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Dear Haynes,

Here is a copy of my ancient (1967) preprint on "pensions" and on derived functors of the mod-p symmetric algebra functor, etc.

Best regards,

Pete

Operations on Derived Functors of Non-additive Functors

by

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(1967)

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§1. Introduction

In their generalization of the classical theory [4] of derived functors, Dold and Puppe have used semi-simplicial methods to define [7] derived functors for non-additive functors between abelian categories. Such derived functors have been computed in a few cases of topological interest ([3] and [7]), and in these computations the suspension map [7] has played a central role. We shall show that the suspension map belongs to a large family of operations which we call pensions. The pensions will be used to compute various derived functors and elucidate their structure. As one application we determine (functorially) the mod-p homology of the symmetric products of any polyhedron (§8).

We consider only covariant functors on the category of R-modules, where R is a fixed commutative ring with identity. Such a functor T has derived functors [7] $L_qT(\cdot,n)$ for $q,n \geq 0$. To each homology element $\alpha \in \tilde{H}_i(R,j;R)$ there corresponds a pension (§2)

$$\alpha: \quad L_qT(\cdot,n) \to L_{q+i}T(\cdot,n+j),$$

and these operations form an algebra under composition. One such pension is the suspension map

$$\sigma: \quad L_qT(\cdot,n) \to L_{q+1}T(\cdot,n+1)$$

which is an isomorphism [7] for $q < 2n$. To illustrate the higher pensions, let SP^r be the r-fold symmetric tensor product functor on abelian groups. There is a pension (§3)

$$\epsilon_r: \quad L_qSP^r(\cdot,n) \to L_{q+2r}SP^r(\cdot,n+2)$$

which is a monomorphism (4.3) for $n \geq 2$, $q \geq 0$, and is an isomorphism (3.2) for $q > r(n+1) - n$.

Using the stability property of the suspension map, Dold has constructed [6] stable derived functors $L^s{}_n T$, $n \geq 0$, which are equivalent to classical left derived functors when T is additive. We construct a theory of stable pensions (§9) which describes the natural operators on stable derived functors.

We consider particularly the case of a functor T on the category of Z_p-modules. The stable pensions give $\sum_{n=0}^{\infty} L^s{}_n T(Z_p)$ the structure of a graded module over the algebra $A_*(p)$ dual to the mod-p Steenrod algebra. If $T \circ T'$ is the composition of two functors on Z_p-modules, there is an isomorphism (10.3) of $A_*(p)$-modules

$$\sum_{n=0}^{\infty} L^s{}_n(T \circ T')(Z_p) \approx (\sum_{n=0}^{\infty} L^s{}_n T(Z_p)) \otimes_{Z_p} (\sum_{n=0}^{\infty} L^s{}_n T'(Z_p)).$$

The mod-p symmetric algebra functor (5.5), SP, has a composition map

$$SP \circ SP \rightarrow SP$$

which gives $\sum_{n=0}^{\infty} L^s{}_n SP(Z_p)$ the structure of an algebra over the Hopf algebra $A_*(p)$. We compute (§12) this stable algebra and other related ones. We remark that another example of such a stable algebra is given by the E^1 term of the Adams spectral sequence constructed in [3].

The paper is divided into two parts dealing respectively with unstable and stable derived functors. The stable case is perhaps of more intrinsic interest than the unstable, and much of the material in the first part has been included because of its relevance to the stable case.

The unstable persion operations were introduced in the author's thesis [1], and he wishes to thank D.M. Kan who served as his thesis advisor and contributed useful ideas.

§2. The Pensions

Let $\mathcal{A}_R$ be the category of R-modules, where R is a commutative ring with identity. For $M \in \mathcal{A}_R$ let RM denote the free R-module generated by the elements of M with the relation $1[0] = 0$. If

$$T: \mathcal{A}_R \to \mathcal{A}_R$$

is a covariant functor with $T(0) = 0$, there is a homomorphism for $M, N \in \mathcal{A}_R$

$$E: RM \otimes_R TN \to T(M \otimes_R N)$$

defined as follows. The restriction of E to $1[m] \otimes TN \approx TN$ is given by

$$T(m \otimes \circ): TN \to T(M \otimes_R N).$$

If X is a semi-simplicial (abbreviated "s.s.") R-module, we prolong E to

$$E: RK(R,n) \otimes_R TX \to T(K(R,n) \otimes_R X),$$

where K(R,n) is the unique s.s. R-module whose normalization [12, p. 236] NK(R,n) satisfies

$$(NK(R,n))_i = \begin{cases} R \text{ for } i = n \\ 0 \text{ for } i \neq n. \end{cases}$$

In view of the Eilenberg-Zilber theorem [12, p. 238], E induces a pairing

$$E_*: \pi_* RK(R,n) \otimes_R \pi_* TX \to \pi_* T(K(R,n) \otimes_R X),$$

where the homotopy functor π_* is defined for an s.s. R-module Y by

$$\pi_* Y = H_*(NY).$$

In particular note that

$$\pi_* RK(R,n) = \widetilde{H}_*(R,n;R).$$

For convenience we replace E_* by a pairing

$$E_*': \widetilde{H}_*(R,n;R) \otimes_R \pi_* TX \to \pi_* TS^n X$$

where $S^n X$ is the n-fold suspension [7, 5.3] of X and E_*' is defined as follows. For $u \in \widetilde{H}_m(R,n;R)$ and $v \in \pi_i\, TX$ let

$$E_*'(u \otimes v) = (-1)^{(m-n)i} (T(\psi)_* \circ E_*)(u \otimes v)$$

where

$$\psi: K(R,n) \otimes_R X \to S^n X$$

is the homotopy equivalence of s.s. R-modules defined as in [7, 5.32].

<u>2.1. The pension algebra.</u> The <u>pension algebra</u> P(R) consists of the R-module

$$\sum_{m,n \geq 0} \widetilde{H}_m(R,n;R)$$

with multiplication given by

$$E_*': \widetilde{H}_p(R,i;R) \otimes_R \pi_q RK(R,j) \to \pi_{p+q} RK(R,i+j)$$

where we have taken $T(\cdot) = R(\cdot)$ and $X = K(R,j)$. <u>Thus P(R) is an associative bigraded R-algebra with identity; and P(R) is anti-commutative in the sense that</u>

$$\alpha \circ \beta = (-1)^{mp} (-\mu)^{nq} \beta \circ \alpha$$

for $\alpha \in \widetilde{H}_{m+n}(R,n;R)$ and $\beta \in \widetilde{H}_{p+q}(R,q;R)$, where $\mu = 1[-1] \in \widetilde{H}_0(R,0;R) = R(R)$. A ring homomorphism $R \to R'$ induces a homomorphism of algebras $P(R) \to P(R')$. We shall use the same notation for an element $\alpha \in P(Z)$ and its image $\alpha \in P(R)$ under the map induced by $Z \to R$.

2.2. The pensions. If X is an s.s. R-module then

$$\sum_{p,q\geq 0} \pi_p TS^q X$$

is a module over P(R) with multiplication given by the pairing

$$E_*': \widetilde{H}_m(R,n;R) \otimes_R \pi_p TS^q X \to \pi_{m+p} TS^{n+q} X.$$

Each element $\alpha \in H_m(R,n;R)$ determines a pension

$$\alpha: \pi_p TX \to \pi_{p+m} TS^n X$$

defined as multiplication by α. If X is taken as an s.s. projective resolution [7, 4.1] of (G,q) for $G \in \mathcal{A}_R$, then we obtain the pension

$$\alpha: L_p T(G,q) \to L_{p+m} T(G,q+n)$$

acting on derived functors [7, 4.6] of T.

2.3. The suspension map. Let $\iota_n \in \widetilde{H}_n(R,n;R)$ be the canonical element, and let σ denote ι_1. Then $\iota_n = (\sigma)^n$ in P(R) and the pension

$$\iota_n = (\sigma)^n: \pi_q TX \to \pi_{q+n} TS^n X$$

equals the n-fold suspension map of [7, 5.9]. If either $\alpha \in \widetilde{H}_{m+n}(R,n;R)$ or $\beta \in H_{p+q}(R,q;R)$ is a homology suspension, then 2.1 implies

$$\alpha \cdot \beta = (-1)^{mp} \beta \cdot \alpha$$

in P(R), since $\sigma \cdot \mu = -\sigma$ in P(R). In particular, for any $\alpha \in H_{m+n}(R,n;R)$

$$\alpha \cdot \sigma = \sigma \cdot \alpha$$

in P(R), and thus each pension commutes with the suspension map.

Remark 2.4. The pension algebra $P(R)$ can also be constructed topologically from the canonical pairing $(K(R), K(R)) \to K(R)$ of the Eilenberg-MacLane spectrum $K(R)$ (see [14, §6]).

§3. The Pensions ϵ_r

For $r \geq 1$ let

$$\epsilon_r \in \widetilde{H}_{2r}(Z,2;Z) \approx Z$$

denote the generator dual to the r^{th} power of the fundamental cohomology class. If R is a commutative ring with identity, then $\epsilon_r \in P(R)$ and clearly $\epsilon_1 = \sigma^2$. We shall show that the pensions ϵ_r satisfy a stability theorem.

A functor

$$T: \mathcal{A}_R \to \mathcal{A}_R$$

is of <u>degree</u> $\leq$ r [9, p. 83] if its r+1-fold cross-effect functor is zero.

An s.s. R-module X is <u>trivial above</u> n if X is homotopy equivalent (in the category of s.s. R-modules) to some Y whose normalization NY has $(NY)_i = 0$ for $i > n$

<u>Theorem 3.1</u>. *Let $T: \mathcal{A}_R \to \mathcal{A}_R$ be of degree $\leq r$, and let X be an s.s. R-module trivial above n. If $r \geq 1$ and $n \geq 0$ then*

$$\epsilon_r: \pi_i TX \longrightarrow \pi_{i+2r} TS^2X$$

is an isomorphism for $i > n(r-1) + 1$ and a monomorphism for $i = n(r-1) + 1$.

<u>Corollary 3.2</u>. *If $T: \mathcal{A}_R \to \mathcal{A}_R$ is of degree $\leq r$ and $G \in \mathcal{A}_R$ is of projective dimension d, then*

$$\epsilon_r: L_i T(G,n) \longrightarrow L_{i+2r} T(G,n+2)$$

is an isomorphism for $i > (n+d)(r-1) + 1$ and monomorphism for $i = (n+d)(r-1) + 1$.

<u>Remark 3.3</u>. Under the hypotheses of 3.1, $\pi_i TX = 0$ for $i > rn$ [7]; however, for $i \leq rn$ the sequence $\pi_i TX \xrightarrow{\epsilon_r} \pi_{i+2r} TS^2X \xrightarrow{\epsilon_r} \pi_{i+4r} TS^4X \xrightarrow{\epsilon_r} \ldots$ stabilizes to groups which

are frequently non-trivial. For example, let

$$SP^r: \mathcal{A}_Z \to \mathcal{A}_Z$$

be the r-fold symmetric tensor product functor (see §4). The sequence

$$L_{0-i}SP^r(Z,0) \xrightarrow{\epsilon_r} L_{2r-i}SP^r(Z,2) \xrightarrow{\epsilon_r} \cdots$$

is

$$Z \xrightarrow{r!} Z \xrightarrow{1} Z \xrightarrow{1} \cdots$$

for $r \geq 1$, $i = 0$; and is the 0 sequence for $r \geq 1$, $i = 1$. The sequence

$$L_{r-i}SP^r(Z,1) \xrightarrow{\epsilon_r} L_{3r-i}SP^r(Z,3) \xrightarrow{\epsilon_r} \cdots$$

is the 0 sequence for $r \geq 1$, $i = 0$; and is the sequence

$$0 \to Z_2 \xrightarrow{1} Z_2 \xrightarrow{1} \cdots$$

for $r > 1$, $i = 1$.

These facts follow by 3.2 and [7].

We devote the rest of §3 to proving 3.1.

Let $U(M_1,\ldots,M_r)$ be a functor from $\mathcal{A}_R$ to $\mathcal{A}_R$ such that $U(M_1,\ldots,M_r) = 0$ if any $M_i = 0$.

Lemma 3.4. *If X is an s.s. R-module, then*

$$\epsilon_r: \pi_* U(X,\ldots,X) \to \pi_{*+2r} U(S^2X,\ldots,S^2X)$$

equals the composition

$$\pi_* U(X,\ldots,X) \to \pi_{*+2} U(X,\ldots,S^2X) \to \cdots$$
$$\cdots \to \pi_{*+2r-2} U(X,S^2X,\ldots,S^2X) \to \pi_{*+2r} U(S^2X,\ldots,S^2X)$$

of double suspensions in each variable.

