"The Homology Groups of Eilenberg-Mac Lane",
course by Henri Cartan, U. of Chicago, Summer, 1953.

We will consider a space $X \ni \pi_i(X) = 0$ for $i \neq n$, $\pi_n(X) = \Pi$. The cohomology and homology depends only on Π and n (proven in 1943 by E-M.L.)

Instead of considering the singular ~~simp~~ complex $S(X)$, we will use cubes rather than simplices.

Let I be the unit interval, $0 \leq \lambda \leq 1$, I^n the unit n-cube.

A continuous map $u : I^n \to X$ (i.e. $u(\lambda_1, \dots, \lambda_n)$) is a <u>singular cube</u> of the space X.

The maps $F_i^0 : (\lambda_1, \dots, \lambda_{n-1}) \to (\lambda_1, \dots, \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_n)$
$F_i^1 : (\lambda_1, \dots, \lambda_{n-1}) \to (\lambda_1, \dots, \lambda_{i-1}, 1, \lambda_{i+1}, \dots, \lambda_n)$
are called <u>face operations</u> and give the faces of u, namely uF_i^0, uF_i^1.

There also is a <u>degeneracy operation</u> $D_i : (\lambda_1, \dots, \lambda_{n+1}) \to (\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_n)$ for $1 \leq i \leq n+1$, and an $n+1$-cube of the form $u D_i$ is called <u>degenerate</u>.

Let $C'(X)$ be the free (graded) group generated by these cubes, and define a <u>bdry</u> of a singular cube (defining a homom. of degree -1) by

$$du = \sum_1^n (-1)^i (uF_i^0 - uF_i^1). \qquad \underline{ddu = 0}.$$

If u is degenerate, du is a sum of degenerate cubes.

Let $D(X)$ be the subcomplex generated by the degenerate cubes (stable by above remarks) and define the <u>singular cubical complex</u> $C(X) = C'(X)/D(X)$ [at times, I will write $C(X)$ when

I mean $C'(X)$].

Define a map : $S(X) \to C(X)$ by, given s, get $s\phi = u$ by a fixed map ϕ of $I^n \to (\mu_0, \dots, \mu_n)$, $\mu_i \geq 0$, $\Sigma \mu_i = 1$.

$$\begin{tikzcd} \Delta \arrow[dr, "s"'] & \square \arrow[l, "\phi"'] \arrow[d, "u"] \\ & X \end{tikzcd}$$

where $\phi(\lambda_1, \dots, \lambda_n) =$ $(1-\lambda_1, \lambda_1(1-\lambda_2), \lambda_1\lambda_2(1-\lambda_3), \dots, \lambda_1 \cdots \lambda_{n-1}(1-\lambda_n), \lambda_1 \cdots \lambda_n)$.

This has been proven a chain equivalence by E-ML

We will now define & discuss the concept of a <u>minimal subcomplex</u> M of $C(X)$ as first done by Eilenberg - Zilber, Annals of Math, vol. 51, page 499.

<u>Def</u>: 2 n-cubes u, v are <u>compatible</u> if $u(\lambda_1, \dots, \overset{0}{\underset{i}{}}, \dots, \lambda_n) = v(\lambda_1, \dots, \overset{0}{\underset{i}{}}, \dots, \lambda_n)$ for all i.

2 compatible n-cubes u, v are <u>homotopic</u> if $\exists$ $(n+1)$-cube $w \ni wF_1^0 = u$, $wF_1^1 = v$, wF_i^0, wF_i^1 $(i \neq 1)$ are the degeneracies of u & v.

<u>Def</u>: A subcomplex $M \subset C(X)$ is called a <u>minimal subcomplex</u> if

(1) M is stable with respect to F, D,

(2) $u \in C(X) \ni uF_i^0$ and $uF_i^1 \in M \Rightarrow \exists v \in M$, v compatible & homotopic to u and v is unique.

One can construct such by induction.

0-cube: only 1 (assume X is arcwise connected), a base pt. x_0.

1-cubes: by (1), must be a loop, take 1 in each homotopy class, making sure that constant loop is taken in identity class. Induction step is similar.

We have a projection: $C(X) \to M$ which is 1 on M, (and is a chain-equivalence when divided by degenerate cubes).

Given another M',
$$M' \to C(X) \to M$$
$$M \to C(X) \to M'$$
and the composition of these 2 maps is 1, $\therefore$ any 2 minimal subcomplexes are isomorphic.

Def: $X = K(\Pi, n)$ means X is a space $\ni \pi_i(X) = 0$ for $i \neq n$ and $\pi_n(X) = \Pi$ ($i \geq 0$, $n \geq 1$).

[We will consider only $n > 1$].

Let us look at M, & see that it depends only on Π and n.

Pick a base pt x_0. Look at q-cubes in M.

$q = 0, 1, \ldots, n-1$: only 1 q-cube and that's the degenerate 1 at the base pt.

$q = n$: the bdry goes $\to x_0$, $\therefore$ defines an elt. of $\pi_n(X, x_0)$, and the n-cubes in M are 1 representative from each homotopy class subject to the condition that the one from the identity class is the degenerate one.

$q = n+1$: each n-face represents an elt. of Π, and by a well-known(?) theorem, the elt. of Π represented by du is 0, i.e. we have an when cocycle 0 on all but 2 opposite faces, must be deg. cube.

n- cocycle (coeffs. in Π) of I^{n+1}! Conversely, given an n-cocycle on I^{n+1}, we get an $n+1$-cube by the same theorem.

$q = n+k$, ($k > 1$): as in $n+1$, we get an n-cocycle with coeffs. in Π, and this is again a 1-1 correspondence (can get back again as $\pi_i(X) = 0$ for $i > n$.)

same remarks as above

Hence each basis cube in M is determined by an n-cocycle.

The face operation on this cocycle is the obvious induced one.

Degeneracy operation: $(\lambda_1, \dots, \lambda_{q+1}) \xrightarrow{D_i} (\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{q+1}) \xrightarrow{u} X$: if λ_i appears in the n-face, it all goes into x_0; if $\lambda_i = 0$ or 1, then goes onto n-face under D_i + gives elt of Π, i.e. the new cocycle gives same value to top + bottom + 0 to sides

Hence, from our $K(\Pi, n)$, we have a complex M depending only on Π and n.

Def: A space X is an H-space if $\exists$ a multiplication in X, i.e. a conts. map $f: X \times X \to X$ with an elt $e \ni ee = e$, and the maps $x \to ex$, $x \to xe$ are homotopic to the identity map.

Remark: Assuming we can always construct a $K(\Pi, n)$ (any n), then we can ~~assume~~ always get one which is an H-space.

Proof: Let $Y = K(\Pi, n+1)$. Let X = space of loops in Y at y_0, and x_0 = constant loop.

An m-cube of X, $u(\lambda_1, \dots, \lambda_m) = v(t) =$
$v(t, \lambda_1, \dots, \lambda_m)$, a loop in Y, + is an $(m+1)$-cube of Y, and if bdry of $u \to x_0$, bdry of $v \to y_0$,
+ $\therefore$ $\pi_m(X) \approx \pi_{m+1}(Y)$ + $\therefore$
$\pi_n(X) \approx \Pi$, $\pi_i(X) = 0$ for $i \neq n$, or $X = K(\Pi, n)$.

But X is an H-space, f is composition of loops.

We now define a multiplication of cubes: (pick e as base pt.)
u a p-cube, v a q-cube, define
$w = u \circ v\ (\lambda_1, \dots, \lambda_{p+q}) = f((u(\lambda_1, \dots, \lambda_p), v(\lambda_{p+1}, \dots, \lambda_{p+q})))$, a $p+q$-cube of X. Extend linearly and note

$$d(u \circ v) = (du) \circ v + (-1)^p u \circ (dv).$$

We have a multiplication in $C(X)$, $\therefore$ we have

$$C(X) \otimes C(X) \to C(X)$$
$$u \otimes v \to u \circ v$$

and this gives a multiplication in M,

$$M \otimes M \to C(X) \otimes C(X) \to C(X) \to M;$$

explicitly, $u, v \in M$ (n-cocycles), $I^p \times I^q \to X$, and looking at an n-face of this cube:

Cases I. Of form $I^k \times I^{n-k}$, $0 < k < n$, then goes into $e \circ e = e$ ~~[illegible]~~

II. $0, n$, $e\,x$ which is homotopic to x + hence the cocycle gives same value in Π.

III. $n, 0$, like case II.

Notice then that this multiplication in

M is independent of the H-space X.
Also note that if a cube is degenerate, then the product is degenerate so we can pass to the quotient.
We denote M by $K(\Pi, n)$ and note that $K(\Pi, n)$ is a differential algebra [i.e. product of basis elts. is a basis elt., unit, associative, and $d(uv) = (du)\cdot v + (-1)^p u\cdot(dv)$.]
Assigning $K(\Pi, n)$ to Π is a covariant functor from the category of abelian groups to that of chain complexes.

We now wish to look at the multiplicative structure in the cohomology (of first X & then $K(\Pi, n)$).
The diagonal map $X \to X \times X$ induces $C(X) \to C(X \times X)$. We also have $a: C(X) \otimes C(X) \to C(X \times X)$ by $u(\lambda_1, \dots, \lambda_p) \otimes v(\lambda_{p+1}, \dots, \lambda_{p+q}) \to (u, v)$ which is a chain equivalence.
Let G be a ring, we get
$$\mathrm{Hom}(C(X), G) \xleftarrow{\text{diag}^*} \mathrm{Hom}(C(X \times X), G) \xrightarrow{a^*} \mathrm{Hom}(C(X) \otimes C(X), G) \leftarrow \mathrm{Hom}(C(X), G) \otimes \mathrm{Hom}(C(X), G)$$
Passing to homology here,
$$H(\mathrm{Hom}(C(X), G)) \leftarrow H(\mathrm{Hom}(C(X \times X), G)) \xrightarrow{\simeq} H(\mathrm{Hom}(C(X) \otimes C(X), G)) \leftarrow H(\mathrm{Hom}(C(X), G) \otimes \mathrm{Hom}(C(X), G)) \leftarrow H(\mathrm{Hom}(C(X), G)) \otimes H(\mathrm{Hom}(C(X), G))$$
& this gives a map $H^*(X, G) \otimes H^*(X, G) \to H^*(X, G)$ which is the multiplication we want.

Now do a similar construction for $K(\Pi, n)$. We have the diagonal map giving $K(\Pi, n) \to K(\Pi \times \Pi, n)$, and we want a map $h: K(\Pi, n) \otimes K(\Pi, n) \to K(\Pi \times \Pi, n)$ which is a chain equivalence.

$f: \Pi \to \Pi \times \Pi$ by $x \to (x, e)$

$g: \Pi \to \Pi \times \Pi$ by $x \to (e, x)$ give 2 maps $: K(\Pi, n) \to K(\Pi \times \Pi, n)$ + define h by $u \otimes v \to f(u) \cdot g(v)$

And using an H-space $X = \mathcal{K}(\Pi, n)$, we see that h is homotopic to 1 $\left((x, x') \to (x, e)(e, x') = (xe, ex') \right)$, + then use the same construction as above.

<u>Suspension homom</u>: Let $Y = \mathcal{K}(\Pi, n+1)$, $y_0 \in Y$, $X = \mathcal{K}(\Pi, n)$ the space of loops at $y_0 \subset Z =$ space of paths of Y starting at y_0.

Z is contractible to z_0 = constant path.

$p: Z \to Y$ assigning endpoint, $X = p^{-1}(y_0)$, Z is a fibre space over Y.

$$\begin{array}{ccccccc} H_{q+1}(Z) & \to & H_{q+1}(Z, X) & \xrightarrow[\approx]{\partial} & H_q(X) & \to & H_q(Z) \\ \| & & \downarrow & & & & \| \\ 0 & & H_{q+1}(Y, y_0) & & & & 0 \\ & & \| & & & & \\ & & H_{q+1}(Y) & & & & \end{array}$$

From fibre space arguments, get $\approx$ for $q < 2n$

$\therefore$ we have a map $H_q(X) \to H_{q+1}(Y)$ which is the suspension homom: $H_q(K(\Pi, n)) \to H_{q+1}(K(\Pi, n+1))$ for $q > 0$.

We can assume Y is an H space with y_0 the unit ($y_0 y_0 = y_0$), then we can multiply 2 paths $t \to f(t) \ni f(0) = y_0 = g(0)$, $t \to g(t)$, then

$t \to f(t)\cdot g(t)$ is $\ni f(0)\cdot g(0) = y_0$, i.e. Z is an H-space, X an H-subspace of Z.

