_Y, D_r(Y, X))$ and $G_r(\ , 1) = \iota_r$ gives commutativity of the diagram in Lemma 4.5.

W_Y has a natural contraction

$$\gamma : W_Y \times [0, 1] \longrightarrow W_Y$$

with $\gamma(\ , 1) =$ identity and $\gamma(\ , 0) =$ base point where γ is defined by multiplying each real coordinate and each "cone coordinate" by t at time t. Consider the natural quotient map $p_1 : F(Y, r) \times X^r \longrightarrow D_r(Y, X)$. Let $p_2 : F(Y, r) \longrightarrow W_Y$ be the composite $F(Y, r) \longrightarrow B(Y, r) \xrightarrow{e_r} W_Y$. Define a map λ as the composite

$$F(Y, r) \times X^r \xrightarrow{\Delta \times 1} F(Y, r) \times F(Y, r) \times X^r \xrightarrow{p_2 \times p_1} W_Y \times D_r(Y, X) \quad ,$$

where Δ is the diagonal.

We combine λ and γ to obtain a composite map θ.

$$\begin{array}{ccccc}
F(Y,r) \times X^r \times [0,1] & \xrightarrow{\lambda \times 1} & W_Y \times D_r(Y,X) \times [0,1] & \xrightarrow{\text{switch}} & W_Y \times [0,1] \times D_r(Y,X) \\
 & & & & \downarrow{\scriptstyle \gamma \times 1} \\
 & \searrow{\scriptstyle \theta} & & & W_Y \times D_r(Y,X) \\
 & & & & \downarrow{\scriptstyle \text{inclusion}} \\
 & & & & C(W_Y, D_r(Y,X))
\end{array}$$

θ passes to quotients and clearly gives the homotopy G_r.

We also require the following which is the subject of section 6.

<u>Lemma 4.6</u>: $C(W_Y, \ )$ is a model for Q.

<u>Proof of Theorem 4.3</u>: We prove the theorem by induction on r. If $r = 1$, the result follows since $\alpha_1(Y, X)$ is just the standard inclusion of $D_1(Y, X)$ into $C(W_Y, D_1(Y, X))$ and $C(W_Y, \ )$ is a model for Q by Lemma 4.5. Assume that Theorem 4.2 for all $k \leq r-1$. Then diagram (4.2) and Lemma 4.5 give the following commutative diagram in the stable category:

$$\begin{array}{ccccc}
\{S(F_{r-1}C(Y, X))\} & \longrightarrow & \{S(F_rC(Y, X))\} & \longrightarrow & \{S(D_r(Y, X))\} \\
\Big\downarrow \{\alpha_{r-1}\} & & \Big\downarrow \{\alpha_r\} & & \Big\downarrow \{\iota_r\} \\
\{S(\bigvee_{j=1}^{r-1} D_j(Y, X))\} & \longrightarrow & \{S(\bigvee_{j=1}^{r} D_j(Y,X))\} & \longrightarrow & \{S(D_r(Y, X))\}
\end{array} .$$

By Lemma 4.4(A), both rows are of the form $\{S(\ )\}$ applied to a cofibration and the result follows by the induction hypothesis.

<u>Remark</u>: The strict commutativity of diagram (4.2) together with Lemma 4.4(D) permit us to define

$$\alpha(Y, X) : C(Y, X) \longrightarrow C(W_Y, \bigvee_{j=1}^{\infty} D_j(Y, X))$$

as $\lim_{r \to \infty} \alpha_r(Y, X)$. In the stable category we have the equation $\lim_{r \to \infty} \{\alpha_r\} = \{\alpha\}$. Direct limits preserve equivalence in the stable category and so if X is in $\mathcal{J}$, Theorem 4.3 implies that α is a stable equivalence and Theorem 1.1 follows.

§5:

Let $\mathcal{C}_n(j)$ denote the Boardman and Vogt's space of j little n-cubes. May [12] in construction 2.4 defines the functor $C_nX = \coprod_{j\geq 0} \mathcal{C}_n(j) \times_{\Sigma_j} X^j/\approx$ and on p. 39 he defines a natural map $\alpha_n : C_nX \longrightarrow \Omega^n\Sigma^nX$. There is a natural map $C_nX \longrightarrow C(R^n, X)$ which we define in §6 and show to be a weak equivalence if X is in $\mathcal{J}$. Apply Theorem 1.1 and our map $C_nX \longrightarrow C(R^n, X)$ to get

<u>Theorem 5.1</u>: For X in $\mathcal{J}$ there is a natural stable weak equivalence

$$C_nX \longrightarrow C(W_{R^n}, \bigvee_{j=1}^{\infty} D_j(R^n, X)) \quad .$$

Since $D_j(R^n, X)$ has the same homotopy type as May's $\mathcal{C}_n(j) \times_{\Sigma_j} X^{[j]}/\mathcal{C}_n(j) \times *$ (Lemma 6.1), we have recovered Snaith's splittings [16].