Proof. Let $\Delta: K(R,2) \to K(R,2) \wedge \cdots \wedge K(R,2)$ be the diagonal map to the r-fold smash product. Then

$$R(\Delta)_*\colon \pi_* RK(R,2) \to \pi_* \otimes^r RK(R,2)$$

gives $R(\Delta)_*(\epsilon_r) = \xi(\epsilon_1 \otimes \cdots \otimes \epsilon_1)$ where

$$\xi\colon \otimes^r \pi_* RK(R,2) \to \pi_* \otimes^r RK(R,2)$$

is induced by the Eilenberg-Zilber map. The lemma now follows since

$$E\colon RK(R,2) \otimes U(X,\ldots,X) \to U(K(R,2)\otimes X,\ldots,K(R,2)\otimes X)$$

equals the composition

$$RK(R,2)\otimes U(X,\ldots,X) \xrightarrow{R(\Delta)\otimes 1} \otimes^r RK(R,2)\otimes U(X,\ldots,X)$$

$$\xrightarrow{1\otimes E} \otimes^{r-1} RK(R,2) \otimes U(X,\ldots,K(R,2)\otimes X) \to \cdots \to U(K(R,2)\otimes X,\ldots,K(R,2)\otimes X).$$

An immediate consequence of 3.4 is:

__Lemma 3.5.__ _If U is additive in each variable and X is an s.s. R-module, then_

$$\epsilon_r\colon \pi_* U(X,\ldots,X) \to \pi_{*+2r} U(S^2X,\ldots,S^2X)$$

is an isomorphism.

To prove 3.1 we shall need the _cross-effect exact sequence_ [11, 16.1].

__Lemma 3.6.__ (Kan-Whitehead). _Let_ $V\colon \mathcal{A}_R \to \mathcal{A}_R$ _be a functor with_ $V(0) = 0$ _and_

$$(*) \qquad 0 \to M' \xrightarrow{j} M \xrightarrow{p} M'' \to 0$$

be a splittable exact sequence in $\mathcal{A}_R$. _Then there is an exact sequence_

$$\cdots \xrightarrow{\partial_{i+1}} V_i(M',\ldots,M') + V_{i+1}(M,M',\ldots,M') \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} VM' + V_2(M,M') \xrightarrow{\partial_1} VM \xrightarrow{V(p)} VM'' \to 0$$

which is natural in (*) and where the restriction of ∂_1 _to_ VM' _is_ $V(j)$.

3.7. Proof of 3.1. The proof is by induction on n. The case n = 0 is trivial, so suppose $n \geq 1$. We may assume $(NX)_i = 0$ for $i > n$. There then exists a dimensionwise splittable short exact sequence of s.s. R-modules

$$0 \to Y' \to Y \to X \to 0$$

with Y' and Y trivial above n-1. Using 3.6 we obtain exact sequences of s.s. R-modules:

(I)

$$\cdots \to K(R,2r) \otimes T_s(Y', \ldots, Y') + K(R,2r) \otimes T_{s+1}(Y, Y', \ldots, Y') \to \cdots$$

$$\cdots \to K(R,2r) \otimes TY + K(R,2r) \otimes T_2(Y, Y') \to K(R,2r) \otimes TY \to K(R,2r) \otimes TX \to 0$$

(II)

$$\cdots \to T_s(K(R,2) \otimes Y', \ldots, K(R,2) \otimes Y') + T_{s+1}(K(R,2) \otimes Y, K(R,2) \otimes Y', \ldots, K(R,2) \otimes Y') \to \cdots$$

$$\cdots \to T(K(R,2) \otimes Y) + T_2(K(R,2) \otimes Y, K(R,2) \otimes Y') \to T(K(R,2) \otimes Y) \to T(K(R,2) \otimes X) \to 0.$$

We construct a map F: (I) → (II) as follows. Let $e_r \colon K(R,2r) \to RK(R,2)$ be an s.s. R-module map representing $\epsilon_r \in \pi_{2r} RK(R,2)$. If $s \geq 1$ and $Y_1, \ldots, Y_s$ are s.s. R-modules, let f_s denote the composition

$$K(R,2r) \otimes T_s(Y_1, \ldots, Y_s) \xrightarrow{e_r \otimes 1} RK(R,2) \otimes T_s(Y_1, \ldots, Y_s) \to T_s(K(R,2) \otimes Y_1, \ldots, K(R,2) \otimes Y_s)$$

where the second map is the obvious generalization of E. The map F: (I) → (II) is built from such maps f_s.

For 3.1 we must show

$$f_{1*} \colon \pi_i(K(R,2r) \otimes TX) \to \pi_i T(K(R,2) \otimes X)$$

is iso for $i > 2r + n(r-1) + 1$ and mono for $i = 2r + n(r-1) + 1$. This is proved by applying the following facts to F: (I) → (II).

(i) $T_s(\cdot,\ldots,\cdot) = 0$ for $s > r$.

(ii) $f_{r_*}: \pi_*(K(R,2r) \boxtimes T_r(Y',\ldots,Y')) \to \pi_* T_r(K(R,2) \boxtimes Y',\ldots,K(R,2) \boxtimes Y')$ is iso by 3.5, since T_r is additive.

(iii) For $s \geq 1$,

$f_{s*}: \pi_i(K(R,2r) \boxtimes T_s(Y_1,\ldots,Y_s)) \to \pi_i T_s(K(R,2) \boxtimes Y_1,\ldots,K(R,2) \boxtimes Y_s)$ is iso for $i > 2r + (n-1)(r-1) + 1$ and mono for $i = 2r + (n-1)(r-1) + 1$ where $Y_1,\ldots,Y_s$ are s.s. R-modules trivial above n-1. This follows from the inductive hypothesis.

§4. An Application of the Pensions ϵ_r

For special functors, the pensions ϵ_r frequently satisfy a monomorphism theorem in addition to the stability theorem (3.1). This will be shown for the following.

(A) The <u>symmetric algebra functor</u>

$$SP = \sum_{r=0}^{\infty} SP^r : \mathcal{A}_Z \to \mathcal{A}_Z$$

where SP(M) is the quotient of the tensor algebra $\otimes(M) = \sum_{r=0}^{\infty} \otimes^r(M)$ by the two sided ideal with generators $x\otimes y - y\otimes x$ for $x,y \in M$.

(B) The <u>exterior algebra functor</u>

$$\Lambda = \sum_{r=0}^{\infty} \Lambda^r : \mathcal{A}_Z \to \mathcal{A}_Z$$

where $\Lambda(M)$ is the quotient of $\otimes(M)$ by the two sided ideal with generators $x\otimes x$ for $x \in M$.

(C) The <u>gamma functor</u>

$$\Gamma = \sum_{r=0}^{\infty} \Gamma^r : \mathcal{A}_Z \to \mathcal{A}_Z$$

where $\Gamma(M)$ is the commutative algebra with divided power operations described in [9, §18]. We shall use the notation of [2, 2.1].

<u>Remark 4.1</u>. It was shown in [2, §7] that if X is an s.s. free abelian group, there are natural isomorphisms

$$\alpha: \pi_* \Lambda^r(X) \to \pi_{*+r} SP^r(SX)$$

$$\beta: \pi_* \Gamma^r(X) \to \pi_{*+2r} SP^r(S^2X).$$

The groups $\pi_* SP^r X$ are of special interest since they determine [5] the homology of symmetric products of polyhedra.

Theorem 4.2. *Let X be an s.s. free abelian group and $r \geq 1$. Then the maps*

(i) $\epsilon_r\colon \pi_* SP^r X \to \pi_{*+2r} SP^r(S^2 X)$

(ii) $\epsilon_r\colon \pi_* \Lambda^r X \to \pi_{*+2r} \Lambda^r(S^2 X)$

(iii) $\epsilon_r\colon \pi_* \Gamma^r X \to \pi_{*+2r} \Gamma^r(S^2 X)$

are monomorphisms onto direct summands provided in (i) that $\pi_0 X = 0$ and $\pi_1 X$ free abelian, and in (ii) that $\pi_0 X$ free abelian.

Corollary 4.3. *If $G \in \mathcal{A}_Z$ and $r \geq 1$, then the maps*

(i) $\epsilon_r\colon L_* SP^r(G,n) \to L_{*+2r} SP^r(G,n+2)$

(ii) $\epsilon_r\colon L_* \Lambda^r(G,n) \to L_{*+2r} \Lambda^r(G,n+2)$

(iii) $\epsilon_r\colon L_* \Gamma^r(G,n) \to L_{*+2r} \Gamma^r(G,n+2)$

are monomorphisms onto direct summands provided in (i) that $n \geq 2$, in (ii) that $n \geq 1$, and in (iii) that $n \geq 0$.

4.4. Proof of 4.2. It suffices to prove 4.2 (ii) since the isomorphisms α and β of 4.1 are compatible with pensions, i.e., for $\omega \in \mathcal{H}_p(Z,m;Z)$

$$\alpha \circ \omega = (-1)^{(p-m)r} \omega \circ \alpha\colon \pi_* \Lambda^r X \to \pi_{*+r+p} SP^r(S^{m+1} X)$$

$$\beta \circ \omega = \omega \circ \beta\colon \pi_* \Gamma^r X \to \pi_{*+2r+p} SP^r(S^{m+2} X).$$

Since Λ^r is r-homogeneous [2, 4.6] there is a natural map

$$h: \Gamma^r M \otimes \Lambda^r N \to \Lambda^r(M \otimes N)$$

adjoint to the composition

$$\Gamma^r M \xrightarrow{\Gamma^r(i)} \Gamma^r \mathrm{Hom}(N, M\otimes N) \xrightarrow{\phi} \mathrm{Hom}(\Lambda^r N, \Lambda^r(M\otimes N))$$

where $i: M \to \mathrm{Hom}(N, M\otimes N)$ is adjoint to $1: M\otimes N \to M\otimes N$ and where ϕ is the r-homogeneous structure [2, 3.1] of Λ^r. The map

$$E: ZM\otimes \Lambda^r N \to \Lambda^r(M\otimes N)$$

equals the composition

$$ZM\otimes \Lambda^r N \xrightarrow{\gamma\otimes 1} \Gamma^r M \otimes \Lambda^r N \xrightarrow{h} \Lambda^r(M\otimes N)$$

where

$$\gamma: ZM \to \Gamma^r M$$

is the homomorphism with $\gamma([m]) = \gamma_r(m)$. We claim that

$$\gamma_*: \pi_{2r} ZK(Z,2) \to \pi_{2r}\Gamma^r K(Z,2)$$

maps ϵ_r to a generator of $\pi_{2r}\Gamma^r K(Z,2) \approx Z$. This is easily verified from the commutative diagram

$$\begin{array}{ccccc} \pi_{2r} ZK(Z,2) & \xrightarrow{\gamma_*} & \pi_{2r}\Gamma^r K(Z,2) & \underset{\approx}{\xrightarrow{\beta}} & \pi_{4r} SP^r K(Z,4) \\ \uparrow \epsilon_r & & \uparrow \epsilon_r & & \approx \uparrow \epsilon_r \\ \pi_0\, ZK(Z,0) & \xrightarrow{\gamma_*} & \pi_0\, \Gamma^r K(Z,0) & \underset{\approx}{\xrightarrow{\beta}} & \pi_{2r} SP^r K(Z,2) \end{array}$$

by using 3.3 and 4.1. Let

$$a: SP^r M \to \Gamma^r M$$

be the homomorphism with $a(m_1 \ldots m_r) = \gamma_1(m_1)\ldots\gamma_1(m_r)$. It is easily shown that

$$a_*: \pi_{2r} SP^r K(Z,2) \to \pi_{2r}\Gamma^r K(Z,2)$$

is an isomorphism.

To prove 4.2 (ii) it now suffices to show that the composition

$$\pi_*(SP^rK(Z,2)\boxtimes \Lambda^rX) \xrightarrow{(a\boxtimes 1)_*} \pi_*(\Gamma^rK(Z,2)\boxtimes \Lambda^rX) \xrightarrow{h_*} \pi_* \Lambda^r(K(Z,2)\boxtimes X)$$

is a monomorphism onto a direct summand, where X is an s.s. free abelian group with π_0X free abelian. We may assume X = ZK where K is an s.s. set with basepoint. Consider the double s.s. abelian group [7, §2]

$$B = \Lambda^r(K(Z,2)\hat{\boxtimes}ZK).$$

We construct two subobjects of B. Let

$$B^1 = (SP^rK(Z,2))\hat{\boxtimes} \Lambda^rZK$$

with the containment $B^1 \subset B$ induced by the transformation

$$h\circ(a\boxtimes 1): SP^rM\boxtimes \Lambda^rN \to \Lambda^r(M\boxtimes N)$$

which is monic for M and N free abelian. Let $B^2 \subset B$ be the subobject such that $B^2_{p,q}$ is generated by the subgroups

$$\Lambda^r(K(Z,2)_p\boxtimes ZL) \subset \Lambda^r(K(Z,2)_p\boxtimes ZK_q) = B_{p,q}$$

where L ranges over the subsets of K_q involving the basepoint and at most r-1 other elements. By [7, §2] it now suffices to prove that the containment $B^1 \oplus B^2 \subset B$ induces a homology isomorphism for the associated total complexes. For this it suffices to show that the containment of s.s. abelian groups $B^1_{*,q} \oplus B^2_{*,q} \subset B_{*,q}$ induces homotopy isomorphisms for each q, and this is straightforward to prove.