Due to mult. in $K(\Pi, n)$, we have
$i : H_q(X) \otimes H_{q'}(X) \to H_{q+q'}(X)$

Th: $i(\beta \otimes \gamma) = \alpha$ with $\deg \beta > 0$, $\deg \gamma > 0 \Rightarrow$ susp. $\alpha = 0$.
Proof: $\beta = \partial \beta'$, $\gamma = \partial \gamma'$ (β', γ' in $H(Z, X)$)
Then $\partial(\beta' \gamma) = \beta\gamma + (-1)\ \underset{0}{\beta'(\partial\gamma)} = \beta\gamma$.

& then $p_*(\beta'\gamma) = p_*(\beta') \cdot \underset{0}{p_*(\gamma)} = 0$ as γ is in X.

Since Z is contractible, $C(Z)$ is acyclic & we have $s: C_q(Z) \to C_{q+1}(Z) \ni ds\alpha + s d\alpha = \alpha$ if $\deg.\alpha > 0$, by $s\alpha : I^{q+1} \to Z$ being the cube I^q contracted to a pt, i.e. $I \times I^q \to Z$.

For $\alpha \in C(X)$, α a cycle, the map $\alpha \to p(s\alpha)$ gives the suspension for $d\alpha \to$ $p\, s\, d\alpha = p\alpha - p\, d\, s\alpha = -p\, d\, s\alpha$ so the map anticommutes with d & so map on homology is the suspension.

The suspension for $K(\Pi, n)$ turns out to be as follows:
$f_q \in K(\Pi, n)$ is an n-cocycle of I^q. $f' \in K(\Pi, n) = Sf$ is an $(n+1)$-cocycle of I^{q+1} which has values
$f'(0 \times \sigma) = 0$, $f'(1 \times \sigma) = 0$, $f'(I \times n\text{-face}) =$
$f(n\text{-face})$.

We will now define a new complex $L(\Pi, n)$.

$y_0 \in Y = K(\Pi, n+1)$, Z = space of paths of Y from y_0,
$Z \supset X$ = space of loops at y_0.

$p : Z \to Y$.

A minimal subcomplex of X is $K(\Pi, n)$.

Add another condition to def. of minimal subcomplex $N \subset C(Z)$, namely

(3) Any q-dim. cube of N ($q \leq n$) is in X (in (2), allowing only homotopies in X up to n) or equivalently

(3) up to dim n, N is a minimal subcomplex of X.

Since $\pi_n(Z) = 0$, there is a 1-1 correspondence between q-cubes in N $(q>n)$ and n-cochains with coeffs in Π of I^q.

Define $L(\Pi, n) = N$, same as $K(\Pi, n)$ for $q \leq n$, and generated by n-cochains of I^q for $q > n$.

$L(\Pi, n)$ is acyclic & as before we have a homotopy operator s $\quad u : I^q \to Z$, the obvious retraction gives $h : I \times I^q \to Z$, & the cochain defined by su is: $\quad t = 1$, get same elt of Π,
$t = 0$, get 0 obviously (everything at y_0)
$I \times n\text{-}1$ face, again get 0, & this shows $L(\Pi, n)$ is acyclic.

Assume Y is an H-space, then we get a multiplicative structure on $L(\Pi, n)$ just like on $K(\Pi, n)$,

on $I^p \times I^q$ an n-cochain

ka, n-ka $(k>0)$ gets 0,

$0 \ , \ n$ gets same elt of Π
$n \ , \ 0$ "

$L(\Pi, n)$ is a differential graded acyclic algebra.
Also, $K(\Pi, n) \subset L(\Pi, n)$, algebraically + geometrically, + this identification preserves everything.

Define by $p: Z \to Y$, a map $L(\Pi, n) \to K(\Pi, n+1)$ and it turns out to be the cobdry of the n-cochain.
Proof: First make clear the isom: $\pi_{n+1}(Y) \cong \pi_n(X)$.
$u: I^{n+1} \to Y$ + faces $\to y_0$. Consider $(0, \lambda_1, \dots, \lambda_n)$, this goes $\to y_0$, $\therefore$ define $v(0, \lambda_1, \dots, \lambda_n) = x_0 \in X \subset Z$, use the covering homotopy theorem + we have

$$\begin{array}{ccc} & & Z \\ & \nearrow^{v} & \downarrow p \\ I^{n+1} & \xrightarrow{u} & Y \end{array}$$

$u = pv$, but u of bdry is y_0, $\therefore$ v of bdry of I^{n+1} is in X + defines an elt. of $\pi_n(X)$, translate this into algebra + get the result.
This gives a homom. of the graded differential algebras + is onto obviously as a cube is acyclic.

$K(\Pi, n) \to L(\Pi, n) \to K(\Pi, n+1)$, image $\subset$ kernel obviously as cobdy of cocycle $= 0$.

Look at $L(\Pi, n)$ for $n=0$. In forming products, we multiply (add) elts. at product vertices, this being the only place where composition in Π enters in.
We have $f: L(\Pi, 0) \xrightarrow{\text{onto}} K(\Pi, 1)$.

Choose $f: K(\Pi,1) \to L(\Pi,0) \ni pf = 1$ by assigning to each 1-cocycle a 0-cochain whose cobdry is this 1-cocycle, starting with 0 of Π at first vertex, this f being multiplicative because first vertex × first vertex is first vertex of product cube.

We have also $Z(\Pi) \otimes K(\Pi,1) \to L(\Pi,0)$ by

$\sigma \otimes u \to$ cochain whose cobdry is u + whose first vertex gets σ ($Z(\Pi)$ = group ring),

+ this is 1-1, onto, hence we get a differential operator in $Z(\Pi) \otimes K(\Pi,1)$, + letting $Z(\Pi) = K(\Pi,0)$, + composing this map with p, we get

$$K(\Pi,0) \to K(\Pi,0) \otimes K(\Pi,1) \to K(\Pi,1)$$

$$\sigma \to \sigma \otimes 1$$

$$\sigma \otimes u \to u$$

which is compatible with differential structure.

More generally, we have this problem for n + $n+1$ rather than 0, 1, + will do later.

We now go onto the algebraic considerations which we need for computations.

<u>Def:</u> A is a <u>graded differential algebra with augmentation</u> over Λ if:

Λ = commutative ring with unit,

$A = \sum_n A_n$, $A_n = 0$ for $n < 0$, A_n are sub-Λ-modules,

a unit, associative multiplication,

$\deg(ab) = \deg(a) + \deg(b)$,

a differential $d \ni \deg(da) = \deg(a) - 1$,
$dd = 0$, $d(ab) = (da)b + (-1)^\alpha a(db)$, $\alpha = \deg a$,
$\exists$ an augmentation $\varepsilon : A \to \Lambda$, $\varepsilon(1) = 1$,
$\sigma : \Lambda \to A$ by $\sigma(1) = 1$, $\varepsilon\sigma = id$, $\varepsilon d = 0$,
$\varepsilon(a) = 0$ if $\deg(a) > 0$, and A is Λ-free.

z. B. $\Lambda(\pi)$, define $\varepsilon x = 1$ for $x \in \pi$.

Def: A graded, differential left A-module M with augmentation:
$M = \sum_n M_n$, M_n are Λ-modules, $M_n = 0$, $n < 0$
$\deg(am) = \deg(a) + \deg(m)$ for $a \in A$, $m \in M$
$\varepsilon : M \to \Lambda \ni \varepsilon(am) = (\varepsilon a)(\varepsilon m)$, d of degree $-1 \ni dd = 0$, $d(am) = (da)\cdot m + (-1)^\alpha a(dm)$

Define $\Lambda \otimes_A M$ by
$0 \to I \to A \xrightarrow{\varepsilon} \Lambda \to 0$, and consider the subset IM, i.e. linear combinations am, $a \in I$, $m \in M$, coeffs. in Λ, and define
$\bar{M} = \Lambda \otimes_A M = M/IM$, and this is a Λ-module with $\bar{d}$ since $d(am) = (da)m + \underbrace{(-1)^\alpha a(dm)}_{\in IM}$ + $\varepsilon(da) = 0 \in I$, $\therefore \in IM$.

Def: A construction over A is the following:
we have an A-module M (as above),
$i : \bar{M} \to M$, a Λ-homom. $\ni$ $\bar{M} \to M \xrightarrow[\text{natural}]{} \bar{M}$ is 1, and
(a) $\bar{M}$ is Λ-free and $A \otimes_\Lambda \bar{M} \to M$ by

$a \otimes \bar{m} \to a(i\bar{m})$ is an isom. onto, and
(b) M is acyclic, i.e.
$\to M_n \xrightarrow{d} M_{n-1} \to \cdots \xrightarrow{d} M_0 \xrightarrow{\varepsilon} \Lambda \to 0$ is exact or
$H_n(M) = 0$ for $n > 0$, $H_0(M) \overset{\varepsilon}{\approx} \Lambda$.

We will show existence of constructions later.

Th. 1: M over A, M' over A' 2 constructions, a homom. $f: A \to A'$ which preserves everything $\Rightarrow$ $\exists\, g: M \to M'$ of degree $0 \ni$
(1) $g d_M = d_{M'} g$, $\varepsilon = \varepsilon' g$, $g(am) = f(a) g(m)$
and any 2 such homoms. g_1, g_2 are homotopic, i.e. $\exists\, s: M \to M'$ of degree $+1 \ni$
(2) $s(am) = (-1)^{\alpha} f(a) s(m)$
(3) $dsm + sdm = g_1 m - g_2 m$ for $m \in M$.
Proof: (Similar to construction of complex in cohomology of groups)
Let m_j be an A-basis of M (use hyp. (a), i.e. a Λ-basis of $\bar{M}$ + that isom.), m_j are homogeneous
To define g, it suffices to define $g(m_j)$ + extend linearly by (1), + it also suffices to show $g d_M = d_{M'} g$ on m_j. Define by induction on degree of m_j:
$\varepsilon m_j = \varepsilon' g(m_j)$ must hold, so take for $g(m_j)$ an elt with known augmentation, + here $gd = dg = 0$.
For induction step, take m_j of degree q, we know $g d_M m_j$ + pick $g m_j \ni d_{M'} g m_j = g d_M m_j$ which we can do because $d_{M'} g d_M m_j = g d_M d_M m_j = 0$, i.e. a cycle + $\therefore$ a bdry as M' is acyclic.

We will define $s(m_j)$ + again extend ~~by~~ linearly (by (2)), + it suffices to show (3) holds for $s(m_j)$ + then it will hold generally, i.e. to show $ds(am_j) + sd(am_j) = g_1(am_j) - g_2(am_j)$.

$ds(am_j) + sd(am_j) = d((-1)^\alpha f(a) s(m_j)) + s((da)m_j + (-1)^\alpha a(dm_j)) =$
$\cancel{(-1)^\alpha (dfa) s(m_j)} + (fa)(dsm_j) + \cancel{(-1)^{\alpha-1}(fda)(sm_j)} + (fa)(sdm_j) = (fa)[g_1(m_j) - g_2(m_j)] =$
$= g_1(am_j) - g_2(am_j)$.

We define $s(m_j)$ by induction on degree of m_j.
For $\deg(m_j)=0$, $ds(m_j)$ must $= g_1(m_j) - g_2(m_j)$, but can define $s(m_j) \ni$ that is true by acyclicity, + ~~similarly~~ if $\deg(m_j) = n$,
$ds(m_j) = g_1(m_j) - g_2(m_j) - sd(m_j)$ + can do for $d(g_1 - g_2 - sd)(m_j) = dg_1 m_j - dg_2 m_j - (g_1 dm_j - g_2 dm_j - sddm_j) = 0$ + use acyclicity. Q.E.D.

<u>Notation</u>: DGA means differential, graded, with augmentation.

<u>Corollary</u>: Note $\bar{M} = M/IM$ is again a DGA-module over A, $a\bar{m} = \varepsilon(a)\bar{m}$, so we consider it only as ~~an~~ Λ-module. We can pass to the quotient + get $\bar{g}: \bar{M} \to \bar{M}'$ since $g(am) = f(a)g(m)$ + f preserves augmentation, i.e. $\varepsilon' f(a) = \varepsilon(a) = 0$ for $a \in I$, + $\bar{g}$ commutes with $\bar{d}$ + hence we get $\bar{g}_*: H_n(\bar{M}) \to H_n(\bar{M}')$. Also, because of

(2), we can pass to the quotient of s & get
$\bar{s} : \bar{M} \to \bar{M}'$ $\ni$ $\bar{d}\bar{s} + \bar{s}\bar{d} = \bar{g}_1 - \bar{g}_2$, i.e.
$\bar{g}_{1*} = \bar{g}_{2*}$ and we get a map of $H(\bar{M}) \to H(\bar{M}')$
depending only on $f : A \to A'$.
Further if $A = A'$, $f = 1$, with M, M' we
get maps $H_n(\bar{M}) \to H_n(\bar{M}')$ & $H_n(\bar{M}') \to H_n(\bar{M})$
with compositions $= 1$ since we could take identity
$: M \to M$, & $\therefore$ both maps are isoms.