Snaith also produced some unstable splittings. To obtain these, recall our functor β_r from §4. The image of β_r is contained in $F_\ell(\coprod_{j=1}^{r} B(R^n, j), \bigvee_{j=1}^{r} D_j(R^n, X))$. $B(R^n, j)$ is an nj-dimensional manifold, so it embeds in R^{2nj} instead of some huge space W_{R^n}. We show

<u>Theorem 5.2</u>: For X in $\mathcal{J}$ we have a natural (up to homotopy) weak equivalence

$$\Sigma^{2nr}F_rC_nX \longrightarrow \Sigma^{2nr}(\bigvee_{j=1}^{r} D_j(Y, X))$$

for all $r < \infty$.

Remark: The value $2nr$ in Theorem 5.2 can be improved a bit, but P. Kirley [10] has shown that the splitting of Theorem 5.1 cannot be realized after a finite number of suspensions.

Proof: We shall exhibit an embedding

$$\varepsilon_r : \coprod_{j=1}^{r} B(\mathbb{R}^n, j) \longrightarrow \mathbb{R}^{2nr} .$$

Consider the composite induced by $\beta_r(\mathbb{R}^n, X)$ and ε_r which gives a functorial map

$$\widetilde{\beta}_r : F_r C_n X \longrightarrow F_\ell C(\mathbb{R}^{2nr}, \bigvee_{j=1}^{r} D_j(\mathbb{R}^n, X))$$

where $\ell = 2^r - 1$. There is a natural homotopy class of maps of $F_r C(\mathbb{R}^n, X)$ to $\Omega^n \Sigma^n X$ by Theorem 6.2 and so $\widetilde{\beta}_r$ induces a map

$$\overline{\beta}_r : F_r C_n X \longrightarrow \Omega^{2nr} \Sigma^{2nr} \bigvee_{j=1}^{r} D_j(\mathbb{R}^n, X) .$$

Adjoint $\overline{\beta}_r$ to a map $adj(\overline{\beta}_r)$. A homological argument similar to the argument given in the proof of Theorem 4.2 shows that $adj(\overline{\beta}_r)$ is a homology isomorphism. But all spaces are suspensions of connected spaces and so the result follows from the Hurewicz theorem.

To define ε_r, fix a homeomorphism a_j, of $\mathbb{R}^{2nr}$ with the subspace of $\mathbb{R}^{2nr}$ given by those points whose first coordinate are in the open interval $(j, j+1)$. Since $B(\mathbb{R}^n, j)$ embeds in $\mathbb{R}^{2nr}$, $j \leq r$, we define ε_r as the composite

$$\coprod_{j=1}^{r} B(\mathbb{R}^n, j) \xrightarrow{\coprod \text{embedding}} \coprod_{1}^{r} \mathbb{R}^{2nr} \xrightarrow{\coprod a_j} \mathbb{R}^{2nr} .$$

Remark: We include an observation which may be useful in calculations with the maps $\overline{\beta}_r$. Using the methods above, it is easy to check that $\overline{\beta}_r$ extends to all of C_nX to give a map

$$\overline{\beta}_r : C_nX \longrightarrow \Omega^{\ell}\Sigma^{\ell}\left(\bigvee_{j=1}^{r} D_j(\mathbb{R}^n, X)\right)$$

where ℓ is the embedding dimension of $B(\mathbb{R}^n, r)$.

§6:

We finish by proving Lemma 4.6 which states that $C(W_Y, \ )$ is a model for Q. We can do this most efficiently by comparing May's approximation $C_n X$ for $\Omega^n\Sigma^n X$ with $C(\mathbb{R}^n, X)$ and by comparing $C(\mathbb{R}^\infty, X)$ with $C(W_Y, X)$ where $\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$. We also prove an unstable analogue for $\Omega^n\Sigma^n X$.