§5. Derived Functors of Quadratic Functors

Using mod-2 pensions we shall compute derived functors for quadratic functors on the category of Z_2-modules.

For $r \geq 1$ let

$$\sigma_r \in \widetilde{H}_r(Z_2,1;Z_2) \approx Z_2$$

denote the non-zero element. Thus $\sigma_1 = \sigma$. It is not hard to show in $P(Z_2)$ that $\sigma_r \cdot \sigma_r = \epsilon_r$; and the pension σ_r has properties very similar to ϵ_r. In particular:

Theorem 5.1. *Let* $T: \mathcal{A}_{Z_2} \to \mathcal{A}_{Z_2}$ *be of degree* $\leq r$, *and let* X *be an s.s.* Z_2*-module trivial above* n. *If* $r \geq 1$ *and* $n \geq 0$ *then*

$$\sigma_r: \pi_i TX \to \pi_{i+r} TSX$$

is an isomorphism for $i > n(r-1) + 1$ *and a monomorphism for* $i = n(r-1) + 1$.

The proof is essentially the same as for 3.1.

Now let $T: \mathcal{A}_{Z_2} \to \mathcal{A}_{Z_2}$ be a *quadratic* functor, i.e., T is of degree ≤ 2.

Proposition 5.2. (i) $L_qT(Z_2,n) = 0$ *for* $q > 2n$.

(ii) *The suspension*

$$\sigma: L_qT(Z_2,n) \to L_{q+1}T(Z_2,n+1)$$

is an isomorphism for $q < 2n$ *and epimorphism for* $q = 2n$.

(iii) *The pension*

$$\sigma_2: L_qT(Z_2,n) \to L_{q+2}T(Z_2,n+1)$$

is an isomorphism for $q > n+1$ *and a monomorphism for* $q = n+1$.

This follows from [7] and 5.1.

We now compute all $L_qT(Z_2,n)$ from a knowledge of the cross-effect diagonal and codiagonal maps [7, 5.23]:

$$\Delta: T(Z_2) \to T_2(Z_2,Z_2)$$

$$\nabla : T_2(Z_2,Z_2) \to T(Z_2).$$

Since $\nabla\circ\Delta = 0$ we can define:

$$G_0 = T(Z_2)$$

$$G_1 = T(Z_2)/\text{image}\,\nabla$$

$$G_2 = \text{kernel}\ \nabla$$

$$G_3 = \text{kernel}\,\nabla/\text{image}(\Delta\circ\nabla)$$

$$G_4 = \text{kernel}(\Delta\circ\nabla)$$

$$G_5 = \text{kernel}(\Delta\circ\nabla)/\text{image}(\Delta\circ\nabla)$$

Combined with 5.2 the following proposition determines all groups $L_qT(Z_2,n)$ and their behavior under σ and σ_2.

Proposition 5.3. There are canonical isomorphisms:

(i) $L_0T(Z_2,0) \approx G_0$

(ii) $L_1T(Z_2,1) \approx G_1$

(iii) $L_2T(Z_2,1) \approx G_2$

(iv) $L_3T(Z_2,2) \approx G_3$

(v) $L_4T(Z_2,2) \approx G_4$

(vi) $L_5T(Z_2,3) \approx G_5$.

Furthermore:

(a) *For* $i = 0, 1,$ *and* 2, *the suspension*

$$\sigma: L_{2i}T(Z_2,i) \to L_{2i+1}T(Z_2,i+1)$$

is given by the quotient map $G_{2i} \to G_{2i+1}$.

(b) *For* $i = 0$ *and* 1, *the pension*

$$\sigma_2: L_iT(Z_2,i) \to L_{i+2}T(Z_2,i+1)$$

is given by the map $G_i \to G_{i+2}$ *induced by*

$$\Delta: T(Z_2) \to \text{kernel}\,\nabla \subset T_2(Z_2,Z_2).$$

(c) *For* $i = 1$ *and* 2, *the pension*

$$\sigma_2: L_{i+1}T(Z_2,i) \to L_{i+3}T(Z_2,i+1)$$

is given by the map $G_{i+1} \to G_{i+3}$ *induced by the inclusion*

$$\text{kernel}\nabla \subset \text{kernel}(\Delta\circ\nabla).$$

The proof of 5.3 is a straightforward application of Lemma 5.4.

The pension

$$\sigma_2: \pi_iT_2(K(Z_2,q),K(Z_2,q)) \to \pi_{i+2}T_2(K(Z_2,q+1),K(Z_2,q+1))$$

is an isomorphism for all $i,q \geq 0$ by the mod-2 analogue of 3.5. Let ∂ denote the composition

$$\pi_{i+2}T(K(Z_2,q+1)) \xrightarrow{\Delta_*} \pi_{i+2}T_2(K(Z_2,q+1),K(Z_2,q+1)) \xrightarrow{(\sigma_2)^{-1}} \pi_iT(K(Z_2,q),K(Z_2,q))$$

where Δ is the diagonal.

Lemma 5.4. (Dold-Puppe) *There is a long exact sequence*

$$\cdots \to \pi_{i+1}TK(Z_2,q) \xrightarrow{\sigma} \pi_{i+2}TK(Z_2,q+1) \xrightarrow{\partial} \pi_i T_2(K(Z_2,q),K(Z_2,q))$$

$$\xrightarrow{\nabla_*} \pi_i TK(Z_2,q) \xrightarrow{\sigma} \pi_{i+1}TK(Z_2,q+1) \to \cdots$$

where ∇ is the codiagonal.

For proof see [7, 6.6].

We shall consider examples based on the following functors.

5.5. Let p be any prime.

(A) The mod-p symmetric algebra functor

$$SP = \sum_{r=0}^{\infty} SP^r : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

may be defined as the restriction of the symmetric algebra functor (§4) to $\mathcal{A}_{Z_p} \subset \mathcal{A}_Z$.

(B) The mod-p truncated symmetric algebra functor

$$SP_p = \sum_{r=0}^{\infty} SP^r{}_p : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

is defined by letting $SP_p(M)$ be the quotient of the algebra $SP(M)$ by the ideal with generators x^p for $x \in M = SP^1 M$.

(C) The mod-p gamma functor

$$\Gamma_p = \sum_{r=0}^{\infty} \Gamma^r{}_p : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

is defined by letting $\Gamma_p(M) = Z_p \otimes_Z \Gamma(M)$ for $M \in \mathcal{A}_{Z_p}$ where Γ is the gamma functor (§4). Thus $\Gamma_p(M)$ is a commutative algebra with divided power operations.

For $M \in \mathcal{A}_{Z_p}$ there are natural algebra homomorphisms:

$$SP(M) \xrightarrow{f} SP_p(M) \xrightarrow{g} \Gamma_p(M)$$

determined uniquely by the condition that they restrict to the identity maps on $SP^1(M) = SP^1{}_p(M) = \Gamma^1{}_p(M) = M$.

We now compute derived functors for the quadratic functors

$$SP^2,\ SP^2{}_2,\ \Gamma^2{}_2: \mathcal{A}_{Z_2} \to \mathcal{A}_{Z_2}.$$

Proposition 5.6. A Z_2-basis for

$$\sum_{n=0}^{\infty} L_*\Gamma^2{}_2(Z_2,n)$$

is given by the elements $(\sigma)^j(\sigma_2)^i(\alpha)$ for $i,j \geq 0$, where $\alpha \in L_0\Gamma^2{}_2(Z_2,0) = Z_2$ is the non-zero element. The map

$$g : \sum_{n=0}^{\infty} L_*SP^2{}_2(Z_2,n) \to \sum_{n=0}^{\infty} L_*\Gamma^2{}_2(Z_2,n)$$

is a monomorphism with image generated by $\sigma_2(\alpha)$ under the action of σ, σ_2. The map

$$g_*\circ f_*: \sum_{n=1}^{\infty} L_*SP^2(Z_2,n) \to \sum_{n=1}^{\infty} L_*\Gamma^2{}_2(Z_2,n)$$

is a monomorphism with image generated by $(\sigma_2)^2(\alpha)$ under the action of σ, σ_2.

This is a simple application of 5.2 and 5.3.

For future reference we observe:

Proposition 5.7. If $T: \mathcal{A}_{Z_2} \to \mathcal{A}_{Z_2}$ is quadratic and X is an s.s. Z_2-module, then $\sigma_i = 0: \pi_*X \to \pi_{*+i}TSX$ for $i > 2$.

Proof. As in 4.4 the map (§2)

$$E\colon Z_2M\otimes TN \to T(M\otimes N)$$

factors as a composition

$$Z_2M\otimes TN \xrightarrow{\gamma\otimes 1} \Gamma^2{}_2M\otimes TN \xrightarrow{h} T(M\otimes N).$$

Clearly $\gamma_* = 0\colon \pi_i Z_2K(Z_2,1) \to \pi_i\Gamma^2{}_2K(Z_2,1)$ for $i > 2$ since (5.6) $\pi_i\Gamma^2{}_2K(Z_2,1) = 0$ for $i > 2$.

§6. Derived Functors of p-Homogeneous Functors

Let p be an odd prime. It is not hard to show that a functor $T: \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$ is p-homogeneous [2, 3.1] if and only if T is of degree $\leq p$ and $T(nf) = n^p T(f)$ for each $n \in Z_p$ and map f in $\mathcal{A}_{Z_p}$. Thus SP^p, $SP^p{}_p$, and $\Gamma^p{}_p$ (5.5) are p-homogeneous.

Let

$$T: \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

be a p-homogeneous functor. We shall show that the groups $L_*T(Z_p,n)$ are determined for all n when they are known for $n \leq 2$.

Proposition 6.1. (i) $L_qT(Z_p,n) = 0$ for $q > pn$.

(ii) The suspension

$$\sigma: L_qT(Z_p,n) \to L_{q+1}T(Z_p,n+1)$$

is an isomorphism for $q < pn$ and an epimorphism for $q = pn$.

(iii) The pension

$$\epsilon_p: L_qT(Z_p,n) \to L_{q+2p}T(Z_p,n+2)$$

is an isomorphism for $q > n+1$ and a monomorphism for $q = n+1$.

Proof. Part (i) follows by [7, 4.23]. Part (ii) follows by the mod-p version of [2, 5.9]. For (iii) observe that

$$\pi_q T_r(K(Z_p,n),\ldots,K(Z_p,n)) = 0$$

for $r \geq 2$ and $q \neq pn$, since $T_r(X,\ldots,X)$ is equal to a sum of direct summands of $T_p(X,\ldots,X)$.

Furthermore for $r \geq 2$

$$\epsilon_p: \pi_{pn}T_r(K(Z_p,n),\dots,K(Z_p,n)) \to \pi_{p(n+2)}T_r(K(Z_p,n+2),\dots,K(Z_p,n+2))$$

is an isomorphism. Thus (iii) follows by a simple modification of 3.7.

Now let I denote the image of the codiagonal

$$\nabla: T_p(Z_p,\dots,Z_p) \to T(Z_p)$$

and let K denote the kernel of the diagonal

$$\Delta: T(Z_p) \to T_p(Z_p,\dots,Z_p).$$

__Proposition 6.2.__ __There are natural exact sequences__

(i) $0 \to I \to L_0T(Z_p,0) \xrightarrow{\sigma} L_1T(Z_p,1) \to 0$

(ii) $0 \to K \to L_0T(Z_p,0) \xrightarrow{\epsilon_p} L_{2p}T(Z_p,2)$

(iii) $0 \to I/I \cap K \to L_{2p}T(Z_p,2) \xrightarrow{\sigma} L_{2p+1}T(Z_p,3) \to 0$

(iv) $0 \to L_2T(Z_p,1) \xrightarrow{\epsilon_p} L_{2p+2}T(Z_p,3) \to I \cap K \to 0.$

__Proof.__ Part (i) follows by [7, 6.11] since I equals the image of $\nabla: T_2(Z_p,Z_p) \to T(Z_p)$. For part (ii) consider the commutative diagram

$$\begin{array}{ccc}
\pi_{2p}TK(Z_p,2) & \xrightarrow{\Delta'_*} & \pi_{2p}T_p(K(Z_p,2),\dots,K(Z_p,2)) \\
\uparrow \epsilon_p & & \uparrow \epsilon'_p \\
\pi_0\ TK(Z_p,0) & \xrightarrow{\Delta_*} & \pi_0\ T_p(K(Z_p,0),\dots,K(Z_p,0))
\end{array}$$

where Δ' is the diagonal Δ, and ϵ'_p is the pension ϵ_p. Using [7, 6.7] one shows Δ'_* is a monomorphism. Since ϵ'_p is an isomorphism, part (ii) follows. For parts (iii) and (iv) consider the commutative diagram

$$\begin{array}{ccc}
\pi_{2p}T_2(K(Z_p,2),K(Z_p,2)) & \xrightarrow{\nabla'_*} & \pi_{2p}TK(Z_p,2) \\
\uparrow \epsilon'_p & & \uparrow \epsilon_p \\
\pi_0\ T_2(K(Z_p,0),K(Z_p,0)) & \xrightarrow{\nabla_*} & \pi_0\ TK(Z_p,0)
\end{array}$$

where ∇' is the codiagonal ∇, and ϵ'_p is the pension ϵ_p. Then

$$I/I \cap K \approx \text{Image } \nabla'_*$$

$$I \cap K \quad (\text{Kernel } \nabla'_*)/\epsilon'_p(\text{Kernel } \nabla_*)$$

since ϵ'_p is an isomorphism, since (Kernel ϵ_p) = K, and since (Image ∇_*) = I. Now (ii) and (iii) follow using [7, 6.7] and a modification of 3.7.