As a special case of our thm, we take a Λ-complex
M, a DGA-complex which is also acyclic, assume
also that M has a Λ-basis of homogeneous elts &
a map $\sigma : \Lambda \to M_0 \ni \varepsilon\sigma = 1$ (i.e. we let $A = \Lambda$),
& to get a contracting homotopy:

$$\begin{array}{ccccccccc}
\to M_n & \xrightarrow{d} & \cdots \to & M_1 & \xrightarrow{d} & M_0 & \xrightarrow{\varepsilon} & \Lambda & \to 0 \\
\downarrow 0 & & & \downarrow 0 & & \downarrow \sigma\varepsilon & & & \\
\to M_n & \xrightarrow{d} & \cdots \to & M_1 & \xrightarrow{d} & M_0 & \xrightarrow{\varepsilon} & \Lambda & \to 0
\end{array}$$

let $g_1 = 1$, $g_2 =$ this map (we have commut. as $\sigma\varepsilon d = 0$)
& hence $\exists\ s : M \to M \ni$ for degree 0, $dsm = m - \sigma\varepsilon m$,
& degree > 0, $dsm + sdm = m$ or we write
$dsm + sdm + \sigma\varepsilon m = m$ in general, called a
contracting homotopy w.r.t. σ.

Given a DGA-algebra A with a Λ-basis of
homogeneous elts & including 1 & we wish to prove the
existence of a DGA-module M over $A \ni$
M has an A-basis of homogeneous elts, & M is
acyclic. (Bar construction of Eil.-M. L.)
__Proof:__ $\sigma : \Lambda \to A$ is 1-1 as $\varepsilon\sigma = 1$.

Consider the cokernel of σ, $\hat{A} = A/\text{image}\ \sigma$, we have a d on $\hat{A}$ as $d=0$ on the scalars.

$a \in A \to [a] \in \hat{A}$

Define $B_n(A) = A \otimes \underbrace{\hat{A} \otimes \cdots \otimes \hat{A}}_{n\text{-times}}$, a left A-module

We have $s_n : B_n(A) \to B_{n+1}(A)$ by

$s_n(a \otimes [a_1] \otimes \cdots \otimes [a_n]) = 1 \otimes [a] \otimes [a_1] \otimes \cdots \otimes [a_n]$

(Notation: Write $a \otimes [a_1] \otimes \cdots \otimes [a_n] = a[a_1, \dots, a_n]$)

Then $0 \to \Lambda \xrightarrow{\sigma} A \xrightarrow[s_0]{} B_1(A) \xrightarrow[s_1]{} \cdots \to B_n(A) \xrightarrow[s_n]{} \cdots$

is exact, as $[\text{scalar}] = 0$.

Define $B_\bullet(A) = \sum_n B_n(A)$, $\bar{B}_n(A) = \underbrace{\hat{A} \otimes \cdots \otimes \hat{A}}_{n\text{-times}}$, i.e.

$B_n(A) = A \otimes \bar{B}_n(A)$,

$\bar{B}_\bullet(A) = \sum_n \bar{B}_n(A)$ & we have $B(A) = A \otimes \bar{B}(A)$.

$B(A)$ will be our DGA-module M.

Define a grading on $B(A)$ by

$\deg(a[a_1, \dots, a_n]) = \deg a + \sum_{1 \le i \le n} (\deg a_i + 1)$

Define augmentation, 0 on elts of degree >0 & those of degree 0 (namely A_0), as given ε.

Define d $\ni$

(1) $d(ax) = (da)x + (-1)^\alpha a(dx)$ for $x \in B(A)$

(2) $dsx + sdx + \sigma\varepsilon x = x$ ($s = s_n$ on $B_n(A)$)

(3) d of degree -1.

Put $d = d' + d''$, we will define d' and then show d'' is defined by conditions on d.

d' on $A \otimes \hat{A} \otimes \cdots \otimes \hat{A}$ is

$d \quad -d \quad \cdots \quad -d$ or

$d'(a[\,]) = da$

$d'(a[b]) = da[b] + (-1)^{\alpha+1} a[db]$, ($+1$ is for $-d$ on b)

$d'(a[b,c]) = da[b,c] + (-1)^{\alpha+1} a[db,c] + (-1)^{\alpha+\beta+1+1} a[b,dc]$

$\therefore \quad d'd' = 0, \quad d's + sd' = 0, \quad d'(ax) = (da)x +$
$(-1)^{\alpha} a(d'x)$, for

$s(a[a_1, \dots, a_n]) = [a, a_1, \dots, a_n]$

$d's(\qquad) = -[da, a_1, \dots, a_n] + (-1)^{\alpha+2}[a, da_1, \dots] + \cdots$

$sd'(\qquad) = [da, a_1, \dots, a_n] + (-1)^{\alpha+1}[a, da_1, \dots] + \cdots$

$\quad \therefore d's + sd' = 0$ for example.

Put $d = d' + d''$ and express conditions on d''

(1'') $d''(ax) = (-1)^{\alpha} a(d''x)$

(2'') $d''sx + sd''x = x - \sigma\varepsilon x$

(3'') d'' is of degree -1.

By (1''), we need only define d'' on $\bar{\mathcal{B}}_n(A)$, for

$$d''(\sum_j a_j \otimes x_j) = \sum_j (-1)^{\alpha_j} a_j (d''x_j)$$

Define by induction : $d'' = 0$ on $\bar{\mathcal{B}}_0(A) = \Lambda$

If we have d'' on $\mathcal{B}_q$ for $q < n$, we will define $d''y$ for $y \in \bar{\mathcal{B}}_n(A)$, $y = [a_1, \dots, a_n] = s_{n-1}(a_1[a_2, \dots, a_n])$, i.e. take $x \ni s(x) = y$ and by (2''), $d''y = x - \sigma\varepsilon x - sd''x$, & this shows existence.

To show uniqueness, show $sx = 0$ for $x \in \mathcal{B}_{n-1}(A)$ ($\forall \therefore x \in \bar{\mathcal{B}}_{n-1}(A)$), then $sd''x = x - \sigma\varepsilon x$.

$n = 1$: $x \in \bar{\mathcal{B}}_0(A) = \Lambda$, then $sd''x = 0 = x - \sigma\varepsilon x$ for $\sigma\varepsilon = 1$ on Λ.

$n > 1$: (by induction) $x = sz$, $z \in \mathcal{B}_{n-2}(A)$ by exactness, $\therefore sd''sz = s(z - \sigma\varepsilon z - sd''z)$ ↑by induction
$= sz$,

$\therefore$ uniqueness.

We have an explicit formula for d''

$d''[a] = a - \sigma\varepsilon a$ (let $x = a$, $sa = [a]$)

& in general we get (by recursion)

$d''[a_1, \dots, a_n] = a_1[a_2, \dots, a_n] + \sum_{i=1}^{n-1} (-1)^{\alpha_1 + \cdots + \alpha_i + i}$

$[a_1,\dots,a_{i-1},a_ia_{i+1},\dots,a_n] + (-1)^{\alpha_1+\dots+\alpha_{n-1}+n}[a_1,\dots,a_{n-1}]\varepsilon(a_n).$

$\hat{A}$ has a Λ-basis of homogeneous elts.
$\bar{B}_n(A)$ " " " ", & hence
$B_n(A)$ has an A-basis " " ".

From properties (1), (2), & (3), we prove $\varepsilon d=0$ & $dd=0$.

$\varepsilon d = 0$: we prove $x \in \bar{B}(A)$ for by (1)

$$\varepsilon d(ax) = \underset{0}{(\varepsilon d a)}\cdot\varepsilon(x) + (-1)^{\alpha}(\varepsilon a)\underset{0}{(\varepsilon d x)} = 0,$$

$$x = sy, \text{ then } \varepsilon d s y = \underbrace{\varepsilon y - \varepsilon\sigma\varepsilon y}_{0} - \underset{0 \text{ by degree}}{\varepsilon s dy}$$

$dd = 0$: it suffices to show for $x \in \bar{B}(A)$

by (1) for $dd(ax) = d[(da)x + (-1)^{\alpha}a(dx)]$

$$= \underset{0}{(dda)}x + \underbrace{(-1)^{\alpha-1}(da)dx + (-1)^{\alpha}(da)(dx)}_{0} + (-1)^{2\alpha-1}a\,\underset{0}{ddx}$$

$x = sy$, $ddsy = dy - dsdy$, but

$$dsdy + s\underset{0 \text{ by induction}}{ddy} = dy, \quad \therefore ddsy = 0.$$

This is our M. Q.E.D.

$$\bar{M} = \Lambda \otimes \bar{B}(A)$$
$$\uparrow \varepsilon$$
$$A \otimes \bar{B}(A)$$

, and our $\bar{d}$ on $\bar{M}$ is $\bar{d} = \bar{d}' + \bar{d}''$ where $\bar{d}'$ is the same as there are no A-operators & $\bar{d}''$ is changed in the first term,

$$\bar{d}''[a_1,\dots,a_n] = (\varepsilon a_1)[a_2,\dots,a_n] + \sum_{i=1}^{n-1}(-1)^{\alpha_1+\dots+\alpha_i+i}[a_1,\dots,a_ia_{i+1},\dots,a_n] + (-1)^{\alpha_1+\dots+\alpha_{n-1}+n}[a_1,\dots,a_{n-1}](\varepsilon a_n).$$

& $\bar{d}''[a] = 0$, $\bar{d}'[a_1,\dots,a_n] = \sum_{i=1}^{n}(-1)^{\alpha_1+\dots+\alpha_{i-1}+i}[a_1,\dots,da_i,\dots,a_n]$

as $0 \to I \to A \xrightarrow{\varepsilon} \Lambda \to 0$ gives

$$0 \to I \otimes \bar{\mathcal{B}}(A) \to \underset{\mathcal{B}(A)}{\underset{\parallel}{A \otimes \bar{\mathcal{B}}(A)}} \to \underset{\bar{\mathcal{B}}(A)}{\underset{\parallel}{\Lambda \otimes \bar{\mathcal{B}}(A)}} \to 0$$

This submodule $\bar{\mathcal{B}}(A)$ of $\mathcal{B}(A)$ which is the image and kernel of s has the property:
$x \in \mathcal{B}(A) \ni dx = 0$ if $\deg x > 0$, $\varepsilon x = 0$ if $\deg x = 0$, then
$\exists!\ y \in \bar{\mathcal{B}}(A) \ni x = dy$.

Proof: $d s x + \underset{0}{\underset{\parallel}{s d x}} = x - \underset{0}{\underset{\parallel}{\sigma \varepsilon x}}$, $\therefore d(sx) = x$, & put $y = sx$,

& we have existence.

Suppose $dy = 0$, $y \in \bar{\mathcal{B}}(A)$, then $d s y = y$ ($\deg y > 0$)
but $sy = 0$, $\therefore y = 0$ & we have uniqueness.

Define the suspension in this general algebraic situation.
A, M acyclic, $\sigma : \Lambda \to M \ni \varepsilon\sigma = 1$, σ defines $\sigma_A : A \to M$ by $\sigma_A(a) = a\sigma(1)$ and assume σ_A is a monomorphism (e.g. $\sigma(1)$ is an elt. of the A-basis) & we identify A with a submodule of M.

We get for $q > 0$

$$\begin{tikzcd} \underset{0}{\underset{\parallel}{H_{q+1}(M)}} \arrow[r] & H_{q+1}(M/A) \arrow[r, "\approx"] \arrow[d] & H_q(A) \arrow[r] \arrow[dl, "S"] & \underset{0}{\underset{\parallel}{H_q(M)}} \\ & H_{q+1}(\bar{M}/\Lambda) & & \end{tikzcd}$$

$M \to \bar{M}$ maps
$A \to \Lambda$ & $\therefore$
$M/A \to \bar{M}/\Lambda$

$H_{q+1}(\bar{M}/\Lambda) = H_{q+1}(\bar{M})$ as Λ has degree 0

& define $S : H_q(A) \to H_{q+1}(\bar{M})$: explicitly by
$a \ni da = 0$, $a = dm$, $m \in M$, take $\bar{m}$ & this is a cycle of dim $q+1$ & its class is the image.

Remark: a, b both of positive degree, then their product has suspension 0.

Proof: $da = 0$, $db = 0$, $b = dm$, $m \in M$.