Recall the construction $C_n X$ of May [12] and the map $\alpha_n : C_n X \longrightarrow \Omega^n\Sigma^n X$ which is a weak equivalence for path connected X. Note that May actually requires X to be path connected rather than just connected; connectivity is insufficient as can be seen by checking [12], page 66, line 6. The natural map $g : \mathcal{C}_n(j) \longrightarrow F(\mathbb{R}^n, j)$ given in [12, Thm. 5.1] by sending an embedded cube to its center clearly passes to quotients and gives a map of filtered spaces

$$\bar{\rho} : C_n X \longrightarrow C(\mathbb{R}^n, X) \quad .$$

By comparing the definitions of the multiplications in $C_n X$ [12, Lemma 1.9] and in $C(\mathbb{R}^n, X)$ given by Remark 4.1, we see that $\bar{\rho}$ is also an H-map.

We next compare $C(\mathbb{R}^\infty, X)$ and $C(W_Y, X)$. There is a natural map $\iota : \mathbb{R}^\infty \longrightarrow W_Y$ given by taking the weak direct product of maps $\mathbb{R}^1 \longrightarrow \mathbb{R}^1 \times Co(B(Y, j))$ where $r \in \mathbb{R}^1$ goes to $(r,$ cone point). We obtain a map of filtered spaces $\bar{\rho} : C_\infty X \longrightarrow C(W_Y, X)$ given by the composite $C_\infty X \xrightarrow{\bar{\rho}} C(\mathbb{R}^\infty, X) \xrightarrow{C(\iota, 1)} C(W_Y, X)$. As above, ρ is also a map of H-spaces.

Since ρ and $\bar{\rho}$ are H-maps of connected H-spaces, it suffices to show that they give isomorphisms in integral homology in order to show that they are weak equivalences. To prove this, we use standard filtration methods: We recall that $C_n X$ is filtered in the analogous way in which

$C(\mathbb{R}^n, X)$ is filtered and that

$$F_{r-1}C_nX \longrightarrow F_rC_nX \longrightarrow e[\mathcal{C}_n(r), X, \Sigma_r]$$

is a cofibration where $e[\mathcal{C}_n(r), X, \Sigma_r] = \mathcal{C}_n(r) \times_{\Sigma_r} X^{[r]} / \mathcal{C}_n(r) \times_{\Sigma_r} *$ [12, Prop. 2.6]. To prove that $\bar{\rho}$ is a weak equivalence, observe that $X \simeq F_1C_nX \longrightarrow F_2C_nX$ and $X \simeq F_1C(\mathbb{R}^n, X) \longrightarrow F_2C(\mathbb{R}^n, X)$ are given by the natural inclusions and so by the evident induction it suffices to prove that the map induced by g,

$$\bar{g} : e[\mathcal{C}_n(r), X, \Sigma_r] \longrightarrow D_r(\mathbb{R}^n, X)$$

gives a homology isomorphism. Similar remarks apply to the map ρ with ∞ replacing n and W_Y replacing $\mathbb{R}^n$. To prove that ρ and $\bar{\rho}$ are weak equivalences, we only need

<u>Lemma 6.1</u>: (A) $\bar{g}$ is a homotopy equivalence for $1 \leq n \leq \infty$, (B) $D_r(\iota, 1) : D_r(\mathbb{R}^\infty, X) \longrightarrow D_r(W_Y, X)$ gives an isomorphism in homology.

<u>Proof</u>: To prove part (A), first observe that by [12, Thm. 5.1], $g : \mathcal{C}_n(r) \longrightarrow F(\mathbb{R}^n, r)$ is an equivariant homotopy equivalence and the result follows.

To prove part (B), consider the commutative diagram

$$\begin{array}{ccccc}
F(\mathbb{R}^\infty, r) \times_{\Sigma_r} * & \longrightarrow & F(\mathbb{R}^\infty, r) \times_{\Sigma_r} X^{[r]} & \longrightarrow & D(\mathbb{R}^\infty, X) \\
\downarrow{\scriptstyle \iota\times 1} & & \downarrow{\scriptstyle \iota\times 1} & & \downarrow{\scriptstyle D(\iota,1)} \\
F(W_Y, r) \times_{\Sigma_r} * & \longrightarrow & F(W_Y, r) \times_{\Sigma_r} X^{[r]} & \longrightarrow & D(W_Y, X)
\end{array} .$$

Since $(X^{[r]}, *)$ is evidently an equivariant NDR-pair, each of the rows above is a cofibration. To show that $D(\iota, 1)$ gives a homology isomorphism, it suffices to show that the two left-hand maps give homology isomorphisms. That these last two maps give homology isomorphisms follows from standard covering space arguments provided we show that $F(W_Y, r)$ is contractible. This last fact is proven in Proposition 6.3.