Remark 6.3. The groups $L_*T(Z_p,n)$ for $n \leq 2$ are usually easy to compute ad hoc, and for $n \geq 3$ they are determined by 6.1 and 6.2.

Before giving examples, we introduce additional mod-p pensions. Recall that the cohomology ring $H^*(Z_p,1;Z_p)$ is the tensor product of the exterior algebra generated by $\iota^1 \in H^1(Z_p,1;Z_p)$ with the polynomial algebra generated by $\beta\iota^1 \in H^2(Z_p,1;Z_p)$, where β is the Bockstein operator.

Notation 6.4. For $r \geq 1$ let $\theta_r \in \tilde{H}_{2r-1}(Z_p,1;Z_p)$ be dual to $(\iota^1)(\beta\iota^1)^{r-1}$, and let $\phi_r \in \tilde{H}_{2r}(Z_p,1;Z_p)$ be dual to $(\beta\iota^1)^r$.

In describing $L_*T(Z_p,n)$ we shall use the pensions σ, ϵ_p, and ϕ_1.

Lemma 6.5. *In $P(Z_p)$ the elements σ, ϵ_p, and ϕ_1 commute with each other, and $\phi_1 \cdot \phi_1 = 0$.*

Proof. Use the commutativity formula in 2.1. In particular

$$\phi_1 \cdot \phi_1 = \mu \cdot \phi_1 \cdot \phi_1 = -\phi_1 \cdot \phi_1$$

so $\phi_1 \cdot \phi_1 = 0$.

The pension

$$\phi_1 : L_* T(Z_p, n) \to L_{*+2} T(Z_p, n+1)$$

may be computed in special examples with the aid of the Bockstein operator which we now describe.

Suppose that:

(i) $U: \mathcal{A}_Z \to \mathcal{A}_Z$ is a p-homogeneous functor [2] with U(M) free abelian for M free abelian.

(ii) $j: T(Z_p \otimes M) \to Z_p \otimes U(M)$ is a natural map for $M \in \mathcal{A}_Z$, which is an isomorphism for M free abelian.

Then the coefficient sequence

$$0 \to Z_p \to Z_{p^2} \to Z_p \to 0$$

induces a Bockstein operator

$$\beta: \pi_q(Z_p \otimes UK(Z,n)) \to \pi_{q-1}(Z_p \otimes UK(Z,n))$$

which is equivalent under j to an operator

$$\beta: L_q T(Z_p, n) \to L_{q-1} T(Z_p, n).$$

For convenience we use the operator

$$\overline{\beta}: L_q T(Z_p, n) \to L_{q-1} T(Z_p, n)$$

defined by $\overline{\beta}(u) = (-1)^q \beta(u)$.

Lemma 6.6. *If* $u \in L_q T(Z_p, n)$, *then*

(i) $\quad \overline{\beta}\sigma(u) = \sigma\overline{\beta}(u)$

(ii) $\quad \overline{\beta}\epsilon_p(u) = \epsilon_p\overline{\beta}(u)$

(iii) $\quad \overline{\beta}\phi_1(u) + \phi_1\overline{\beta}(u) = \sigma(u)$.

Proof. Parts (i) and (ii) are straightforward. For (iii) let $\Gamma^r: \mathcal{A}_Z \to \mathcal{A}_Z$ as in §4. For any $r \geq 1$ and $M \in \mathcal{A}_Z$ the homomorphism

$$t: Z_p \otimes \Gamma^r M \to Z_p \otimes \Gamma^r(Z_p \otimes M)$$

induced by $M \to Z_p \otimes M$ is an isomorphism. Let k denote the composition

$$Z_p(Z_p \otimes M) = Z_p \otimes Z(Z_p \otimes M) \xrightarrow{1 \otimes \gamma} Z_p \otimes \Gamma^r(Z_p \otimes M) \xrightarrow{t^{-1}} Z_p \otimes \Gamma^r M$$

where γ is as in 4.4.

For $M, N \in \mathcal{A}_Z$ there is a commutative diagram

$$\begin{array}{ccc} Z_p(Z_p \otimes M) \otimes T(Z_p \otimes N) & \xrightarrow{E} & T(Z_p \otimes M \otimes N) \\ \downarrow{\scriptstyle k \otimes j} & & \downarrow{\scriptstyle j} \\ (Z_p \otimes \Gamma^p M) \otimes (Z_p \otimes U(N)) & \to & Z_p \otimes U(M \otimes N) \end{array}$$

where the bottom map is induced by the map

$$h: \Gamma^p M \otimes UN \to U(M \otimes N)$$

defined as in 4.4 from the p-homogeneous structure of U. A straightforward computation shows that

$$k_*: \pi_* Z_p K(Z_p, 1) \to \pi_*(Z_p \otimes \Gamma^p K(Z,1))$$

gives $\beta k_*(\phi_1) = k_*(\sigma)$. A simple argument taking $M = K(Z,1)$ and $N = K(Z,n)$ in the above diagram then proves part (iii).

As an application we now compute derived functors of the p-homogeneous functors (5.5)

$$SP^p,\ SP^p{}_p,\ \Gamma^p{}_p : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

and describe their behavior under the maps (5.5)

$$SP^p \xrightarrow{f} SP^p{}_p \xrightarrow{g} \Gamma^p{}_p.$$

Proposition 6.7. *A Z_p-basis for*

$$\sum_{n=0}^{\infty} L_*\Gamma^p{}_p(Z_p,n)$$

is given by the elements $(\phi_1)^k(\sigma)^j(\epsilon_p)^i(\nu)$ *for* $i \geq 0$, $j \geq 0$, *and* $0 \leq k \leq 1$, *where*

$$\nu \in L_0\Gamma^p{}_p(Z_p,0) = \Gamma^p{}_p(Z_p)$$

is the element corresponding to $\gamma_p(1) \in \Gamma^p{}_p(Z_p)$. *The map*

$$g_*: \sum_{n=0}^{\infty} L_*SP^p{}_p(Z_p,n) \to \sum_{n=0}^{\infty} L_*\Gamma^p{}_p(Z_p,n)$$

is a monomorphism with image generated by $\phi_1(\nu)$ *and* $\epsilon_p(\nu)$ *under the action of* ϕ_1, ϵ_p, *and* σ. *The map*

$$g_* \circ f_*: \sum_{n=1}^{\infty} L_*SP^p(Z_p,n) \to \sum_{n=1}^{\infty} L_*\Gamma^p{}_p(Z_p,n)$$

is a monomorphism with image generated by $\epsilon_p(\nu)$ *under the action of* ϕ_1, ϵ_p, *and* σ.

Proof. For $M \in \mathcal{A}_{Z_p}$ there are natural exact sequences

$$0 \to M \to SP^p(M) \xrightarrow{f} SP^p{}_p(M) \to 0$$

$$0 \to SP^p{}_p(M) \xrightarrow{g} \Gamma^p{}_p(M) \to M \to 0.$$

By [7] $L_*SP^p(Z_p,1) = 0$, $L_{2p}SP^p(Z_p,2) \approx Z_p$, and $L_iSP^p(Z_p,2) = 0$ for $i \neq 2p$. The above exact sequences then imply

$$L_i\Gamma^p{}_p(Z_p,1) = \begin{cases} Z_p & \text{for } i = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

$$L_i\Gamma^p{}_p(Z_p,2) = \begin{cases} Z_p & \text{for } i = 2,\ 3,\ 2p \\ 0 & \text{otherwise.} \end{cases}$$

By 6.2 (i) the element $\sigma(\nu)\in L_1\Gamma^p{}_p(Z_p,1)$ is a generator. Since $\Gamma^p{}_p(Z_p\otimes M)\approx Z_p\otimes\Gamma^p(M)$ for $M\in\mathcal{A}_Z$, 6.6 applies to show $\bar{\beta}\phi_1(\nu) = \sigma(\nu)$. Hence $\phi_1(\nu)\in L_1\Gamma^p{}_p(Z_p,1)$ is a generator. For the functor $\Gamma^p{}_p$, $I = 0$ and $K = 0$ in 6.2, so the first part of 6.7 follows from 6.2 and 6.1. The remainder of 6.7 then follows easily from the above exact sequences.

For future reference we observe:

<u>Proposition 6.8</u>. *If* $T\colon \mathcal{A}_{Z_p}\to\mathcal{A}_{Z_p}$ *is p-homogeneous and* X *is an s.s.* Z_p*-module, then*

(i) $\epsilon_i = 0\colon \pi_*TX \to \pi_{*+2i}TS^2X$ *for* $1 < i < p$ *and* $i > p$.

(ii) $\theta_i = 0\colon \pi_*TX \to \pi_{*+2i-1}TSX$ *for* $i > 1$.

(iii) $\phi_i = 0\colon \pi_*TX \to \pi_{*+2i}TSX$ *for* $i > 1$.

The proof is the same as 5.7, using 6.7 in place of 5.6.

§7. Composition Products and Their Pensions

In order to compute $\pi_* SPX$ where X is an s.s. Z_p-module, we introduce composition products and describe their behavior under pensions.

Let R be a commutative ring with identity and

$$T, T': \mathcal{A}_R \to \mathcal{A}_R$$

be functors with $T(0) = T'(0) = 0$. If X is an s.s. R-module we shall define a map

$$c: L_i T(R,j) \times \pi_j T'X \to \pi_i \; T \circ T'X$$

where $T \circ T'$ is the composed functor. For $u \in L_i T(R,j)$ and $v \in \pi_j T'X$ let

$$c(u,v) = T(\bar{v})_*(u)$$

where

$$\bar{v}: K(R,j) \to T'X$$

represents v. We denote c(u,v) by

$$u \circ v \in \pi_i T \circ T'X$$

and call it the <u>composition product</u> of u and v.

<u>Lemma 7.1</u>. <u>Let</u> $t,u \in L_i T(R,j)$, $v,w \in \pi_j T'X$, <u>and</u> $x \in L_{i-1} T(R,j-1)$. <u>Then</u>

(i) $(t+u) \circ v = t \circ v + u \circ v$

(ii) $(\sigma x) \circ (v+w) = (\sigma x) \circ v + (\sigma x) \circ w$

(iii) $\sigma(u \circ v) = (\sigma u) \circ (\sigma v)$.

The proof is straightforward.

Thus composition products are compatible with suspensions. For the pensions ϵ_s (§3), σ_s (§5), and ϕ_s (6.4), the results are more complicated.

Theorem 7.2. *If $x \in L_{i-1}T(R,j-1)$ and $v \in \pi_j T'X$, then for $s \geq 1$*

$$\epsilon_s((\sigma x)\circ v) = \sum_{r=1}^{s} \sum_{m_1+\ldots+m_r=s} (\epsilon_{m_1}\ldots\epsilon_{m_r}(\sigma x))\circ(\epsilon_r v)$$

where the second sum ranges over all $(m_1,\ldots,m_r)$ with $m_1,\ldots,m_r \geq 1$ and $m_1+\ldots+m_r=s$.

Theorem 7.3. *If $R = Z_2$, $x \in L_{i-1}T(Z_2,j-1)$, and $v \in \pi_j T'X$, then for $s \geq 1$*

$$\sigma_s((\sigma x)\circ v) = \sum_{r=1}^{s} \sum_{m_1+\ldots+m_r=s} (\sigma_{m_1}\ldots\sigma_{m_r}(\sigma x))\circ(\sigma_r v).$$

Theorem 7.4. *If $R = Z_p$, p an odd prime, $x \in L_{i-1}T(Z_p,j-1)$ and $v \in \pi_j T'x$, then for $s \geq 1$*

$$\phi_s((\sigma x)\circ v) = (-1)^{i-j} \sum_{r=1}^{s} \sum_{m_1+\ldots+m_r=s} (\epsilon_{m_1}\ldots\epsilon_{m_r}(\sigma x))\circ(\phi_r v)$$

$$+ \sum_{r=0}^{s-1} \sum_{m+m_1+\ldots+m_r=s} (\phi_m\epsilon_{m_1}\ldots\epsilon_{m_r}(\sigma x))\circ(\Theta_{r+1}v).$$

Before proving these formulae, we show how they simplify when v is a suspension.

Lemma 7.5. (i) *In $P(R)$, $\sigma\cdot\epsilon_r = 0$ unless r is a prime power, and $p\cdot\sigma\cdot\epsilon_{p^j} = 0$ for p prime and $j \geq 1$.*

(ii) In $P(Z_2)$, $\sigma \cdot \sigma_r = 0$ unless $r = 2^j$ with $j \geq 0$.