$d(am) = (-1)^\alpha a\, dm = (-1)^\alpha ab$, i.e. $ab = d((-1)^\alpha am)$ & $(-1)^\alpha am \to 0$ in $\bar{M}$ as $\deg a > 0$ & $\therefore \epsilon a = 0$.

Remark: S is natural in that $f: A \to A'$, a DGA-homom, $g: M \to M'$, compatible with f & σ, then

$$\begin{array}{ccc} H_q(A) & \xrightarrow[S]{} & H_{q+1}(\bar{M}) \\ \downarrow f_* & & \downarrow \bar{g}_* \\ H_q(A') & \xrightarrow[S]{} & H_{q+1}(\bar{M}') \end{array}$$

is commutative.

As usual, we can compute S with a contracting homotopy (there is 1 if M has a homogeneous A-basis), i.e. $\exists$ a $\mathbb{1}$-~~homo~~ endom. s of $M \ni ds + sd = 1 - \sigma\epsilon$ (1).

$A \xrightarrow[s|A]{} M \xrightarrow[\text{nat.}]{} \bar{M}$ gives $\bar{s}: A \to \bar{M} \ni$

(2) $\bar{d}\bar{s}a + \bar{s}da = 0$, & $\bar{s}$ defines $\bar{s}_*: H_q(A) \to H_{q+1}(\bar{M})$ $(q > 0)$ & $\bar{s}_* = S$; for if $da = 0$, by (1) $dsa = a$, $\therefore$ take $m = sa$, & $\bar{m} = \bar{s}a$.

Returning to the case where M is the bar construction, $\bar{s}(a) = [a]$, $\bar{s}: A \to \bar{\mathcal{B}}(A)$ defines the susp. $S: H_q(A) \to H_{q+1}(\bar{\mathcal{B}}(A))$.

Bar construction is a covariant functor of $A \to \bar{M}$, in fact $f: A \to A'$, a DGA-homom. gives $g: \mathcal{B}(A) \to \mathcal{B}(A')$

by $g(a[a_1,\dots,a_n]) = (fa)[fa_1,\dots,fa_n]$ & defines
$\bar g : \bar{\mathcal{B}}(A) \to \bar{\mathcal{B}}(A')$ by $\bar g[a_1,\dots,a_n] = [fa_1,\dots,fa_n)$ &
$\therefore$ also $\bar g_* : H(\bar{\mathcal{B}}(A)) \to H(\bar{\mathcal{B}}(A'))$ which by Th. 1 depends only on f.

Th. 2: (by E – M.L.) Assume $f_* : H(A) \to H(A')$ is an isom. onto, then $\bar g_* : H(\bar{\mathcal{B}}(A)) \to H(\bar{\mathcal{B}}(A'))$ is also (like homotopy & homology groups).

Proof: We have 2 parts of the degree:

$$\deg'[a_1,\dots,a_n] = \sum_{i=1}^{n} \deg a_i$$

$$\deg''[a_1,\dots,a_n] = n.$$

$$\bar d' \qquad \bar d''$$

$$(-1,0) \qquad (0,-1).$$

Define $F_q(\bar{\mathcal{B}}(A)) = F_q = \sum_{n\le q} \bar{\mathcal{B}}_n(A)$,

$F_q/F_{q-1} \approx \bar{\mathcal{B}}_q(A)$, with differential $\bar d'$ as $\bar d'' : F_q \to F_{q-1}$,

& $F'_q/F'_{q-1} \approx \bar{\mathcal{B}}_q(A')$ with $\bar d'$.

f defines $\bar{\mathcal{B}}_q(A) \to \bar{\mathcal{B}}_q(A')$ by $\hat A \otimes \cdots \otimes \hat A \to \hat A' \otimes \cdots \otimes \hat A'$
But $H(\hat A) \to H(\hat A')$ is an isom. onto by the 5 lemma:

$$0 \to \Lambda \to A \to \hat A \to 0 \quad \text{gives}$$

$$\begin{array}{ccccccccc}
\Lambda & \to & H(A) & \to & H(\hat A) & \to & \Lambda & \to & H(A) \\
\downarrow\approx & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow\approx \\
\Lambda & \to & H(A') & \to & H(\hat A') & \to & \Lambda & \to & H(A')
\end{array}$$

$\therefore \approx$ & hence

$H(\hat A \otimes \cdots \otimes \hat A) \to H(\hat A' \otimes \cdots \otimes \hat A')$ is an isom. onto, i.e.
$H(F_q/F_{q-1}) \to H(F'_q/F'_{q-1})$ " " "
& by the 5 lemma & recursion, $H(F_q) \approx H(F'_q)$ for:

$$\begin{array}{ccccccccc}
H(F_q/F_{q-1}) & \to & H(F_{q-1}) & \to & H(F_q) & \to & H(F_q/F_{q-1}) & \to & H(F_{q-1}) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
H(F'_q/F'_{q-1}) & \to & H(F'_{q-1}) & \to & H(F'_q) & \to & H(F'_q/F'_{q-1}) & \to & H(F'_{q-1})
\end{array}$$

$\therefore \simeq$.

& then pass to the limit.

Q.E.D.

<u>Corollary 2'</u>: $f : A \to A'$, both with homogeneous Λ-basis conty 1, M over A, M' over A' (not necessarily bar constructions), both acyclic, homogeneous A (or A') basis with $\bar{g}_* : H(\bar{M}) \to H(\bar{M}')$ defined by f. If $f_* : H(A) \to H(A')$ is an isom. onto, so is $\bar{g}_*$.

<u>Proof</u>: $A \xrightarrow{1} A \xrightarrow{f} A' \xrightarrow{1} A'$

$$M \underset{g}{\to} B(A) \xrightarrow{h} B(A') \underset{g'}{\to} M',$$ by Th 1, we get

g & g' compatible with 1_A, $1_{A'}$ & passing to homology we get 3 isoms, $\therefore$ composition is an isom. & is the map induced $1_{A'} f 1_A = f$.

Q.E.D.

<u>General def. of construction</u>: given A, a DGA-algebra over Λ,

(1) a DGA-module N (a Λ-module) with a homogeneous basis, a specified elt. $n_0 \in N$ in the basis, the only 1 of degree 0 in the basis $\ni \varepsilon(n_0) = 1$, & $\Lambda \to N$ by $1 \to n_0$, a monomorphism $\ni$ $\varepsilon(\text{this map}) = 1$

(2) $A \otimes_\Lambda N = M$, graded + augmented naturally, and assume a $d \ni$ with it, M is a DGA-module over A + $\ni$ $A \otimes N \to \Lambda \otimes N$ (by $\varepsilon \otimes 1$) is compatible with d,

(3) M is acyclic.

Consider 2 such A A'

N N'

$M = A \otimes N$ $M' = A' \otimes N'$, by thm 1,

$f: A \to A'$ gives $g: M \to M'$, and note g is determined if we know it on elts of form $1 \otimes n_j$; in general, the image of such are not in the submodule N'; if there is such a g, it is called <u>special</u>.

<u>Remark</u>: If N' is a bar construction, $\exists$ a unique special g, i.e. a $g: M \to \mathcal{B}(A') \ni N$ goes into $\overline{\mathcal{B}}(A')$.

<u>Proof</u>: We need $k: N \to \overline{\mathcal{B}}(A') \ni$ (1) $g(a \otimes n) = f(a) \otimes k\, n$ gives a g compatible with d, i.e. (2) $gd = dg$ + $g\varepsilon = \varepsilon$. As usual, if (2) is satisfied for $m = 1 \otimes n_j$ (n_j a basis elt. of N), it is in general.

We define uniquely k by recursion.

$k(n_0) = 1 \in \Lambda = \overline{\mathcal{B}}_0(A')$ is our only choice $\ni$ $g\varepsilon = \varepsilon$, + this is only basis elt of degree 0.

For induction step, note we must have $dg(1 \otimes n_j) = gd(1 \otimes n_j)$, but g is defined by k on right side, + we must choose an elt. of $\overline{\mathcal{B}}(A')$ whose bdry is a given cycle, + by a previous remark, there is 1 (+ only 1) such, hence k + g

exist & are unique. Q.E.D.

<u>Def.</u> of tensor product of 2 DGA-algebras A, A', each having a homogeneous Λ-base only 1.

$A \otimes_\Lambda A'$

a grading is defined as usual

Augm: $\varepsilon(a \otimes a') = (\varepsilon a)(\varepsilon' a')$

Mult: $(a \otimes a')(b \otimes b') = (-1)^{\alpha'\beta} ab \otimes a'b'$

Unit: $1 \otimes 1$, again part of homogeneous base

Diff: $d(a \otimes a') = (da) \otimes a' + (-1)^\alpha a \otimes (d'a')$

& I will check for example that d acts right on the products: $d((a \otimes a')(b \otimes b')) = (-1)^{\alpha'\beta} d(ab \otimes a'b')$

$= (-1)^{\alpha'\beta} d(ab) \otimes a'b' + (-1)^{\alpha'\beta + \alpha + \beta} ab \otimes d'(a'b')$

$= (-1)^{\alpha'\beta}[(da)b + (-1)^\alpha a(db)] \otimes a'b' + (-1)^{\alpha'\beta + \alpha + \beta} ab \otimes [(d'a') b' + (-1)^{\alpha'} a'(d'b')]$, while $d(a \otimes a') \cdot (b \otimes b') + (-1)^{\alpha + \alpha'}(a \otimes a') \cdot d(b \otimes b') = (-1)^{\alpha'\beta}(da) b \otimes a'b' + (-1)^{\alpha + (\alpha'-1)\beta}(ab) \otimes (d'a') b' + (-1)^{\alpha + \alpha'}[(-1)^{\alpha'(\beta - 1)} a(db) \otimes a'b' + (-1)^{\beta + \alpha'\beta} ab \otimes a'd'b']$ & these are =.

<u>Def.</u> of tensor product of 2 constructions:

A	A'
N	N'
$M = A \otimes N$	$M' = A' \otimes N'$

Let $A'' = A \otimes A'$ & will define a construction over A''. Define $N'' = N \otimes N'$, and must define a differentiation on $A'' \otimes N'' = A \otimes A' \otimes N \otimes N' \approx A \otimes N \otimes A' \otimes N' = M \otimes M'$, the isom. being given by

$a \otimes a' \otimes n \otimes n' \to (-1)^{\alpha'\nu} a \otimes n \otimes a' \otimes n'$, and on $M \otimes M'$ we already have a differentiation $d''(m \otimes m') =$

$(d\,m) \otimes m' + (-1)^{\mu}\, m \otimes (d'm')$, so this defines one on $A'' \otimes N''$, & we won't show all the properties (we need $(-1)^{\alpha\nu}$ to get mult. property). We show $M \otimes M'$ is acyclic for:

$\exists\ s, s' \ni d\,s\,m + s\,d\,m = m - \sigma\varepsilon\, m$, &

$d's'm' + s'd'm' = m' - \sigma'\varepsilon' m'$ as M, M' are acyclic & have basis of homogeneous elts, & let $s'' = s \otimes 1 + (\sigma\varepsilon) \otimes s'$, & we verify that $d''s'' + s''d'' = 1 - (\sigma\varepsilon) \otimes (\sigma'\varepsilon')$;

$d''s''(m \otimes m') + s''d''(m \otimes m') = d''(s\,m \otimes m' + (\sigma\varepsilon m) \otimes (s'm'))$
$+ s''(d\,m \otimes m' + (-1)^{\mu}\, m \otimes d'm') = d\,s\,m \otimes m' + (-1)^{\mu+1} s\,m \otimes d'm'$
$+ \underset{0}{d\sigma\varepsilon m} \otimes s'm' + (-1)^{0}\sigma\varepsilon m \otimes d's'm' + s\,d\,m \otimes m' +$
$\underset{0}{\sigma\varepsilon d\,m} \otimes s'm' + (-1)^{\mu}\, \cancel{s\,m \otimes d'm'} + (-1)^{\mu=0}\sigma\varepsilon m \otimes s'd'm'$
$= m \otimes m' - \sigma\varepsilon m \otimes m' + \sigma\varepsilon m \otimes m' - \sigma\varepsilon m \otimes \sigma'\varepsilon' m'$, O.K.
& this shows $M \otimes M'$ is acyclic.