We now show

Lemma 4.6: $C(W_Y, \;)$ is a model for Q.

Proof: Set $MX = C_\infty X$. May's map $\alpha : C_\infty X \longrightarrow QX$ and our map $\rho : C_\infty X \longrightarrow C(W_Y, X)$ are weak equivalences and $\alpha : C_\infty X \longrightarrow QX$ clearly satisfies hypothesis (B) in the definition of "model for Q."

We also obtain

Theorem 6.2: If $r < \infty$ and $1 \leq n \leq \infty$, then there exists a unique homotopy class of maps $f : F_r C(\mathbb{R}^n, X) \longrightarrow \Omega^n\Sigma^n X$ such that the diagram

$$\begin{array}{ccc} F_r C_n X & \xrightarrow{\alpha_n} & \Omega^n\Sigma^n X \\ \Big\downarrow \bar{\rho} & \nearrow f & \\ F_r C(\mathbb{R}^n, X) & & \end{array}$$

homotopy commutes.

Proof: Since $e[\mathcal{C}_n(r), X, \Sigma_r] \longrightarrow D_r(\mathbb{R}^n, X)$ is a homotopy equivalence by Lemma 6.1(A), it follows that $[D_r(\mathbb{R}^n, X), A] \longrightarrow [e[\mathcal{C}_n(r), X, \Sigma_r], A]$ is a set theoretic isomorphism (where $[\;,\;]$ denotes homotopy classes of maps).

We want to show that $[F_r C(\mathbb{R}^n, X), A] \longrightarrow [F_r C_n X, A]$ is a set theoretic isomorphism. If A is a loop space we use the Barratt-Puppe sequence and an induction similar to that used in the proof above that $\bar{\rho}$ is a weak equivalence. Theorem 6.2 follows by setting $A = \Omega^n\Sigma^n X$.

To finish the business of section 6, we need only prove

<u>Proposition 6.3</u>: Let W be any contractible space such that there exists an embedding $b : W \times [-1, 1] \longrightarrow W$ and an isotopy $h_t : W \longrightarrow W$ such that h_1 is the identity and h_0 is the composite $W = W \times 0 \subset W \times [-1, 1] \xrightarrow{b} W$. Then $F(W, k)$ is contractible.

Before we give the proof, we give some examples.

1) If W satisfies our condition, then so does $W \times Y$ for any contractible space Y.

2) If W is homeomorphic to E^∞ for any topological vector space E of dimension greater than zero, then W satisfies our conditions. To see this define $b((\vec{x}_1, \ldots), s) = (s\vec{e}, \vec{x}_1, \ldots)$ for some fixed non-zero vector $\vec{e} \in E$. Define $h_t(\vec{x}_1, \ldots, \vec{x}_k, \ldots) = (t\vec{x}_1, \ldots, t\vec{x}_k + (1-t)\vec{x}_{k-1}, \ldots)$. It is easy to check the required properties.

3) The Hilbert cube satisfies our conditions.

4) W_Y satisfies our condition since $W_Y = R^\infty \times$ (some contractible space).

<u>Proof of 6.3</u>: Let $i : F(W, k) \longrightarrow W^k$ denote the inclusion. Define $e : W^k \longrightarrow F(W, k)$ by $e(v_1, \ldots, v_k) = (b(v_1, 1), \ldots, b(v_k, 1/k))$. Define $R_t : W^k \longrightarrow W^k$ by $R_t = s_t^1 \times \ldots \times s_t^k$ where

$$s_t^i(v) = \begin{cases} b(v, t/i) & 0 \leq t \leq 1 \\ h_{-t}(v) & -1 \leq t \leq 0 \end{cases}$$

R_{-1} is the identity and $R_1 = i \circ e$.

Observe that if $\vec{v} \in W^k$ is in the image of i, then $R_t(\vec{v})$ is also in the image of i. Hence we can define a map $r_t : F(W, k) \longrightarrow F(W, k)$ so that

$$\begin{array}{ccc} F(W, k) & \xrightarrow{r_t} & F(W, k) \\ \downarrow i & & \downarrow i \\ W^k & \xrightarrow{R_t} & W^k \end{array}$$

commutes, r_{-1} is the identity and r_1 is $e \circ i$. We have proved that $F(W, k)$ and W^k have the same homotopy type.

Since W is contractible, it follows that $F(W, k)$ is contractible.//

<u>Joke</u>: Hilbert cube aficionados denote the Hilbert cube by Q. For X in $\mathfrak{T}$, $C(Q, X)$ and QX have the same weak homotopy type.

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