(iii) In $P(Z_p)$ with p an odd prime, $\sigma \cdot \theta_r = 0$ for $r > 0$, $\sigma \cdot \phi_r = 0$ unless $r = p^j$ with $j \geq 0$, and $\phi_r \cdot \phi_r = 0$ for r positive and odd.

The proof uses standard facts concerning the homology of Eilenberg-MacLane spaces.

In 7.2, 7.3, and 7.4 assume $X = SY$ and $v = \sigma y$ for $y \in \pi_{j-1}T'Y$. Then these theorems respectively imply:

Corollary 7.6. For p prime and $s \geq 0$

$$\epsilon_p s((\sigma x) \circ (\sigma y)) = \sum_{r=0}^{s} ((\epsilon_p s-r)^{p^r}(\sigma x)) \circ (\epsilon_p r(\sigma y)).$$

Corollary 7.7. For $s \geq 0$

$$\sigma_2 s((\sigma x) \circ (\sigma y)) = \sum_{r=0}^{s} ((\sigma_2 s-r)^{2^r}(\sigma x)) \circ (\sigma_2 r(\sigma y)).$$

Corollary 7.8. For $s \geq 0$

$$\phi_p s((\sigma x) \circ (\sigma y)) = (-1)^{i-j} \sum_{r=0}^{s} ((\epsilon_p s-r)^{p^r}(\sigma x)) \circ (\phi_p r(\sigma y)) + (\phi_p s(\sigma x)) \circ (\sigma^2 y).$$

Remark 7.9. These results stabilize to formulae involving the diagonal in the dual to the Steenrod algebra — a fact which will be exploited in Part II.

We devote the rest of §7 to proving 7.2, 7.3, and 7.4.

Let

$$T, T' : \mathcal{A}_R \to \mathcal{A}_R$$

as above and let X be an s.s. R-module. The composition of

$$R\circ RK(R,q)\boxtimes T\circ T'X \xrightarrow{E} T(RK(R,q)\boxtimes T'X) \xrightarrow{T(E)} T\circ T'(K(R,q)\boxtimes X) \xrightarrow{T\circ T'(\psi)} T\circ T'(S^qX)$$

(where ψ and E are as in §2) induces in homotopy a map

$$G: \pi_* R\circ RK(R,q)\boxtimes\pi_* T\circ T'X \to \pi_* T\circ T'(S^qX).$$

For $M \in \mathcal{A}_R$ let

$$d: R(M) \to (R\circ R)(M)$$

be the homomorphism such that

$$d(1[m]) = 1[1[m]]$$

for $m \in M$. Thus

$$d_*: \pi_* RK(R,q) \to \pi_* R\circ RK(R,q).$$

<u>Lemma 7.10</u>. <u>Let</u> $y \in \pi_i RK(R,p)$, $z \in \pi_p RK(R,q)$, $u \in \pi_k TK(R,j)$ <u>and</u> $v \in \pi_j T'X$. <u>Then</u>

(i) $z(u\circ v) = (-1)^{(p-q)k} G((d_* z)\boxtimes(u\circ v))$ <u>in</u> $\pi_{p+k} T\circ T'(S^qX)$.

(ii) $G((y\circ z)\boxtimes(u\circ v)) = ((-1)^{(i-p)k} y(u))\circ((-1)^{(p-q)j} z(v))$ <u>in</u> $\pi_{i+k} T\circ T'(S^qX)$.

(iii) <u>If</u> $\delta \in \pi_* R\circ RK(R,q)$ <u>lies in the image of the codiagonal</u>

$$\nabla_*: \pi_* R_2(RK(R,q),RK(R,q)) \to \pi_* R(RK(R,q))$$

<u>and</u> $w \in \pi_{k-1} TK(R,j-1)$, <u>then</u>

$$G(\delta \boxtimes ((\sigma w)\circ v)) = 0.$$

<u>Proof</u>. Part (i) follows since the map

$$E: RM\boxtimes T\circ T'N \to T\circ T'(M\boxtimes N)$$

equals the composition

$$RM\otimes T\circ T'N \xrightarrow{d\otimes 1} R\circ RM\otimes T\circ T'N \xrightarrow{E} T(RM\otimes T'N) \xrightarrow{T(E)} T\circ T'(M\otimes N).$$

Part (ii) follows from the commutative diagram

$$\begin{array}{ccc} RK(R,p)\otimes TK(R,j) & \xrightarrow{R(\overline{z})\otimes T(\overline{v})} & R\circ RK(R,q)\otimes T\circ T'X \\ \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} \\ T(K(R,p)\otimes K(R,j)) & \xrightarrow{T(\overline{z}\otimes\overline{v})} & T(RK(R,q)\otimes T'X) \end{array}$$

where $\overline{z}$: $K(R,p) \to RK(R,q)$ represents z and $\overline{v}$: $K(R,j) \to T'X$ represents v.

To prove (iii) consider the commutative diagram

$$\begin{array}{ccccc} R_2(M,M)\otimes TN & \xrightarrow{\nabla\otimes 1} & RM\otimes TN & \xrightarrow{E} & T(M\otimes N) \\ \downarrow{\scriptstyle i\otimes\Delta} & & & & \uparrow{\scriptstyle \nabla} \\ R(M+M)\otimes T_2(N,N) & & \xrightarrow{E_2} & & T_2(M\otimes N, M\otimes N) \end{array}$$

where i: $R_2(M,M) \subset R(M+M)$ is the cross-effect inclusion, and E_2 is the obvious generalization of E. Since

$$\Delta_*\colon \pi_*TK(R,j) \to \pi_*T_2(K(R,j),K(R,j))$$

gives $\Delta_*(\sigma w) = 0$, a simple argument using the above diagram with $M = RK(R,q)$ and $N = K(R,j)$ proves part (iii).

Clearly 7.2 follows from 7.10 using

<u>Lemma 7.11</u>. <u>The map</u>

$$d_*\colon \pi_*RK(R,2) \to \pi_*R\circ RK(R,2)$$

gives

$$d_*(\epsilon_s) = \delta + \sum_{r=1}^{s} \sum_{m_1+\dots+m_r=s} (\epsilon_{m_1}\dots\epsilon_{m_r})\circ\epsilon_r$$

for $s \geq 1$, *where* δ *lies in the image of*

$$\nabla_*: \pi_* R_2(RK(R,2), RK(R,2)) \to \pi_* R(RK(R,2)).$$

Proof. It suffices to suppose $R = Z$. For $r \geq 1$ let

$$i_r: K(Z,2r) \to ZK(Z,2)$$

$$j_r: ZK(Z,2) \to K(Z,2r)$$

be s.s. homomorphisms such that i_r represents ϵ_r and $j_r \circ i_r = 1$. Then $d_*(\epsilon_s)$ equals the image of ϵ_s under

$$\pi_* ZK(Z,2) \xrightarrow{d_*} \pi_* ZZK(Z,2) \xrightarrow{b_*} \pi_* ZZK(Z,2)$$

where

$$b = Z(\sum_{r=1}^{s} i_r \circ j_r).$$

Hence

$$d_*(\epsilon_s) = \delta + \sum_{r=1}^{s} u_r$$

where δ is in the image of the codiagonal ∇_*, and where u_r is the image of ϵ_s under

$$\pi_* ZK(Z,2) \xrightarrow{d_*} \pi_* ZZK(Z,2) \xrightarrow{Z(i_r \circ j_r)_*} \pi_* ZZK(Z,2).$$

Hence $u_r \in \pi_{2s}(Z \circ Z)K(Z,2)$ is a composition product $u_r = v_r \circ \epsilon_r$ where v_r is the image of ϵ_s under

$$\pi_* ZK(Z,2) \xrightarrow{d_*} \pi_* ZZK(Z,2) \xrightarrow{Z(j_r)_*} \pi_* ZK(Z,2r).$$

Thus v_r is the image of ϵ_s under

$$Z(k_r)_*: \pi_* ZK(Z,2) \to \pi_* ZK(Z,2r)$$

where

$$k_r: K(Z,2) \to K(Z,2r)$$

is given by $k_r(y) = j_r(1[y])$ for $y \in K(Z,2)$. But k_r is homotopic as a map of s.s. sets to

$$K(Z,2) \xrightarrow{\Delta} K(Z,2)\wedge\ldots\wedge K(Z,2) \xrightarrow{w} \boxtimes^r K(Z,2) \xrightarrow{\psi} K(Z,2r)$$

where Δ is the diagonal to the r-fold smash product, where $w(y_1\wedge\ldots\wedge y_r) = y_1\boxtimes\ldots\boxtimes y_r$, and where ψ is as in §2. Hence $v_r \in \pi_{2s} ZK(Z,2r)$ is the image of ϵ_s under

$$\pi_* ZK(Z,2) \xrightarrow{Z(\Delta)_*} \pi_* \boxtimes^r ZK(Z,2) \xrightarrow{Z(w)_*} \pi_* Z\boxtimes^r K(Z,2) \xrightarrow{Z(\psi)_*} \pi_* ZK(Z,2r).$$

It follows that

$$v_r = \sum_{m_1+\ldots+m_r=s} \epsilon_{m_1}\cdots\epsilon_{m_r}$$

in P(Z). Since

$$d_*(\epsilon_s) = \delta + \sum_{r=1}^{s} v_r \circ \epsilon_r$$

the lemma follows.

Clearly 7.3 follows from 7.10 using

Lemma 7.12. The map

$$d_*: \pi_* Z_2 K(Z_2,1) \to \pi_* Z_2 \circ Z_2 K(Z_2,1)$$

gives

$$d_*(\sigma_s) = \delta + \sum_{r=1}^{s} \sum_{m_1+\ldots+m_r=s} (\sigma_{m_1}\cdots\sigma_{m_r})\circ\sigma_r$$

for $s \geq 1$, *where* δ *lies in the image of the codiagonal* ∇_*.

The proof is similar to 7.11.

Clearly 7.4 follows from 7.10 using

Lemma 7.13. *The map*

$$d_*: \pi_* Z_p K(Z_p,1) \to \pi_* Z_p Z_p K(Z_p,1)$$

gives

$$d_*(\phi_s) = \delta + \sum_{r=1}^{s} \sum_{m_1+\dots+m_r=s} (\epsilon_{m_1}\dots\epsilon_{m_r})\circ\phi_r$$

$$+ \sum_{r=0}^{s-1} \sum_{m+m_1+\dots+m_r=s} (\phi_m\cdot\epsilon_{m_1}\dots\epsilon_{m_r})\circ\theta_{r+1}$$

for $s \geq 1$, *where* δ *lies in the image of the codiagonal* ∇_*.

The proof is similar to 7.11.

§8. The mod-p Homology of Symmetric Products

As an application of pensions we compute functorially the mod-p homology of symmetric products of polyhedra.

8.1. Symmetric smash products and symmetric products. If K is an s.s. set with basepoint, the r-fold symmetric smash product of K, $SP^r_\wedge K$, is formed from the r-fold smash product of K, $K\wedge\cdots\wedge K$, by identifying $y_1\wedge\cdots\wedge y_r$ with $y_{\tau(1)}\wedge\cdots\wedge y_{\tau(r)}$ for each permutation τ on r elements where $y_1,\ldots,y_r \in K_n$, $n \geq 0$. Clearly

$$\tilde{H}_*(SP^r_\wedge K;Z_p) \approx \pi_* SP^r(Z_pK)$$

where

$$SP^r: \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

is as in 5.5. It is well known (see [5]) that

$$\tilde{H}_*(SP^r{}_x K;Z_p) \approx \sum_{i=1}^{r} \tilde{H}_*(SP^i_\wedge K;Z_p)$$

not quite.

where $SP^r{}_x K$ is r-fold symmetric product of K. We shall compute $\pi_* SP^r X$ functorially in terms of $\pi_* X$, where X is an s.s. Z_p-module. This will therefore determine the groups $\tilde{H}_*(SP^r{}_x K;Z_p)$ in terms of $\tilde{H}_*(K;Z_p)$.

8.2. Semi-simplicial commutative Z_p-algebras. Let $\mathcal{C}_p$ denote the category of s.s. objects over the category of associative commutative Z_p-algebras (without 1). For example, if X is an s.s. Z_p-module, then

$$\sum_{r=1}^{\infty} SP^r X \in \mathcal{C}_p,$$

if Y is an s.s. abelian group, then

$$Z_p Y \in \mathcal{C}_p.$$

If $B \in \mathcal{C}_p$ *then* $\pi_* B$ *is an associative, anticommutative, graded* Z_p*-algebra (without 1). Furthermore if* $u \in \pi_n B$, $n > 0$, *then* $u^2 = 0$ *when* n *odd, and* $u^p = 0$ *when* n *even.* The multiplication in $\pi_* B$ is induced by the composition

$$\pi_i B \otimes \pi_j B \xrightarrow{\xi} \pi_{i+j}(B \otimes B) \xrightarrow{m_*} \pi_{i+j} B$$

where ξ is the Eilenberg-Zilber map and

$$m: B \otimes B \to B$$

is the multiplication map of B.

We now suppose p = 2 and will defer the case p odd to 8.8.