Do this now for the bar construction:

$$\begin{array}{ll} A & A' \\ \bar{\mathcal{B}}(A) & \bar{\mathcal{B}}(A') \\ \mathcal{B}(A) = A \otimes \bar{\mathcal{B}}(A) & \mathcal{B}(A') = A \otimes \bar{\mathcal{B}}(A') \end{array}$$

& we get

$$A \otimes A'$$
$$\bar{\mathcal{B}}(A) \otimes \bar{\mathcal{B}}(A')$$
$$\mathcal{B}(A) \otimes \mathcal{B}(A') \simeq A \otimes A' \otimes \bar{\mathcal{B}}(A) \otimes \bar{\mathcal{B}}(A')$$

but we also have bar construction for $A \otimes A'$,

$$A \otimes A'$$
$$\bar{\mathcal{B}}(A \otimes A')$$
$$\mathcal{B}(A \otimes A') = A \otimes A' \otimes \bar{\mathcal{B}}(A \otimes A')$$

& by the remarks above, we have a unique special homom: $\mathcal{B}(A) \otimes \mathcal{B}(A') \to \mathcal{B}(A \otimes A')$ which maps $\bar{\mathcal{B}}(A) \otimes \bar{\mathcal{B}}(A')$ into $\bar{\mathcal{B}}(A \otimes A')$

Now assume A is anticommutative, i.e.
$ba = (-1)^{\alpha\beta} ab$. Then the map: $A \otimes A \to A$ by $a \otimes b \to ab$ is a DGA-homom. for it preserves multiplication (it doesn't in general) for
$(a \otimes b)(a' \otimes b') = (-1)^{\beta\alpha'} aa' \otimes bb' \longrightarrow$
$(-1)^{\beta\alpha'} aa'bb' = aba'b$, & this defines a special map : $\mathcal{B}(A \otimes A) \to \mathcal{B}(A)$ & by composing with above, we get special map g: $\mathcal{B}(A) \otimes \mathcal{B}(A) \to \mathcal{B}(A)$, i.e. one sending $\bar{\mathcal{B}}(A) \otimes \bar{\mathcal{B}}(A)$ into $\bar{\mathcal{B}}(A)$.
This map g makes $\mathcal{B}(A)$ & $\bar{\mathcal{B}}(A)$ a DGA-algebra:
$1 \varepsilon \Lambda = \bar{\mathcal{B}}_0(A)$, & to show $g(1 \otimes x) = x$. It suffices to show for homogeneous elts. of $\bar{\mathcal{B}}(A)$ & we do by recursion. Suppose true for x of deg $x < n$, then also for $y = \Sigma a_j \otimes x_j$ with deg $x_j < n$, & to show that $g(1 \otimes x)$ & x are same elts of $\bar{\mathcal{B}}(A)$, we show that they have same bdry, but $dg(1 \otimes x) =$
$gd(1 \otimes x) = g(1 \otimes dx) = dx$, or in case deg $x = 0$, check that they have same augm.
A similar recursion proof showing bdries are $=$ shows associativity & anti-commutativity.

$\therefore$ $\bar{\mathcal{B}}(A)$ is an anti-comm. DGA-algebra, hence we can repeat our process. Define
$\mathcal{B}'(A) = \mathcal{B}(A)$, $\bar{\mathcal{B}}'(A) = \bar{\mathcal{B}}(A)$, & $\mathcal{B}^{k+1}(A) =$
$\mathcal{B}(\bar{\mathcal{B}}^k(A))$, $\bar{\mathcal{B}}^{k+1}(A) = \bar{\mathcal{B}}(\bar{\mathcal{B}}^k(A))$,
denote $H(\bar{\mathcal{B}}^k(A)) = H(A, k)$, which is again an

anti-comm. alg. We also have the suspension, for $q > 0$, $S: H_q(A, n) \to H_{q+1}(A, n+1)$ as described before.

Notice we have ~~not~~ used other coeffs: let Λ be a Z-algebra. Let $B = \Lambda \otimes_Z A$, then $\bar{B}^n(B) = \Lambda \otimes_Z \bar{B}^n(A)$ + $H(B, n) = H(A, n; \Lambda)$ + similarily for other constructions.

In our case, let $C = \Lambda(\Pi)$, for Π an abelian group, in fact, enough to study $A = Z(\Pi)$. Denote:

$H(\Lambda(\Pi), n) = H(\Pi, n; \Lambda)$ +

$H(Z(\Pi), n) = H(\Pi, n)$, this notation being same as previously used for homology of $K(\Pi, n)$ but Eilenberg + MacLane have defined a map $\bar{B}^n(A) \to K(\Pi, n)$ which defines $\cong$ on homology + which commutes with the suspension [We will not prove that here].

We now add more conditions (+ notation change) to the notion of a <u>construction</u>.

$A = A^0_\bullet$, as usual, anti-comm.

$A'_\bullet$, a Λ-algebra, graded, augmented, homogeneous Λ basis cntg $1 = h_0$, anti-comm.

$A^0_\bullet \otimes A'_\bullet$ is graded, augmented, is anti-comm with defined mult + assume a $d \ni A^0_\bullet \otimes A'_\bullet$ is an acyclic DGA-algebra. Identify $A^0_\bullet$ by $a_0 \to a_0 \otimes 1$, + assume d on $A^0_\bullet \otimes A'_\bullet$ agrees with d on $A^0_\bullet$. $A^0_\bullet \otimes A'_\bullet \to A'_\bullet$ by $a_0 \otimes a_1 \to (\varepsilon a_0) a_1$, d passing to the quotient + $A'_\bullet$ becomes a DGA-alg over Λ.

With 2 such constructions, we have a special homom. if along with $f_0 : A^0_\bullet \to C^0_\bullet$, we have $f_1 : A^1_\bullet \to C^1_\bullet \ni f_0 \otimes f_1 : A^0_\bullet \otimes A^1_\bullet \to C^0_\bullet \otimes C^1_\bullet$ is multiplicative & compatible with diff.

In case $C^1_\bullet = \bar{B}(C^0_\bullet)$, $f_0 : A^0_\bullet \to C^0_\bullet$, we know $\exists! f_1 : A^1_\bullet \to \bar{B}(C^0_\bullet)$ which is compatible with diff., but this map is also multiplicative (same sort of recursion & taking bdy proof.)

Iterating these processes, we get special homoms. of iterated constr. into iterated bar constr. & if $f_{0*} : H(A^0_\bullet) \simeq H(C^0_\bullet)$, then $f_{k*} : H(A^k_\bullet) \to H(\bar{B}^k(C^0_\bullet))$ are all $\simeq$, being multiplicative also. In particular, $A^0 = C^0$, we see that any iterated construction gives same homology algebras as the iterated bar construction.

Also, if we have 2, $A^0, A^1, \ldots,\ C^0, C^1, \ldots \ni f_i : A^i \to C^i$, i.e. all special maps, then the following is commutative:

$$\begin{array}{ccc} H(A^n) & \to & H(C^n) \\ \wr\wr & & \wr\wr \\ H(A^0, n) & \to & H(C^0, n) \end{array}$$

We now go into some explicit constructions for different groups & rings.

$\Lambda = \mathbb{Z}$, $\Pi = \mathbb{Z}$ (written multiplicative with generator x)

$A^0 = \mathbb{Z}(\Pi)$, $\varepsilon(x^n) = 1$, deg $= 0$

"ring of polynomials in x with $\pm$ exponents + coeffs. $\in \mathbb{Z}$.

Let $A^1 = E(1)$, meaning the exterior algebra of degree 1, let y be the generator of degree 1, relation is $yy = 0$.

Augmentation: 0 on elts of degree > 0 as always, and identity on scalars

d on $A^0 \otimes A^1$: $dx = 0$ (x is of degree 0), $dy = x - 1$ ($\varepsilon d = 0$ obviously), $\mathbb{Z}$-basis is x^n, $x^n y$ (meaning $(x^n \otimes 1)(1 \otimes y) = x^n \otimes y$) with $d(x^n y) = x^{n+1} - x^n$, acyclic obviously.

Passing to quotient, $dy = 0$, $dx = 0$.

Let $A^2 = P(2)$, the twisted polynomial algebra with 1 generator z of degree 2; twisted meaning that the "powers" of z forming elts of higher degree multiply by $z_p z_q = C(p,q)\, z_{p+q}$ where $C(p,q) = \binom{p+q}{q} = \dfrac{(p+q)!}{p!\,q!}$. $\varepsilon =$ id. on scalars.

d on $A^1 \otimes A^2$: $dy = 0$, $dz = y$, $dz_n = y z_{n-1}$, this d is multiplicative since $C(p,q) = C(p-1,q) + C(p,q-1)$. Acyclic, O.K.

Passing to quotient, $dz_n = 0$ as $\varepsilon(y) = 0$, $\therefore d = 0$ + $A^2 = P(2) = H(\mathbb{Z}, 2; \mathbb{Z})$

Suspension: $S(y) = z$ obviously as $dz = y$ in $A^1 \otimes A^2$.

We continue this later.

$\Lambda = \mathbb{Z}$, $\Pi = \mathbb{Z}_h$, $x =$ gen. of Π.

$\mathbb{Z}(\Pi) = A^0 = \mathbb{Z}[x]/x^h - 1$.

Let $A' = \underset{y}{E(1)} \otimes \underset{z}{P(2)}$

d on $A^0 \otimes A'$: $d = 0$ on A^0, $dy = x-1$,

$dz = (1+x+x^2+\cdots x^{h-1})y$, $dz_k = (1+x+\cdots+x^{h-1})yz_{k-1}$

is acyclic $\quad (d(yz_k) = (x-1)z_k)$

& passing to quotient, $dy = 0$, $dz_k = hyz_{k-1}$

& $A' = E(1) \otimes P(2)$ gives $H(Z_h, 1; Z)$.

In many cases, it is easier to get a construction for $\Lambda = Z_p$, the field — for those already made, we need only take tensor product (top of page 27).

$\Lambda = Z_p$, $\Pi = Z$

$\therefore A' = E(1) \otimes Z_p = E_p(1)$ (coeffs. taken mod p).

& $A^2 = P_p(2)$

For $\Pi = Z_h$, we need only look at h of the form p^f as

$\Pi = \Pi_1 \times \Pi_2 \Rightarrow \Lambda(\Pi) = \Lambda(\Pi_1) \otimes \Lambda(\Pi_2)$ &

we would only need to tensor product of the constructions.

$\Lambda = Z_p$, $\Pi = Z_{p^f}$

$A' = E_p(1) \otimes P_p(2)$ with $d = 0$ as $p \mid h = p^f$

$\Pi = Z_{q^f}$, $q \neq p$.

$A' = E_p(1) \otimes P_p(2)$ which is acyclic as $p \nmid h = q^f$, & so for example $d(\frac{1}{h} z_k) = yz_{k-1}$,

& $\therefore$ $H_\ell(Z_{q^f}, 1; Z_p) = 0$ for $\ell > 0$

& $\therefore$ also $H_\ell(Z_{q^f}, n; Z_p) = 0$ for $\ell > 0$

& any n. (4) fibre space + spectral sequence.

Also note $H_\ell(Z_{p^f}, n; Z_p) = H_\ell(Z_p, n; Z_p)$,

$\therefore$ assume $f = 1$.

To get A^2 for $\Pi = Z_p$, $\Lambda = Z_p$, we make a construction for each factor, $E_p(1)$, $P_p(2)$ and take tensor product. In fact we make a construction for the following 2 cases (which will give us an iterative procedure)

(1) $A^0 = E_p(n-1)$ $\quad$ n even, $d=0$

(2) $A^0 = P_p(n)$ $\quad$ " $\quad$ "

Case (1) Let x = gen. of $E_p(n-1)$

Let $A' = P_p(n)$, y = gen.

d on $A^0 \otimes A' = E_p(n-1) \otimes P_p(n)$

$dx = 0$, $dy = x$, $d\,y_k = x\,y_{k-1}$, is acyclic, passing to quotient get $d=0$ on $P_p(n)$, i.e

$$E_p(n-1) \longrightarrow P_p(n) + S(x) = y.$$

Case (2) $A^0 = P_p(n)$, generator x. This is tougher & we break it up into steps.

Consider $1, x_{pk}, x_{2p^k}, \dots, x_{(p-1)p^k}$.

These generate a subalgebra as

$$x_{ap^k}\, x_{bp^k} = 0 \text{ for } a+b=p \text{ because } C(ap^k, bp^k) \equiv 0 \ (\text{mod } p)$$

& $C(ap^k, bp^k) \not\equiv 0 \ (\text{mod } p)$ if $a+b<p$. This subalgebra is $Q_p(np^k)$ where $Q_p(n) = Z_p[\underset{\text{degree } n}{u}]/(u^p)$

& $\therefore$ $A^0 = P_p(n) \approx Q_p(n) \otimes Q_p(np) \otimes \cdots \otimes Q_p(np^k) \otimes \cdots$

& we shall give a construction for $C^0 = Q_p(n)$, gen. x.

~~Let $C' = E_p(n+1)$, gen. y~~

~~d on $C^0 \otimes C'$: $dx = 0$, $dy = x$~~

Let $C' = \underset{y}{E_p(n+1)} \otimes \underset{z}{P_p(np+2)}$

d on $C^0 \otimes C'$: $dx = 0$, $dy = x$, $d(x_k y) = x_k \cdot x = (k+1)\, x_{k+1}$, ~~d~~ which is acyclic if $p \nmid k+1$, but $d(x_{p-1} y) = 0$, so we put $dz = -x_{p-1} y$ & this makes it acyclic.

Passing to quotient, $d = 0$, $\therefore$

$$Q_p(n) \longrightarrow E_p(n+1) \otimes P_p(pn+2) \text{ with}$$
$$S(x) = y.$$

& finally for case (2),

$$P_p(n) \longrightarrow E_p(n+1) \otimes P_p(pn+2) \otimes E_p(pn+1) \otimes$$
$$P_p(p^2 n + 2) \otimes E_p(p^2 n+1) \otimes P_p(p^3 n+2) \otimes \cdots$$
$$\approx E_p(n+1) \bigotimes_{k>0} [E_p(p^k n+1) \otimes P_p(p^k n+2)] \text{ with } d = 0.$$

& $S(x) = y =$ gen. of $E_p(n+1)$, $S(x_{p^k}) =$ gen. of $E_p(p^k n+1)$, $S(\text{other basis elts}) = 0$, as product of elts of degree >0.

This gives the results:

$H(Z,1;Z_p) = E_p(1)$

$H(Z,2;Z_p) = P_p(2)$

$H(Z,3;Z_p) = E_p(3) \bigotimes_{k>0} [E_p(2p^k+1) \otimes P_p(2p^k+2)]$

$\vdots$

$H(Z_p,1;Z_p) = E_p(1) \otimes P_p(2)$

$H(Z_p,2;Z_p) = P_p(2) \otimes E_p(3) \bigotimes_{k>0} [E_p(2p^k+1) \otimes P_p(2p^k+2)]$

$\vdots$

We now wish to give a combinatorial description of the generators appearing in the above algebras (their direct description gets complicated fast) and with the following properties:

(1) 2 generators, one of which is gotten from the other

by applications of S get the same description
(2) different generators of same n get different description
(3) one of the numbers, n, is the first n in which the generator appears
(4) give its stable degree q, i.e. its degree at any time is $n+q$.

$[n$ is that in $H(Z_p, n; Z_p)]$

Let $\varepsilon = \begin{cases} 0 & \text{odd} \\ 1 & \text{even} \end{cases}$ degree the first time it appears.

Initial class — those with $n=1$, $\varepsilon = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow$ only if $\Pi = Z_p$.

Consider a generator not of this type, appearing in n, it "comes" in construction of some $P_p(m)$ (or else is a suspension) in $H(\Pi, n-1; Z_p)$, its degree being $p^k m + 1 + \varepsilon$ & $k > 0$ if not from suspension & we associate to this a preceding generator, namely that of $E_p(p^{k-1}m+1)$ also coming from $P_p(m)$, and define a number $\lambda = p^{k-1}\frac{m}{2}$. Also define $x =$ difference of stable degrees with the preceding one $= p^k m - p^{k-1} m + \varepsilon = p^{k-1} m(p-1) + \varepsilon = 2(p-1)\lambda + \varepsilon$.

Hence for each generator we have 5 numbers,

$n, q, \varepsilon, \lambda, x$ ($\lambda = 0$, $x = \varepsilon$ for initial gen.)

We have relations:

$n + q = 2p\lambda + 1 + \varepsilon$ (for degree at first appearance is $p^k m + 1 + \varepsilon$)

$x = 2(p-1)\lambda + \varepsilon$

$q - q' = x$, q' is that for preceding generator

$n - n' = 2(\lambda - p\lambda') - \varepsilon'$

Each generator is described by its finite sequence of preceding generators:

$n_i,\ q_i,\ \varepsilon_i,\ \lambda_i,\ x_i$ for $0 \le i \le k$ ($k \ge 0$)

$\ni \lambda_0 = 0,\ n_0 = 1,\ q_0 = \varepsilon_0 = x_0$ ($\varepsilon_0 = 0$ if $\pi = Z$)

with relations

(1) $q_i + n_i = 2p\lambda_i + 1 + \varepsilon_i$

(2) $x_i = 2(p-1)\lambda_i + \varepsilon_i$

(3) $n_{i+1} - n_i = 2(\lambda_{i+1} - p\lambda_i) - \varepsilon_i$

(4) $q_{i+1} - q_i = x_{i+1}$

& from this we get 3 descriptions of all generators by giving all possible such combinations (from which we derive other)

Case I. Give $n_0 = 1,\ 2 \le n_1 \le \cdots \le n_k,\ \varepsilon_k$ ($k \ge 0$) n_1 is odd if $\pi = Z$.

[From this, since $\lambda_0 = 0$, $\lambda_{i+1} = p\lambda_i + \left[\dfrac{n_{i+1} - n_i + 1}{2}\right]$,

$n = n_k,\ q = 2p\lambda_k + 1 + \varepsilon_k - n$, ~~[illegible]~~

& get x_k by (2).]

Case II. Give $\lambda_0, \dots, \lambda_k,\ \varepsilon_0, \dots, \varepsilon_k \ni \lambda_0 = 0$, $\lambda_1 \ge 1,\ \lambda_{i+1} \ge p\lambda_i + \varepsilon_i,\ \varepsilon_0 = 0$ if $\pi = Z$

Case III. Give $x_0, \dots, x_k \ni x_0 = 0$ or 1 but $x_0 = 0$ if $\pi = Z$, $x_i \equiv 0$ or $1 \pmod{2p-2}$, $x_{i+1} \ge p x_i$, $x_1 \ge 2p-2$.

(This one ~~gives~~ shows how many generators of stable degree $q = \Sigma x_i$).

Construction with integral coeffs.

$H(Z, 1; Z) = E(1)$

$H(Z_h, 1; Z) = H(E(1)_x \otimes P(2)_y)$ with differential

$dy = hx$, $dx = 0$, + denote this algebra by $E(Z_h, 1)$, or more generally, for m even, $E(Z_h, m-1) = \underset{x}{E(m-1)} \otimes \underset{y}{P(m)}$ with $dy = hx$, $dx = 0$.

We will make constructions for
(1) $E(m-1) = E(Z, m-1)$, $d = 0$
(2) $P(m)$
(3) $E(Z_h, m-1)$
(4) $P(Z_h, m)$, which will describe later.

(1) $\underset{x}{E(m-1)} \otimes \underset{y}{P(m)}$ with $dy = x$ does it, passing to quotient, $P(m)$ with $d = 0$, $S(x) = y$.

(2) Try $\underset{x}{P(m)} \otimes \underset{y}{E(m+1)}$, $dy = x$, $dx_k = 0$,

$d(x_k y) = (k+1)x_{k+1}$, + hence we have cycles which aren't bdries, + (using a lemma of Cartan) we have in homology, cyclic groups in dim. m, $2m$, $3m$, ...
0, Z_2, Z_3, ...

Start killing x_2. Add $\underset{u}{E(2m+1)}$, $du = x_2$.

But $2x_2 = d(xy)$, $\therefore$ $2u - xy$ is a cycle.
Add $\underset{v}{P(2m+2)}$ with $dv = 2u - xy$
+ after doing this, one gets with homology
0, 0, Z_3, Z_2, Z_5, Z_3, Z_7, ...
(divide by 2 where can).

Now kill Z_3, add $\underset{u'}{E(3m+1)} \otimes \underset{v'}{P(3m+2)}$
$du' = x_3$, $dv' = 3u' - (\ )$ + get

$0, 0, 0, Z_2, Z_5, 0, Z_7, \cdots$ divided by 3.

& in general odd $E(p^k m+1) \otimes P(p^k m+2)$
$\qquad\qquad\qquad\qquad U \qquad\qquad V$

$\ni dU = x_{p^k}$, $dV = pU - (\quad)$
$\qquad\qquad\qquad\qquad\qquad\downarrow$
$\qquad\qquad\qquad\qquad\qquad 0$ in quotient

& passing to quotient gives $dU = 0$, $dV = pU$

& $\therefore$ we get

$$P(m) \longrightarrow E(m+1) \bigotimes_{\substack{p \\ k>0}} E(Z_p, p^k m+1)$$

& $S(x)$ = gen. of $E(m+1)$

$S(x_{p^k})$ = gen. of $E(p^k m+1)$

& taking mod p, we get our previous result,

$$P_p(m) \longrightarrow E_p(m+1) \bigotimes_{k>0} [E_p(p^k m+1) \otimes P_p(p^k m+2)]$$

(3) $E(Z_h, m-1) = E(\overset{u}{m-1}) \otimes P(\overset{m}{v})$, $du = 0$, $dv = hu$.

$d(v_{k+1}) = hu\, v_k$, $d(u v_k) = 0$,

& we have in homology Z_h in degrees $m-1, 2m-1, 3m-1, \ldots$

Introduce x of degree $m \ni dx = u$, then $d(v - hx) = 0$

" $\quad y \quad$ " $\quad$ " $\quad m+1 \ni dy = v - hx$

$E(u) \otimes P(v) \otimes P(x) \otimes E(y)$, change variables,

$z = v - hx$ & get

$$\underbrace{E(\overset{m-1}{u}) \otimes P(\overset{m}{x})}_{\text{acyclic}} \otimes \underbrace{P(\overset{m}{z}) \otimes E(\overset{m+1}{y})}_{\text{similar to (2)}}$$ with $dx = u$, $dy = z$
$\qquad du = 0$, $dz = 0$

& so as in (2), for each prime p & $k>0$, introduce generators $Z_{p,k}$ of degree $p^k m+1$, $Y_{p,k}$ of degree $p^k m+2 \ni dZ_{p,k} = z_{p^k}$,

$dY_{p,k} = p Z_{p,k} - V_{p,k}$ where $d V_{p,k} = p z_{p^k}$

+ passing to quotient$_k$ (we get different from (2) as $z_{p^k} \neq 0$)

$$dZ_{p,k} = (h)^{p^k} x_k, \quad dx = 0, \quad dy = -hx$$
$$dY_{p,k} = p Z_{p,k} \overset{p}{-} h\, W_{p,k}$$

in

$$\underset{m}{P(x)} \otimes \underset{m+1}{E(y)} \bigotimes_{\substack{p\\k>0}} [\underset{p^k m+1}{E(Z_{p,k})} \otimes \underset{p^k m+2}{P(Y_{p,k})}]$$

using the lemmas as before.

This large product is quite complicated, but part of it we simplify:

$p|h$: Let $Z'_{p,k} = Z_{p,k} - \frac{h}{p} W_{p,k}$, $dZ'_{p,k} = 0$,
$dY_{p,k} = p Z'_{p,k}$;

+ ∴ the factors for $p|h$ are $\bigotimes_{k>0} E(Z_p, p^k m+1)$.

+ call $\underset{m}{P(x)} \otimes \underset{m+1}{E(y)} \bigotimes_{\substack{p\nmid h\\k>0}} [\underset{p^k m+1}{E(Z_{p,k})} \otimes \underset{p^k m+2}{P(Y_{p,k})}]$

$$= P(Z_h, m)$$

with above differentiation, + the homology of this is not too bad — from first 2 terms get

$$\underset{m}{Z_h}, \quad \underset{2m}{Z_{2h}}, \quad \underset{3m}{Z_{3h}}, \ldots$$

, + adding the other terms serves to divide the orders by high powers of everything except primes in h, i.e. we get

$$Z_{(h,h^\infty)}, \quad Z_{(2h,h^\infty)}, \quad Z_{(3h,h^\infty)}, \ldots$$

(z.B. $h = 10$,

10, 20, 10, 40, 50, 20, 10, 80, 10, 100, ...)

by using the lemma.

$$\therefore E(Z_h, m-1) \longrightarrow P(Z_h, m) \bigotimes_{p|h,\, k>0} E(Z_p, p^k m+1)$$

& $S(\text{gen. } u)$ = first gen. of $P(Z_h, m)$.

(4) $P(Z_h, m) = \underset{m}{P(x)} \otimes \underset{m+1}{E(y)} \otimes \cdots$ $\quad dy = -hx,\ dx = 0$

Introduce u of degree $m+1$, $du = x$

" v " " $m+2$, $dv = hu + y$ & by lemma, this serves to divide the homology we had: Z_h, $Z_{(2h, h^\infty)}$, $\cdots$ by h, getting

0, $Z_{(2, h^\infty)}$, $Z_{(3, h^\infty)}$, $\cdots$, & $\therefore$ we must

add terms for $p \mid h$ & passing to quotient & we get

$P(Z_h, m) \longrightarrow E(Z_h, m+1) \bigotimes_{\substack{p \mid h \\ k > 0}} E(Z_p, p^k m + 1)$.