8.3. *Homotopy operators on* $\mathcal{C}_2$. Each element

$$y \in L_i SP^r(Z_2, n)$$

determines by composition a natural homotopy operator

$$y: \pi_n B \to \pi_i B, \qquad B \in \mathcal{C}_2,$$

and all primary homotopy operators on $\mathcal{C}_2$ arise in this way.

For $2 \leq t \leq n$ let

$$\alpha_t: \pi_n B \to \pi_{n+t} B, \qquad B \in \mathcal{C}_2$$

be the operator given by the non-zero element (see 5.6)

$$\alpha_t \in L_{n+t} SP^2(Z_2, n) \approx Z_2.$$

Theorem 8.4. *Let* $B \in \mathcal{C}_2$.

(i) <u>If</u> $u, v \in \pi_n B$, <u>then</u>

$$\alpha_t(u+v) = \begin{cases} \alpha_t(u) + \alpha_t(v) \ \underline{\text{for}}\ 2 \le t < n \\ \alpha_t(u) + \alpha_t(v) + u \cdot v \ \underline{\text{for}}\ 2 \le t = n. \end{cases}$$

(ii) <u>If</u> $u \in \pi_i B$, $v \in \pi_j B$, <u>and</u> $2 \le t \le i+j$, <u>then</u>

$$\alpha_t(u \cdot v) = \begin{cases} 0 \text{ for } i, j > 0 \\ u^2 \cdot \alpha_t(v) \text{ for } i = 0. \end{cases}$$

(iii) <u>If</u> $u \in \pi_k B$, $2 + m + n \le k$, <u>and</u> $m, n \ge 0$, <u>then</u>

$$0 = \sum_{i+j=n} \binom{n}{i} \alpha_{2+2m+i}(\alpha_{2+m+j}(u))$$

$$0 = \sum_{i+j=n} \binom{n}{i} \alpha_{3+2m+i}(\alpha_{2+m+j}(u)).$$

This will be proved in 8.13.

If $u \in \pi_n B$ and $I = (i_1, \ldots, i_k)$, $k \ge 0$, let

$$\alpha_I(u) = \begin{cases} u \text{ for } k = 0 \\ \alpha_{i_1}(\ldots(\alpha_{i_k}(u))\ldots) \text{ for } k > 0 \end{cases}$$

when the right hand side is defined.

Call I <u>admissible</u> if $k = 0$ or if $k > 0$, $i_k \ge 2$, and $i_{s-1} \ge 2i_s$ when $2 \le s \le k$.

For I admissible let

$$\text{excess}(I) = \begin{cases} 0 \quad \text{for } k = 0 \\ i_1 \quad \text{for } k = 1 \\ i_1 - i_2 - \ldots - i_k \text{ for } k \ge 2. \end{cases}$$

Remark 8.5. By 8.4, for $u \in \pi_n B$ any well defined $\alpha_I(u)$ may be expressed as a sum of products of terms $\alpha_J(u)$ where J is admissible with excess (J) $\leq$ n.

If X is any s.s. Z_2-module consider

$$\sum_{r=1}^{\infty} SP^r X \in \mathcal{C}_2$$

and identify $\pi_* X$ with $\pi_* SP^1 X$.

Theorem 8.6. *Let* $\{a\}_{a \in A}$ *be a* Z_2-*basis for* $\pi_* X$ *with* $a \in \pi_{d(a)} X$. *Then* $\sum_{r=1}^{\infty} \pi_* SP^r X$ *is the associative, commutative, graded* Z_2-*algebra (without 1) with generators*

$$\{\alpha_I(a) \mid a \in A, \text{ I admissible, excess}(I) \leq d(a)\}$$

and with relations $(\alpha_I(a))^2 = 0$ *when* $d(a) \geq 1$.

This will be proved in 8.14.

Remark 8.7. In 8.6

$$\alpha_I(a) \in \pi_{i+d(a)} SP^{2^k}(X)$$

where $I = (i_1, \ldots, i_k)$ and $i = i_1 + \ldots + i_k$. Thus 8.6 determines the individual groups $\pi_q SP^r X$. This determination is functorial in $\pi_* X$ in view of 8.4.

We now turn to the case of p an odd prime.

8.8. Homotopy operators on $\mathcal{C}_p$. For $B \in \mathcal{C}_p$ we define homotopy operators

$$\nu_t \colon \pi_n B \to \pi_{n+2t(p-1)} B \text{ for } 2 \leq 2t \leq n$$

$$\eta_t \colon \pi_n B \to \pi_{n+1+2t(p-1)} B \text{ for } 3 \leq 2t+1 \leq n$$

corresponding to the generators

$$\sigma^{n-2t}\epsilon_p^t = \nu_t \in L_{n+2t(p-1)}SP^p(Z_p,n) \approx Z_p$$

$$\phi_1\sigma^{n-2t-1}\epsilon_p^t = \eta_t \in L_{n+1+2t(p-1)}SP^p(Z_p,n) \approx Z_p$$

given by the basis for $L_*SP^p(Z_p,n)$ described in 6.7.

<u>Theorem 8.9</u>. <u>Let</u> $B \in \mathcal{C}_p$.

(i) <u>If</u> $u,v \in \pi_n B$ <u>then</u>:

(a) $\eta_t(u+v) = \eta_t(u) + \eta_t(v)$ for $3 \le 2t+1 \le n$.

(b) $\nu_t(u+v) = \nu_t(u) + \nu_t(v)$ for $2 \le 2t < n$.

(c) $\nu_t(u+v) = \nu_t(u) + \nu_t(v) + \sum_{i=1}^{t-1} \nu_i(u)\cdot\nu_{t-i}(v)$ for $2 \le 2t = n$.

not $\sum_{i=1}^{p-1} \frac{u^i v^{p-i}}{i!(p-i)!}$?

(ii) <u>If</u> $u \in \pi_i B$ <u>and</u> $v \in \pi_j B$ <u>then</u>:

(a) $$\eta_t(u\cdot v) = \begin{cases} 0 \text{ for } 3 \le 2t+1 \le i+j \text{ and } i,j > 0 \\ u^p\cdot\eta_t(v) \text{ for } 3 \le 2t+1 \le i+j \text{ and } i = 0. \end{cases}$$

(b) $$\nu_t(u\cdot v) = \begin{cases} 0 \text{ for } 2 \le 2t \le i+j \text{ and } i,j > 0 \\ u^p\cdot\nu_t(v) \text{ for } 2 \le 2t \le i+j \text{ and } i = 0. \end{cases}$$

(iii) <u>If</u> $u \in \pi_k B$ <u>then</u>:

(a) <u>For</u> $m,n \ge 0$, $1 \le r < p$, <u>and</u> $k \ge 2+2m+2n$

$$0 = \sum_{i+j=n} \binom{n}{i} \nu_{r+pm+i}(\nu_{1+m+j}(u)).$$

(b) <u>For</u> $m,n \ge 0$ <u>and</u> $k \ge 4+2m+2n$

$$0 = \sum_{i+j=n} \binom{n}{i} \nu_{p+pm+i}(\nu_{2+m+j}(u)).$$

(c) For $m,n \geq 0$, $1 \leq r \leq p$, and $k \geq 3+2m+2n$

$$0 = \sum_{i+j=n} \binom{n}{i} \nu_{r+pm+i}(\eta_{1+m+j}(u)).$$

(d) For $m,n \geq 0$, $1 \leq r < p$, and $k \geq 2+2m+2n$

$$0 = \sum_{i+j=n} \binom{n}{i} \eta_{r+pm+i}(\nu_{1+m+j}(u)).$$

(e) For $m,n \geq 0$ and $k \geq 5+2m+2n$

$$0 = \sum_{i+j=n} \binom{n}{i} \eta_{p+pm+i}(\nu_{2+m+j}(u)) + \sum_{i+j=n} \binom{n}{i} \nu_{p+pm+i}(\eta_{2+m+j}(u)).$$

(f) For $m,n \geq 0$, $1 \leq r \leq p$, and $k \leq 3+2m+2n$

$$0 = \sum_{i+j=n} \binom{n}{i} \eta_{r+pm+i}(\eta_{1+m+j}(u)).$$

For $k \geq 0$ let $I = (\delta_1, s_1, \delta_2, s_2, \ldots, \delta_k, s_k)$ be a 2k-triple of integers with each $s_i \geq 1$ and $\delta_i = 0,1$. For $u \in \pi_n B$ let

$$\alpha_I(u) = \begin{cases} u \text{ for } k = 0 \\ \alpha_{\delta_1, s_1}(\ldots(\alpha_{\delta_k, s_k}(u))\ldots) \text{ for } k > 0 \end{cases}$$

when the right side is defined, where $\alpha_{0,s} = \nu_s$ and $\alpha_{1,s} = \eta_s$.

Call I admissible if $k = 0$ or if $k > 0$ and $s_{i-1} \geq ps_i + \delta_i$ when $2 \leq i \leq k$. For I admissible let

$$\text{excess}(I) = \begin{cases} 0 \text{ for } k = 0 \\ 2s_1+\delta_1 \text{ for } k = 1 \\ 2s_1+\delta_1 - \sum_{i=2}^{k} (2s_i(p-1)+\delta_i) \text{ for } k \geq 2. \end{cases}$$

Remark 8.10. By 8.9 if $u \in \pi_n B$, any well defined $\alpha_I(u)$ may be expressed as a sum of products of terms $\alpha_J(u)$ where J is admissible with excess $(J) \leq n$.

If X is any s.s. Z_p-module, consider

$$\sum_{r=1}^{\infty} SP^r X \in \mathcal{C}_p$$

and identify $\pi_* X$ with $\pi_* SP^1 X$.

Theorem 8.11. Let $\{a\}_{a \in A}$ be a Z_p-basis for $\pi_* X$ with $a \in \pi_{d(a)} X$. Then $\sum_{r=1}^{\infty} \pi_* SP^r X$ is the associative, anticommutative, graded Z_p-algebra (without 1) with generators

$$\{\alpha_I(a) \mid a \in A, \text{ I admissible, and excess}(I) \leq d(a)\}$$

and with relations $(\alpha_I(a))^2 = 0$ when degree $\alpha_I(a)$ is odd and $(\alpha_I(a))^p = 0$ when degree $\alpha_I(a)$ is positive even.

This will be proved in 8.14 and 8.17.

Remark 8.12. In 8.11

$$\alpha_I(a) \in \pi_{s+d(a)} SP^{p^k}(X)$$

where $I = (\delta_1, s_1, \ldots, \delta_k, s_k)$ and

$$s = \sum_{i=1}^{k} (\delta_i + 2s_i(p-1)).$$

Thus 8.11 determines the individual groups $\pi_q SP^r(X)$, and this determination is functorial in $\pi_* X$ in view of 8.9.

We devote the rest of §8 to proofs.

8.13. Proof of 8.4. Part (i) is proved by a straightforward argument with the universal example

$$B = \sum_{r=1}^{\infty} SP^r(K(Z_2,n) + K(Z_2,n)).$$

For part (ii) the universal example is

$$B = \sum_{r=1}^{\infty} SP^r(K(Z_2,i) + K(Z_2,j))$$

and the case $i = 0$, $j > 0$ follows easily. Now suppose $i,j > 0$. For $M,N \in \mathcal{A}_{Z_2}$ consider the homomorphism

$$h: SP^2(M \boxtimes N) \to SP^2M \boxtimes SP^2N$$

with $h((m_1 \boxtimes m_2)\cdot(m_2 \boxtimes n_2)) = (m_1\cdot m_2)\boxtimes(n_1\cdot n_2)$. Form the induced map

$$h_*: \pi_* SP^2(K(Z_2,i)\boxtimes K(Z_2,j)) \to \pi_*(SP^2K(Z_2,i)\boxtimes SP^2K(Z_2,j)).$$

We claim $h_* = 0$. For a functor in two variables pensions may be applied individually to the left and right variables. Since $h_* = 0$ for $i = j = 1$, we simply apply σ and σ_2 to individual variables to prove inductively that $h_* = 0$ using 5.6. Now part (ii) follows from the universal example since the composition of

$$\pi_* SP^2(K(Z_2,i)\boxtimes K(Z_2,j)) \xrightarrow{h_*} \pi_*(SP^2K(Z_2,i)\boxtimes SP^2K(Z_2,j)) \to \pi_* SP^4(K(Z_2,i)+K(Z_2,j))$$

is zero.

For part (iii) consider the generators (see 5.6)

$$b \in L_4SP^2(Z_2,2) \approx Z_2$$

$$\sigma\sigma(b) \in L_6SP^2(Z_2,4) \approx Z_2$$

$$\sigma\sigma_2(b) \in L_7SP^2(Z_2,4) \approx Z_2.$$

The natural composition map q: $SP^2_{\circ} SP^2 \to SP^4$ induces composition products (§7)

$$q_*((\sigma\sigma\ (b))\circ b) \in L_6SP^4(Z_2,2)$$

$$q_*((\sigma\sigma_2(b))\circ b) \in L_7SP^4(Z_2,2)$$

which are zero since by [7, 12.22] $L_qSP^4(Z_2,2) \approx 0$ for $q \neq 8$. For $x \in L_{i-1}SP^2(Z_2,j-1)$ and $v \in L_jSP^2(Z_2,n)$, 5.7 and 7.3 imply

$$\sigma((\sigma x)\circ v) = (\sigma(\sigma x))\circ(\sigma v)$$

$$\sigma_2((\sigma x)\circ v) = (\sigma_2(\sigma x)\circ(\sigma v) + (\sigma\sigma(\sigma x))\circ(\sigma_2 v)$$

$$\sigma_4((\sigma x)\circ v) = (\sigma_2\sigma_2(\sigma x))\circ(\sigma_2 v).$$

The relations $q_*((\sigma\sigma(b))\circ b) = 0$ and $q_*((\sigma\sigma_2(b))\circ b) = 0$ thus generate new relations for composition products, and these imply 8.4 (iii).