Look at the results for low n.

$\pi = Z$

$n = 1$, $\quad E(1)$

$n = 2$, $\quad P(2)$

$n = 3$, $\quad E(3) \bigotimes_{\substack{p \\ k > 0}} E(Z_p, 2p^k + 1)$

$n = 4$, $\quad P(4) \bigotimes_{k > 0} P(Z_p, 2p^k + 2) \bigotimes_{\substack{p \\ k, l > 0}} E(Z_p, 2p^{k+l} + 2p^l + 1)$

$\vdots$

$\pi = Z_h$

$n = 1$, $\quad E(Z_h, 1)$

$n = 2$, $\quad P(Z_h, 2) \bigotimes_{\substack{p \mid h \\ k > 0}} E(Z_p, 2p^k + 1)$

$n = 3$, $\quad E(Z_h, 3) \bigotimes_{p \mid h,\, k > 0} P(Z_p, 2p^k + 2) \bigotimes_{p \mid h,\, k > 0}$

$E(\mathbb{Z}_p, 2p^{k+l} + 2p^l + 1)$

$\vdots$

We have a similar combinatorial description of generators, which I won't prove:

$n_i, q_i, \varepsilon_i, \lambda_i, x_i \quad, i = 1, \dots, k.$ for each p/h

$\ni q_i + n_i = 2p\lambda_i + 1$

$x_i = 2(p-1)\lambda_i + \varepsilon_i$

$n_{i+1} - n_i = 2(\lambda_{i+1} - p\lambda_i) - \varepsilon_{i+1}$, & we get

I. $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, n_1 odd if $\Pi = \mathbb{Z}$

III. Let $y_0 = \varepsilon_1$, $y_i = x_i + \varepsilon_{i+1} - \varepsilon_i$ for $i < k$, $y_k = x_k - \varepsilon_k$ then $y_0 = 0$ or 1 ($= 0$ if $\Pi = \mathbb{Z}$), $y_1 \geq 2p-2$, $y_{i+1} \geq p y_i$

$y_i \equiv 0$ or $1 \pmod{2p-2}$, $y_k \equiv 0 \pmod{2p-2}$.

Now to determine the multiplicative structure of cohomology.

A as usual, anticomm.

Suppose given $D: A \to A \otimes A$, a DGA-homom.

(in our case $\Lambda(\Pi) \to \Lambda(\Pi) \otimes \Lambda(\Pi) = \Lambda(\Pi \times \Pi)$ by $\Pi \to \Pi \times \Pi$, $x \to (x, x)$)

We have then $\bar{B}^n(A) \to \bar{B}^n(A \otimes A) \xleftarrow{\text{special maps}} \bar{B}^n(A) \otimes \bar{B}^n(A)$

& taking Hom's, we get multiplication in cohomology $H^*(A, n) = H(\mathrm{Hom}(\bar{B}^n(A), \Lambda))$ as before.

Take $A = \Lambda(\Pi)$; D, the diagonal map, & get $H^*(\Pi, n; \Lambda)$ with mult.

E–M. have defined a map $\bar{B}^n(Z(\Pi)) \to K(\Pi, n)$ which defines an isom. for homology & cohomology, $H^*(\Pi, n) \simeq H^*(K(\Pi, n))$, and this is multiplicative, because the following diagram is commutative:

$$\begin{tikzcd}
\bar{B}^n(A) \arrow[r] & \bar{B}^n(A \otimes A) \arrow[d] & \bar{B}^n(A) \otimes \bar{B}^n(A) \arrow[l] \arrow[d] \\
K(\Pi, n) \arrow[r, "\text{diag}"'] & K(\Pi \times \Pi, n) & K(\Pi, n) \otimes K(\Pi, n) \arrow[l] \\
 & (x, 1) & x \arrow[l] \\
 & (1, x) & x \arrow[l]
\end{tikzcd}$$

To compare multiplication induced by other constructions & that of bar construction, we add to the notion of multiplicative construction:

$A \otimes N$, as usual. It will be called <u>perfect</u> if given on $A \otimes N$ there is a contracting homotopy $s \ni$

(1) $sd + ds = 1 - \sigma\varepsilon$

(2) $s = 0$ on scalars

(3) $ss = 0$

(4) $1 \otimes N \subset \operatorname{Im}(s)$

(5) $\operatorname{Im}(s) . \operatorname{Im}(s) \subset \operatorname{Im}(s)$.

<u>Remarks</u>: If $x \in A \otimes N$, $dx = 0$ $\quad \deg x > 0$ $\implies$
$\qquad\qquad\qquad\qquad\quad \varepsilon x = 0$ $\quad \deg x = 0$

$\exists!\, y \ni x = dy$, $y \in \operatorname{Im}(s)$.

<u>Proof</u>: Same as before, page 19.

Note, the bar construction is perfect.

If we have 2 such, A, N, s
A', N', s', put
$s'' = s\otimes 1 + (\sigma\epsilon)\otimes s''$ on $A\otimes N\otimes A'\otimes N'$, & then the tensor product of 2 perfect constructions is perfect.

$$\Big(s''s'' = (s\otimes 1 + (\sigma\epsilon)\otimes s')(s\otimes 1 + (\sigma\epsilon)\otimes s') =$$
$$ss\otimes 1 + s\sigma\epsilon\otimes s' + \sigma\epsilon s\otimes s' + \sigma\epsilon\sigma\epsilon\otimes s's'$$
$$= 0 + \underset{(2)}{0} + 0 + 0 = 0 \Big)$$

Def: (A, N) $\quad (A', N', s')$, perfect, $f: A \to A'$; a homom. $g: A\otimes N \to A'\otimes N'$ is <u>perfect</u> if $g(1\otimes N) \subset \operatorname{Im}(s')$.

<u>Remarks</u>. 1. A special map is perfect.
2. Identity map is perfect (4)
3. Tensor product of 2 homoms. is perfect.

<u>Th</u>: $(A, N), \underset{\text{perfect}}{(A', N', s')}, f: A \to A' \Longrightarrow$

$\exists!\ g: A\otimes N \to A'\otimes N'$ which is perfect.

<u>Proof</u>. Same proof as for perfect into bar construction, see page 23, need remarks on bottom last page mainly. Note, to show g is multiplicative, need (5).

For (A', N') perfect, we have for $f: A \to A'$, the following commutative diagram (passing to quotient)

$$\begin{array}{ccc} N & \longrightarrow & N' \\ \downarrow{\scriptstyle \text{special}} & & \uparrow \leftarrow \text{induced by } 1 \\ \bar{\mathcal{B}}(A) & \longrightarrow & \bar{\mathcal{B}}(A') \end{array}$$

because in

$$\begin{array}{ccc} A\otimes N & & A'\otimes N' \\ \downarrow & & \uparrow \leftarrow \text{perfect} \\ A\otimes \bar{B}(A) & \xrightarrow[\text{special}]{} & A'\otimes \bar{B}(A') \end{array}$$

, $1 \otimes N \to hu(o')$, ∴ is the unique perfect map & is comm.

$A^0 = A,\ A^0 \otimes A^1,\ A^1 \otimes A^2, \ldots$

$A'^0 = A',\ A'^0 \otimes A'^1,\ A'^1 \otimes A'^2, \ldots$ perfect

$f : A \to A'$, & get successive perfect homoms.

$f^i : A^i \otimes A^{i+1} \to A'^i \otimes A'^{i+1}$ & ∴

$$\begin{array}{ccc} A^n & \to & A'^n \\ \downarrow & & \uparrow \\ \bar{B}^n(A) & \to & \bar{B}^n(A') \end{array}$$

is commut.

& ∴ passing to homology, the induced homoms. for a given 2 constructions are "same" as those of bar construction.

In particular, let $A' = A^0 \otimes A^0$, with $A = A^0$, perfect, the product is ∴ perfect, & $D : A \to A \otimes A$ by this result gives $D^n : A^n \to A^n \otimes A^n$, which defines multiplication in cohomology.

Let $\Lambda = Z$, $A = Z(\Pi)$, $\Pi = Z_h$, gen. x.

$A^0 = Z(\Pi)$

$A' = \underset{y}{E(1)} \otimes \underset{z}{P(2)}$, $dy = x - 1$, $dz = (1 + x + \cdots + x^{h-1})y$

& define $s(x^a z_k) = \begin{cases} 0 & \text{if } a = 0 \\ (1 + x + \cdots + x^{a-1})\, y z_k, & 1 \le a \le h-1 \end{cases}$

$$s(x^a y z_k) = \begin{cases} 0 & 0 \le a \le h-2 \\ z_{k+1} & a = h-1 \quad \text{(just verify)} \end{cases}$$

& the diagonal map on π gives rise to the following on A':

$$y z_k \longrightarrow \sum_{i=0}^{k} (z_i \otimes y z_{k-i} + y z_i \otimes x z_{k-i})$$

$$z_k \longrightarrow \sum_{i=0}^{k} z_i \otimes z_{k-i} + \xi \sum_{i=0}^{k-1} y z_i \otimes y z_{k-i-1}$$

where $\xi \in A^0 \otimes A^0 = Z(\pi) \otimes Z(\pi)$

$$\xi = \sum_{0 \le q < r \le h-1} x^q \otimes x^r$$

& passing to quotient, replace x by 1, ξ by $\frac{h(h-1)}{2}$

$$y z_k \longrightarrow (1 \otimes y + y \otimes 1)\left(\sum_{i=0}^{k} z_i \otimes z_{k-i}\right)$$

$$z_k \longrightarrow \sum_{i=0}^{k} z_i \otimes z_{k-i} + \frac{h(h-1)}{2}(y \otimes y)\left(\sum_{i=0}^{k-1} z_i \otimes z_{k-i-1}\right)$$

If we reduce mod p, considering $h = p^f$,

then $\frac{h(h-1)}{2} \equiv 0 \mod p$ unless $p = 2, f = 1$

& passing to cohomology, we get

$$E_p^*(1) \otimes P_p^*(2) = H^*(Z_{p^f}, 1; Z_p) \quad \text{if } p^f \ne 2$$

↑ ordinary polyn. alg.

& if $p^f = 2$, get $P_p^*(1)$

$$H^*(Z, 1; Z_p) = E_p^*(1)$$

More results:

$p = 2$

$$H^*(Z,1;Z_2) = E_2^*(1)$$

$$H^*(Z,2;Z_2) = P_2^*(2)$$

$$H^*(Z,3;Z_2) = \bigotimes_{k\geq 0} P_2^*(2^{k+1}+1)$$

$$H^*(Z,4;Z_2) = \bigotimes_{k,l\geq 0} P_2^*(2^{k+l+1}+2^l+1)$$

$$H^*(Z_{2^f},1;Z_2) = \begin{cases} E_2^*(1) \otimes P_2^*(2) & f \geq 2 \\ P_2^*(1), & f=1 \end{cases}$$

$$H^*(Z_{2^f},2;Z_2) = \bigotimes_{k\geq 0} P_2^*(2^k+1) = P_2^*(2) \otimes P_2^*(3) \otimes P_2^*(5) \cdots$$

$$H^*(Z_{2^f},3;Z_2) = \bigotimes_{k,l\geq 0} P_2^*(2^{k+l}+2^l+1) = P_2^*(3) \otimes P_2^*(4) \cdots$$

---

p odd.

$$H^*(Z,1;Z_p) = E_p^*(1)$$

$$H^*(Z,2;Z_p) = P_p^*(2)$$

$$H^*(Z,3;Z_p) = E_p^*(3) \bigotimes_{k>0} E_p^*(2p^k+1) \otimes P_p^*(2p^k+2)$$

$$H^*(Z_{p^f},1;Z_p) = E_p^*(1) \otimes P_p^*(2)$$

$$H^*(Z_{p^f},2;Z_p) = P_p^*(2) \otimes E_p^*(3) \bigotimes_{k>0} E_p^*(2p^k+1) \otimes P_p^*(2p^k+2)$$

$f : A \to A \otimes A$ defines $\cdot$ in dual.

$A^* = \mathcal{H}om_\Lambda(A, \Lambda)$,

$$A^* \xleftarrow{f^*} \mathcal{H}om(A \otimes A, \Lambda) \leftarrow \mathcal{H}om(A, \Lambda) \otimes \mathcal{H}om(A, \Lambda)$$

$a_1 \otimes a_2$ $\quad f_1 \quad f_2$

$$f(a_1 \otimes a_2) = (f_1 a_1)(f_2 a_2)$$

$\cdot : A^* \otimes A^* \to A^*$

---

p odd. $\quad E_p(m-1) \to P_p(m)$ $\quad$ remember.

but if also, $E_p(m-1) \to E_p(m-1) \otimes E_p(m-1)$

we have $\quad x \to x \otimes 1 + 1 \otimes x$, the construction

will give

$$P_p(m) \to P_p(m) \otimes P_p(m)$$

$$x_k \to \sum_{i=0}^{k} x_i \otimes x_{k-i}$$

& mult. in $P_p^*(m)$, is ordinary mult. of generators.

Starting with $P_p(m)$, get $E_p(m+1) \otimes \underset{k>0}{E_p(p^k m+1)} \otimes P_p(p^k m+1)$

& starting with above, get a mapping here into tensor itself

get

$$E_p(m+1) \to E_p(m+1) \otimes E_p(m+1)$$

$$P_p(p^k m+2) \to P_p(p^k m+2) \otimes P_p(p^k m+2)$$, divides into parts.

+ in cohomology: $E_p^*(m+1) \bigotimes_{k \geq 0} E_p^*(\ ) \otimes P_p^*(\ )$

ordinary mult.

$p = 2$ ~~$E_2(q) \otimes P_2(2q) \to P_2(q)$~~

$P_2(m) \to P_2(m) \otimes P_2(m)$ gives

$\bigotimes_{k \geq 0} P_2(2^k m + 1) \to$ itself twice, which divides up + gives ordinary multi.

on duals in each case.

$P_2^*(m) \to \bigotimes_{k \geq 0} P_2^*(2^k m + 1)$ gives above results.

Suspension: $H_q(\Pi, n; Z_p) \to H_{q+1}(\Pi, n+1; Z_p)$

+ dual $H^{q+1}(\Pi, n+1; Z_p) \to H^q(\Pi, n; Z_p)$

Bockstein operator:

$E_p(p^k m + 1) \otimes P_p(p^k m + 2)$

x y

$dy = px$ in integral

x is Bockstein of y of order p (?)

Pair up, this operator commutes with suspension.

Combinatorial description:

Still good for mod p if p = odd prime in cohom.

$\lambda_0 = 0,\ \varepsilon_0,\ \lambda_1, \varepsilon_1, \dots, \lambda_k, \varepsilon_k$, $\quad \varepsilon_i = \begin{cases} 0 \\ 1 \end{cases}$

$\varepsilon_0 = 0$ if $\Pi = \mathbb{Z}$.

$\ni \lambda_{i+1} \geq p\lambda_i + \varepsilon_i$

$\lambda_1 \geqq 1$ $\qquad (\Pi)$

Change for $p=2$.

$x_1, \dots, x_k \ni x_{i+1} \geqq 2x_i, \quad x_1 \geq \begin{cases} 2 & \text{if } \Pi = \mathbb{Z} \\ 1 & \text{if } \Pi = \mathbb{Z}_{2^f} \end{cases}$

$(q = \Sigma x_i, \quad n+q = 2x_k + 1)$

Over graded algebra, with relations only following from anti-commu.

A general such algebra is of this form.

It is possible to define in unique way the generators by Steenrod operations.

Results of these: $p=2$.

$Sq^i : H^q \longrightarrow H^{q+i}$ $\qquad$ coeffs. mod 2

$Sq^0 = id$; $\quad Sq^1$ = Bockstein hom. of order 2.

$Sq^i = 0$ if $i > q$; $\quad Sq^q u = u^2$

$$Sq^k(uv) = \sum_{0\le i\le k} (Sq^i u)(Sq^{k-i} v)$$

$$Sq^i \delta = \delta Sq^i$$

p odd

$$\mathcal{B}(p) : H^q \to H^{q+1}$$

$$\wp^\lambda : H^q \to H^{q+2\lambda(p-1)} \quad +$$

$$\wp^0 = id, \quad \wp^\lambda = 0 \text{ if } \lambda > \frac{q}{2},$$

q even, $\wp^{q/2} u = u^p$

$$\wp^\lambda(uv) = \sum_{0\le \mu \le \lambda} (\wp^\mu(u))(\wp^{\lambda-\mu}(v))$$

$$\mathcal{B}\delta = \delta\mathcal{B}, \quad \wp^\lambda\delta = \delta\wp^\lambda$$

$\uparrow$ susp.

$(i = x, \lambda = \lambda)$ ~~Ge~~ Get correspondence.

(If $\pi = Z_2 s$, takes $\mathcal{B}(2^{?})$ instead of Sq^1)

For given degree n $\quad Sq^{x_k} \cdots Sq^{x_2} Sq^{x_1}$

applied to fundamental class gives a generator, + this gives our natural choice (may not get ones chosen, but get generators which will do it).

+ show these are independent

Any $\mathcal{S}\varphi = \Sigma$ linear combination as here.

If hold in Eilen.-M.-Z. space, holds in general.

---

p = odd.

$$\mathcal{S}^{\varepsilon_k}_{(p)}\mathcal{P}^{\lambda_k} \ldots \mathcal{S}^{\varepsilon_1}_{(p)}\mathcal{P}^{\lambda_1}\mathcal{S}^{\varepsilon_0}(p^f)(u)$$

& the results are the same.

For given n, taking sequences which exist for that n, give all generators. Again a basis also.

Notes from talk by Cartan on Computation of Homotopy Groups, notes from those of Mac Lane, Aug 4, 1953.

$G_k = \pi_{n+k}(S_n)$, n large.

$\pi_3(S_2) = Z$, $\pi_{n+1}(S_n) = Z_2$, $(n \geq 3)$,

$\pi_5(S_3) = Z_2$, $\pi_6(S_3) = Z_{12}$.

$S(X, k)$ is singular subcomplex of X, $\ni$ $k-1$ skeleton goes into pt.

Let $H(X, k) = H(S(X, k))$.

By Hurewicz, $(i < k \Rightarrow \pi_i = 0) \Rightarrow \pi_k(X) \simeq H_k(X, k)$, $k \neq 1$

Def: $f : Y \to X \ni \pi_i(Y) = 0$ for $i < k$, $f_* : \pi_i(Y) \simeq \pi_i(X)$ $i \geq k$,
then Y kills homotopy groups of X up to $k-1$

Th: $f_* : H_n(Y) \simeq H_n(X, k)$ all n.

Let (X, k) denote such a Y. Z. B. Universal covering space kills π_1. Hopf map $S^3 \to S^2$ kills π_2.

Let $X \subset V$. $Y = \Gamma_{X,V}$ = set of all paths in V, initial pt in X, having same homotopy type as X.

$\Gamma_{X,V} \to V$, end pt., is a fibre map.

$F = \Gamma_{X,a}$ = fibre.

$F \to X$, to origin is a fibre map, Ω = fibre = all loops.

Th: If $\pi_k(V) = 0$ for $k > n$, $\pi_k(X) \simeq \pi_k(V)$ for $k \leq n$ (kill higher groups), then

$\pi_k(F) = 0$ for $k \leq n$, $\pi_k(F) \cong \pi_k(X)$ for $k > n$

by projection.

Proof:

$$\begin{tikzcd}
\pi_i(F) \arrow[r] \arrow[dr] & \pi_i(\Gamma_{X,V}) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_i(V) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_{i-1}(F) \arrow[r] \arrow[dr] & \pi_{i-1}(\Gamma_{X,V}) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_{i-1}(V) \arrow[d, phantom, "\cong"] \\
& \pi_i(X) \arrow[r] & \pi_{i-1}(\Omega) \arrow[ur] & & \pi_{i-1}(X) \arrow[r] & \pi_{i-1}(\Omega) \arrow[d] \\
& & & & & \pi_{i-2}
\end{tikzcd}$$

need commut. $i = k+1$

i.e. V kills to right $\Rightarrow$ F kills to left.

Given X, take V_{n+1} (kills all $k > n+1$)

$\cap$

V_n (" " $k \geq n+1$)

& replace in thm. X by V_{n+1}, V by V_n

& get

Th: $\pi_k(F) = 0$ $k \leq n$, $\pi_{n+1}(F) \cong \pi_{n+1}(V_{n+1}) \cong \pi_{n+1}(X)$

$\pi_k(F) = 0$, $k > n+1$, where

F is a fibre space over given X.

: we have $\underset{\uparrow\text{fibre}}{K(\pi_{n+1}, n+1)} \to \underset{\uparrow\text{space}}{V_{n+1}} \to \underset{\uparrow\text{base}}{V_n}$

& this gives results of Postnikov,

$K(\pi_2(X), 2) \to V_2 \to V_1 = K(\pi_1(X), 1)$

$K(\pi_3(X), 3) \to V_3 \to V_2$

Take $X = (X, n)$. Then $V = K(\pi_n(X), n)$, $\Omega =$

$K(\pi_n(X), n-1)$ & by thm. & construction before it get $(X, n+1) \to (X, n) \to K(\pi_n(X), n-1)$

$K(\pi_n, n-1) \to (X, n+1) \to (X, n)$

z.B. $K(Z, 2) \to (S_3, 4) \to S_3$

& $H(S_3, 4)$ is

1	2	3	4	5	6	7	8	
0	0	0	Z_2	0	Z_3	0	Z_4	0 . . .

by spectral sequence

& gives as consequence

$\pi_{2p}(S_3)$ is first homotopy group of S_3 with $\neq 0$ p-primary component & it is Z_p (Serre's generalization of Hurewicz Th)

Results of spheres: $\pi_6(S_3) = Z_{12}$, $\pi_7(S_4) = Z + Z_{12}$, $\pi_8(S_5) = Z_{24} = G_3$; $\pi_7(S_3) = Z_2$, $\pi_8(S_4) = Z_2 + Z_2$, $\pi_9(S_5) = Z_2$, $\pi_{10}(S_6) = 0 = G_4$; $\pi_8(S_3) = Z_2$, $\pi_9(S_4) = Z_2 + Z_2$, $\pi_{10}(S_5) = Z_2$, $\pi_{11}(S_6) = Z$, $\pi_{12}(S_7) = 0 = G_5$; $G_6 = 0$, $G_7 = Z_{240}$, $G_8 = Z_2 + Z_2$, $G_9 = Z_2 + Z_2 + Z_2$, $G_{10} = Z_2 + Z_9$, $G_{11} = Z_8 + Z_{27} + Z_7$, $G_{12} = 0$, $G_{13} = G_3$

Cartan

Théorème préparatoire

Notations : $P(n)$ désigne l'algèbre (sur l'anneau Z) des polynômes canulés a un générateur u de degré n. Pour tout entier t, on notera $u(t)$ le générateur de degré nt; donc $u(1)=u$. On a

$$u(t).u(t') = C(t,t')\, u(t+t'), \text{ avec } C(t,t') = \frac{(t+t')!}{t!\,t'!}$$

Enoncé du théorème préparatoire.

Hypothèses : on suppose $P(n)$ muni d'un opér. diff. nul, et plongé dans une algèbre (sur Z) différentielle graduée A, de manière que :

(α) $P(n) \longrightarrow H(A)$ est un épimorphisme;

(β) l'image de u dans $H(A)$ est d'ordre fini, et par suite $H(A)$ est un groupe de torsion. On suppose que si t est premier à p, l'ordre de $u(tp^h)$ dans $H(A)$ est :

- premier à p si $h < k$;

- de composante p-primaire p^{h-k+f} si $h \geq k$

(k et f sont des entiers > 0 donnés).

Construction : soit cp^f l'ordre de $u(p^k)$ dans $H(A)$, c étant donc premier à p. Soit $v \in A$ tel que $dv = cp^f u(p^k)$. Introduisons un x de degré np^k+1 et un y de degré np^k+2; sur l'alg. $B = A \otimes E(x) \otimes P(y)$, définissons une différentielle en posant

(2) $\quad dx = c\, u(p^k), \qquad dy = p^f x - v.$

Conclusion : $P(n) \longrightarrow H(B)$ est un épimorphisme; pour $a \in P(n)$, les conditions $a \underset{B}{\sim} 0$ et $p^f a \underset{A}{\sim} 0$ sont équivalentes. (Donc l'ordre de $u(t)$ dans $H(A)$ est le même que dans $H(B)$ si $t \not\equiv 0$ (p^k), est p^f fois son ordre dans $H(B)$ dans le cas contraire).

Appendice : supposons que, pour tout entier $h \geq k$, on ait dans l'algèbre A

$p^{h-k+f}u(p^h) = da_h$, où $a_h \in A$ appartient à l'idéal engendré par les éléments de degré > 0 de $P(n)$. Alors, dans B, pour tout $h \geq k+1$, on a

(5) $\quad p^{h-k}u(p^h) = d(u(p^h - p^k).x) + p\, db_h$,

où b_h appartient à l'idéal de B engendré par les éléments de degré > 0 de $P(n)$.