<u>8.14. Proof of 8.6</u>. For $M \in \mathcal{A}_{Z_2}$ and $r \geq 1$, let

$$t:\ SP^rM \to \otimes^rM$$

be the homomorphism

$$t(m_1,\ldots,m_r) = \sum_{\tau} m_{\tau(1)}\otimes\cdots\otimes m_{\tau(r)}$$

where τ ranges over the permutations on r elements.

For $r,s \geq 1$ let

$$d:\ SP^{rs}M \to SP^r(SP^sM)$$

$$e:\ SP^{r+s}M \to SP^rM\otimes SP^sM$$

be the unique natural homomorphisms such that the diagrams

$$\begin{array}{ccc} SP^{rs}M & \xrightarrow{d} & SP^r(SP^sM) \\ \downarrow t & & \downarrow t\circ t \\ \otimes^{rs}M & \rightarrow & \otimes^r(\otimes^sM) \end{array}$$

$$\begin{array}{ccc} SP^{r+s}M & \xrightarrow{e} & SP^rM\otimes SP^sM \\ \downarrow t & & \downarrow t\otimes t \\ \otimes^{r+s}M & \rightarrow & \otimes^rM\otimes\otimes^sM \end{array}$$

commute, where the lower maps "insert parentheses".

If $j \geq 2$ the map $q\circ d$

$$SP^{2^j}M \xrightarrow{d} SP^2(SP^{2^{j-1}}M) \xrightarrow{q} SP^{2^j}M$$

is an isomorphism where q is the natural composition map.

If $r \neq 2^j$ for any j, the map $m\circ e$

$$SP^rM \xrightarrow{e} SP^{2^k}M\otimes SP^{r-2^k}M \xrightarrow{m} SP^rM$$

is an isomorphism where k is the largest integer such that 2^k divides r, and m is the multiplication map in the symmetric algebra.

If X is an s.s. Z_2-module

$$q_*: \pi_*SP^2(SP^{2^{j-1}}X) \rightarrow \pi_*SP^{2^j}X$$

$$m_*: \pi_*(SP^{2^k}X\otimes SP^{r-2^k}X) \rightarrow \pi_*SP^rX$$

are thus epimorphisms. This easily implies that

$$\sum_{r=1}^{\infty} \pi_*SP^rX$$

is generated by $\pi_*X = \pi_*SP^1X$ under the action of the operators α_t, $t \geq 2$, and of the algebra multiplication and addition.

It then follows by 8.4 and 8.5 that the desired Z_2-basis for $\sum_{r=1}^{\infty} \pi_* SP^r X$ at least generates. Now 8.6 follows for $X = K(Z_2, n)$ by a counting argument using the fact [7, 4.16] that

$$\sum_{r=1}^{\infty} \pi_q SP^r K(Z_2, n) \approx \mathfrak{H}_q(Z, n; Z_2)$$

for $n \geq 1$. For general X, 8.6 follows since X is homotopy equivalent to a sum of complexes $K(Z_2, n)$, $n \geq 0$, and

$$\left(\sum_{r=1}^{\infty} SP^r M_1\right) \otimes \cdots \otimes \left(\sum_{r=1}^{\infty} SP^r M_k\right) \approx \sum_{r=1}^{\infty} SP^r (M_1 \oplus \cdots \oplus M_k)$$

for $M_1, \ldots, M_k \in \mathcal{A}_{Z_2}$.

<u>8.15</u>. <u>Proof of 8.9</u>. Parts (i) and (ii) follow as in 8.14. For part (iii) let $q\colon \Gamma^p_p \circ \Gamma^p_p \to \Gamma^{p^2}_p$ be the natural composition map. We claim that

(1) $\quad q_*((\sigma^{2p} \nu) \circ (\epsilon_p \nu)) = 0.$

(2) $\quad q_*((\sigma^2 \nu) \circ (\phi_1 \nu)) = 0.$

(3) $\quad q_*((\sigma^{2p-2r} \epsilon_p^r \nu) \circ (\epsilon_p \nu)) = 0$ for $1 \leq r < p$.

(4) $\quad q_*((\sigma^{2p-2r-1} \phi_1 \epsilon_p^r \nu) \circ (\epsilon_p \nu)) = 0$ for $1 \leq r < p$

using the notation of 6.7. To prove (1) and (2) consider the commutative diagram

$$\begin{array}{ccc} SP^p_p M & \xrightarrow{j} & SP^{p^2}_p M \\ \downarrow g & & \downarrow g \\ \Gamma^p_p M & \xrightarrow{\gamma_p} & \Gamma^{p^2}_p M \end{array}$$

for $M \in \mathcal{A}_{Z_p}$, where the monomorphism g is as in 5.5, γ_p is the p^{th} divided power operator, and j is the restriction of the function γ_p. Clearly for $y \in \pi_t \Gamma^p{}_p K(Z_p,n)$, $q_*((\sigma^t v)\circ y)$ is the image of y under

$$\gamma_{p*}: \pi_t \Gamma^p_p K(Z_p,n) \to \pi_t \Gamma^{p^2}_p K(Z_p,n).$$

Thus (1) and (2) follow from the diagram using 6.7, since the method of 8.14 shows

$$\pi_{2p} SP^{p^2}_p K(Z_p,2) = 0$$

$$\pi_2 SP^{p^2}_p K(Z_p,1) = 0.$$

The analogues (see (6.7) of (3) and (4) for the composition map $q: SP^p \circ SP^p \to SP^{p^2}$ are easily proved as in 8.13, and these analogues imply (3) and (4) using the map (5.5) $g\circ f: SP^r \to \Gamma_p{}^r$. The relations (1)-(4) imply new relations using 6.8, 7.2, and 7.4. Using 8.16 one then deduces relations for $q: SP^p \circ SP^p \to SP^{p^2}$ which imply 8.9 (iii).

__Lemma 8.16.__ _The map_

$$f_* \circ g_*: \pi_* SP^r K(Z_p,n) \to \pi_* \Gamma_p{}^r K(Z_p,n)$$

is a monomorphism for $r,n \geq 1$.

__Proof.__ For $SP^r, \Gamma^r: \mathcal{A}_Z \to \mathcal{A}_Z$ consider the map (4.4) $a: SP^r \to \Gamma^r$. The composition

$$\pi_* SP^r K(Z,n) \xrightarrow{a_*} \pi_* \Gamma^r K(Z,n) \xrightarrow{\beta} \pi_{*+2r} SP^r K(Z,n+2)$$

equals the pension ϵ_r where β is the isomorphism of 4.1. Thus (4.2) a_* is a monomorphism onto a direct summand, which implies 8.16.

__8.17.__ The proof of 8.11 is a mod-p version of 8.14.

Part II. Stable Derived Functors

§9. Stable Pensions for Stable Derived Functors

9.1. Stable derived functors.

Let $T: \mathcal{A}_R \to \mathcal{A}_R$ be a covariant functor with $T(0) = 0$, where R is a commutative ring with identity. For $n \geq 0$, T has stable derived functors

$$L^s{}_nT: \mathcal{A}_R \to \mathcal{A}_R$$

which are additive functors defined [6, 6.7] as the limit

$$L^s{}_nT(G) = \lim_{q\to\infty} L_{n+q}T(G,q)$$

taken with respect to the suspension map

$$\sigma: L_{n+q}T(G,q) \to L_{n+q+1}T(G,q+1).$$

The following are basic properties (see [6]) of stable derived functors.

(1) If $0 \to T' \to T \to T'' \to 0$ is an exact sequence of functors $\mathcal{A}_R \to \mathcal{A}_R$, there is a natural homomorphism (see 9.2)

$$\bar{\partial}_n: L^s{}_nT'' \to L^s{}_{n-1}T', \qquad n \geq 1$$

such that

$$\cdots \to L^s{}_{n+1}T'' \xrightarrow{\bar{\partial}_{n+1}} L^s{}_nT' \to L^s{}_nT \to L^s{}_nT'' \xrightarrow{\bar{\partial}_n} \cdots \to L^s{}_0T \to L^s{}_0T'' \to 0$$

is exact.

(2) If $0 \to G' \to G \to G'' \to 0$ is exact in $\mathcal{A}_R$, there is a natural homomorphism (see 9.2)

$$\bar{\partial}_n: L^s{}_nT(G'') \to L^s{}_nT(G')$$

such that

$$\cdots \to L^s_{n+1}T(G'') \xrightarrow{\bar\partial_{n+1}} L^s_n T(G') \to L^s_n T(G) \to L^s_n T(G'') \xrightarrow{\bar\partial_n} \cdots$$

$$\cdots \to L^s_0 T(G) \to L^s_0 T(G'') \to 0$$

is exact.

(3) Under the hypotheses of (1) and (2) the diagram

$$\begin{array}{ccc} L^s_{n+1}T''(G'') & \xrightarrow{\bar\partial_{n+1}} & L^s_n T'(G'') \\ \downarrow \bar\partial_{n+1} & & \downarrow \bar\partial_n \\ L^s_n\ T''(G') & \xrightarrow{\bar\partial_n} & L^s_{n-1}T'(G') \end{array}$$

anticommutes.

(4) If $T: \mathcal{A}_R \to \mathcal{A}_R$ is additive, then $L^s_n T$ equals the classical [4] n^{th} left derived functor of T.

(5) If $U(\cdot,\cdot): \mathcal{A}_R \times \mathcal{A}_R \to \mathcal{A}_R$ with $U(X,Y) = 0$ when either $X = 0$ or $Y = 0$ and $T(X) = U(X,X)$, then $L^s_n T = 0$ for $n \geq 0$.

9.2. The boundary operator in 9.1 (1) arises as follows. If $K(G,q)$ is an s.s. projective resolution [7, 4.1] of type (G,q), then the homotopy exact sequence for

$$0 \to T'K(G,q) \to TK(G,q) \to T''K(G,q) \to 0$$

has a boundary operator

$$\partial_n: \pi_{n+q}T''K(G,q) \to \pi_{n-1+q}T'K(G,q).$$

We replace ∂_n by

$$\bar\partial_n = (-1)^q\ \partial_n$$

which commutes with the suspension map. This operator $\bar\partial_n$ stabilizes to give the boundary operator in (1).

For 9.1 (2) let

$$0 \to K(G',q) \xrightarrow{i} K(G,q) \xrightarrow{j} K(G'',q) \to 0$$

be a short exact sequence of s.s. projective resolutions representing

$$0 \to G' \to G \to G'' \to 0.$$

The short exact sequence

$$0 \to (\text{Kernel } T(j)) \to TK(G,q) \xrightarrow{T(j)} TK(G'',q) \to 0$$

gives a boundary map

$$\partial_n\colon \pi_{n+q}TK(G'',q) \to \pi_{n-1+q}(\text{Kernel } T(j)).$$

The map T(i) induces an isomorphism

$$T(i)_*\colon \pi_{n-1+q}TK(G',q) \to \pi_{n-1+q}(\text{Kernel } T(j))$$

for $n \leq q$, as shown by the cross-effect exact sequence (3.6) and the connectivity property of cross-effects [7, 6.10]. Thus for $n \leq q$

$$\partial_n\colon \pi_{n+q}T(G'',q) \to \pi_{n-1+q}T(G',q),$$

and the operator

$$\overline{\partial}_n = (-1)^q \partial_n$$

then stabilizes to the boundary operator in (2).

9.3. Stable pensions. The stable pension algebra $P^s(R)$ is the graded R-algebra with

$$P^s(R)_m = \lim_{q\to\infty} \widetilde{H}_{m+q}(R,q;R)$$

taken with respect to the suspension map and with multiplication in $P^s(R)$ induced by that in P(R). The algebra $P^s(R)$ is anticommutative by 2.1, i.e., if $\alpha \in P^s(R)_m$ and $\beta \in P^s(R)_n$, then

$$\alpha\cdot\beta = (-1)^{mn}\,\beta\cdot\alpha.$$

The action of pensions on derived functors (2.2) stabilizes, so that for each element $\alpha \in P^s(R)_m$ there is a <u>stable pension</u>

$$\alpha: L^s{}_nT \to L^s{}_{m+n}T.$$

For $G \in \mathcal{A}_R$ the stable pensions give

$$\sum_{n=0}^{\infty} L^s{}_nT(G)$$

the structure of a graded module over $P^s(R)$.

<u>The stable pensions commute with boundary maps.</u> Thus for $\alpha \in P^s(R)_m$ the diagram

$$\begin{array}{ccc} L^s{}_nT'' & \xrightarrow{\bar\partial_n} & L^s{}_{n-1}T' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ L^s{}_{m+n}T'' & \xrightarrow{\bar\partial_{m+n}} & L^s{}_{m+n-1}T' \end{array}$$

commutes under the hypotheses of 9.1 (1), and

$$\begin{array}{ccc} L^s{}_nT(G'') & \xrightarrow{\bar\partial_n} & L^s{}_{n-1}T(G') \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ L^s{}_{m+n}T(G'') & \xrightarrow{\bar\partial_{m+n}} & L^s{}_{m+n-1}T(G') \end{array}$$

commutes under the hypotheses of 9.1 (2).

<u>9.4</u>. <u>Stable operators</u>. We shall give an alternative description of $P^s(R)$ as an algebra of operators. Let $op(R)_m$ be the set consisting of the (possibly non-additive) operators

$$\Phi: L^s{}_nT(R) \to L^s{}_{n+m}T(R)$$

defined for each $n \geq 0$ and each $T: \mathcal{A}_R \to \mathcal{A}_R$ with $T(0) = 0$, such that Φ is natural in T and the diagram

$$\begin{array}{ccc} L^s_n T''(R) & \xrightarrow{\bar\partial_n} & L^s_{n-1}T'(R) \\ \downarrow \Phi & & \downarrow \Phi \\ L^s_{m+n}T''(R) & \xrightarrow{\bar\partial_{m+n}} & L^s_{m+n-1}T'(R) \end{array}$$

commutes for each exact sequence

$$0 \to T' \to T \to T'' \to 0.$$

Then addition and scalar multiplication of operators make $op(R)_m$ into an R-module. If

$$op(R) = \sum_m op(R)_m$$

there is a homomorphism

$$P^s(R) \to op(R)$$

sending elements of $P^s(R)$ to the associated pensions. This map carries a product in $P^s(R)$ to a composition in $op(R)$.

<u>Theorem 9.5</u>. *The map $P^s(R) \to op(R)$ is an isomorphism.*

<u>Proof</u>. For $T: \mathcal{A}_R \to \mathcal{A}_R$ there is an exact sequence of functors

$$0 \to \Omega T \to T_2{}^d \xrightarrow{\nabla} T \to T^{ad} \to 0$$

where $T_2{}^d(X) = T_2(X,X)$, ∇ is the cross-effect codiagonal, $\Omega T = \text{Kernel}\,\nabla$, and $T^{ad} = \text{Cokernel}\,\nabla$. The functor T^{ad} is additive so $L^s{}_i T^{ad}(R) = 0$ for $i > 0$ by 9.1 (4); and $L^s{}_i T_2{}^d(R) = 0$ for $i \geq 0$ by 9.1 (5). Thus for each j there is a map

$$e\colon L^s{}_j \Omega T(R) \to L^s{}_{j+1} T(R)$$

such that e commutes with operators in $op(R)$ and is an isomorphism for $j \geq 0$. Let $\Omega^0 T = T$ and $\Omega^n T = \Omega(\Omega^{n-1}T)$ for $n \geq 1$. For $n \geq 0$ and $\Phi \in op(R)_m$ the diagram

$$\begin{array}{ccc} L^s{}_0\Omega^n T(R) & \xrightarrow{e^n} & L^s{}_n T(R) \\ \downarrow{\Phi} & & \downarrow{\Phi} \\ L^s{}_m\Omega^n T(R) & \xrightarrow{e^n} & L^s{}_{m+n} T(R) \end{array}$$

commutes. Thus for Φ, $\Phi' \in op(R)_m$ a necessary and sufficient condition that $\Phi = \Phi'$ is that

$$\Phi = \Phi' ; \ L^s{}_0 T(R) \to L^s{}_m T(R)$$

for each functor T.

Now consider the free R-module functor

$$R(\cdot): \mathcal{A}_R \to \mathcal{A}_R.$$

For any functor T there is an isomorphism

$$Hom(R(\cdot), T(\cdot)) \approx T(R)$$

sending each $f: R(\cdot) \to T(\cdot)$ to the image of $1[1]$ under $f: R(R) \to T(R)$. We note that the inverse of this isomorphism sends $u \in T(R)$ to

$$E(\cdot \otimes u): R(X) \to T(X)$$

where

$$E: R(X) \otimes T(R) \to T(X \otimes R) = T(X)$$

is as in §2.

Now consider the epimorphism

$$\sigma: T(R) = L_0 T(R,0) \xrightarrow{\sigma} L_1 T(R,1) = L^s{}_0 T(R).$$

For the functor $R(\cdot)$, let $1 \in L^s{}_0 R(R)$ denote $\sigma(1[1])$. It follows from the above that for any $u \in L^s{}_0 T(R)$ there is a map $f: R(\cdot) \to T(\cdot)$ such that

$$f_*: L^s{}_0 R(R) \to L^s{}_0 T(R)$$

gives $f_*(1) = u$. Hence for Φ, $\Phi' \in op(R)_m$ a necessary and sufficient condition that $\Phi = \Phi'$ is that

$$\Phi, \Phi' : L^s{}_0 R(R) \to L^s{}_m R(R)$$

satisfy $\Phi(1) = \Phi'(1)$.

By definition $L^s{}_i R(R) = P^s(R)_i$ for $i \geq 0$, and for $\alpha \in P^s(R)_m$ the pension

$$\alpha : L^s{}_0 R(R) \to L^s{}_m R(R)$$

gives $\alpha(1) = \alpha$. This easily implies the theorem.

Let p be a prime.

10.1. The algebra $P^s(Z_p)$.

From the presentation of the Steenrod algebra in [8], it is easily seen that there is an algebra isomorphism $P^s(Z_p) \approx A_*(p)$ where $A_*(p)$ is the dual to the mod-p Steenrod algebra.

Recall [13] that $A_*(2)$ is the Z_2-polynomial algebra on generators ξ_i of degree 2^i-1 for $i \geq 1$. For $i \geq 1$ the element (§5) $\sigma_2 i \in \widetilde{H}_2 i(Z_2,1;Z_2)$ stabilizes to an element $[\sigma_2 i] \in P^s(Z_2)$ and the isomorphism $P^s(Z_2) \approx A_*(2)$ carries $[\sigma_2 i]$ to ξ_i.

For p odd recall that $A_*(p)$ is the tensor product of: (i) the Z_p-polynomial algebra on generators ξ_i of degree $2p^i-2$ for $i \geq 1$ and (ii) the Z_p-exterior algebra on generators τ_i of degree $2p^i-1$ for $i \geq 0$. For $i \geq 1$ the element (§3) $\epsilon_p i \in \widetilde{H}_{2p} i(Z_p,2;Z_p)$ stabilizes to an element $[\epsilon_p i] \in P^s(Z_p)$ and the isomorphism $P^s(Z_p) \approx A_*(p)$ carries $[\epsilon_p i]$ to ξ_i. For $i \geq 0$ the element (6.4) $\phi_p i \in \widetilde{H}_{2p} i(Z_p,1;Z_p)$ stabilizes to an element $[\phi_p i] \in P^s(Z_p)$ and the isomorphism $P^s(Z_p) \approx A_*(p)$ carries $[\phi_p i]$ to τ_i.

We henceforth identify the algebras $P^s(Z_p)$ and $A_*(p)$ for p prime.

10.2. Composed functors.

Let

$$T,T' : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

be covariant functors with $T(0) = T'(0) = 0$, and consider the composed functor

$$T \circ T' : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}.$$

The composition products (§7)

$$L_{i+j+n}T(Z_p,j+n) \times L_{j+n}T'(Z_p,n) \to L_{i+j+n}(T\circ T')(Z_p,n)$$

stabilize to give pairings

$$L^s{}_iT(Z_p)\otimes_{Z_p} L^s{}_jT'(Z_p) \to L^s{}_{i+j}(T\circ T')(Z_p)$$

which induce a homomorphism

$$c:\left(\sum_{i=0}^{\infty} L^s{}_iT(Z_p)\right)\otimes_{Z_p}\left(\sum_{j=0}^{\infty} L^s{}_jT'(Z_p)\right) \to \sum_{n=0}^{\infty} L^s{}_n(T\circ T')(Z_p).$$

As we have seen (9.3), the stable pensions give $\sum_{i=0}^{\infty} L^s{}_iT(Z_p)$ the structure of a module over $A_*(p)$. Since $A_*(p)$ is a Hopf algebra, both sides of the above homomorphism are modules over $A_*(p)$.

<u>Theorem 10.3</u>. <u>The map</u>

$$c:\left(\sum_{i=0}^{\infty} L^s{}_iT(Z_p)\right)\otimes_{Z_p}\left(\sum_{j=0}^{\infty} L^s{}_jT'(Z_p)\right) \to \sum_{n=0}^{\infty} L^s{}_n(T\circ T')(Z_p)$$

<u>is a homomorphism of modules over</u> $A_*(p)$. <u>Furthermore</u>, c <u>is an isomorphism provided either</u> (i) T <u>commutes with direct limits over directed systems or</u> (ii) <u>if</u> $G \in \mathcal{A}_{Z_p}$ <u>is finitely generated then so is</u> $T'(G)$.

<u>Proof</u>. First let p = 2. For $u \in L^s{}_iT(Z_2)$ and $v \in L^s{}_jT'(Z_2)$, the formula 7.7 stabilizes to show for $n \geq 1$

$$\xi_n(c(u\otimes v)) = \sum_{k=0}^{n} c((\xi_{n-k})^{2^k}(u)\otimes\xi_k(v))$$

where $\xi_0 = 1$. Thus c is an $A_*(2)$-homomorphism.

Now let p be odd. For $u \in L^s{}_iT(Z_p)$ and $v \in L^s{}_jT'(Z_p)$ the formulas 7.6 and 7.8 stabilize to show for $n \geq 1$ and $m \geq 0$ that

$$\xi_n(c(u \otimes v)) = \sum_{k=0}^{n} c((\xi_{n-k})^{p^k}(u) \otimes \xi_k(v))$$

$$\tau_m(c(u \otimes v)) = (-1)^i \sum_{k=0}^{m} c((\xi_{m-n})^{p^k}(u) \otimes \tau_k(v)) + c(\tau_m(u) \otimes v).$$

Thus c is an $A_*(p)$-homomorphism for p odd,

The final statement of 10.3 follows since $T'K(Z_p,q)$ is homotopy equivalent to a sum $Y = \sum_{\beta \in B} K(Z_p, d(\beta))$, where B is a basis for $\pi_* T'K(Z_p,q)$ with $\beta \in \pi_{d(\beta)} T'K(Z_p,q)$ for $\beta \in B$. By (i) or (ii), TY can be analyzed using cross-effects and the desired isomorphism is easily proved.

§11. Functor Algebras

The theory of stable derived functors is particularly interesting when applied to functor algebras.

Let $\mathcal{A}$ be any category.

<u>Definition 11.1</u>. A <u>functor algebra</u> over $\mathcal{A}$ consists of:

(i) A covariant functor $T: \mathcal{A} \to \mathcal{A}$

(ii) A natural transformation $m: T\circ T \to T$ such that the diagram

$$\begin{array}{ccc} T\circ T\circ T & \xrightarrow{m\circ 1} & T\circ T \\ \Big\downarrow{\scriptstyle 1\circ m} & & \Big\downarrow{\scriptstyle m} \\ T\circ T & \xrightarrow{m} & T \end{array}$$

commutes.

(iii) A natural transformation $i: I \to T$ where $I: \mathcal{A} \to \mathcal{A}$ is the identity functor and such that the composed maps

$$T = I\circ T \xrightarrow{i\circ 1} T\circ T \xrightarrow{m} T$$

$$T = T\circ I \xrightarrow{1\circ i} T\circ T \xrightarrow{m} T$$

are the identity.

<u>11.2</u>. <u>The functor algebra of adjoint functors</u>. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary categories; let

$$F: \mathcal{A} \to \mathcal{B}$$

$$G: \mathcal{B} \to \mathcal{A}$$

be covariant functors such that F is the left adjoint [10] of G; and let

$$\Theta: I_{\mathcal{A}} \longrightarrow G \circ F$$

$$\phi: F \circ G \to I_{\mathcal{B}}$$

be the adjunction morphisms, where $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are the indicated identity functors.

A functor algebra is now constructed as follows:

(i) Let $T = G \circ F: \mathcal{A} \to \mathcal{A}$

(ii) Let $m: T \circ T \to T$ be the map $G \circ F \circ G \circ F \xrightarrow{1 \circ \phi \circ 1} G \circ I_{\mathcal{B}} \circ F = G \circ F$

(iii) Let $i = \Theta: I_{\mathcal{A}} \to G \circ F = T$.

It is straightforward to verify that (T,m,i) is a functor algebra.

Example 11.3. The mod-p symmetric algebra functor (5.5)

$$SP^{+} = \sum_{r=1}^{\infty} SP^{r}: \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

has the structure of a functor algebra. Let $\mathcal{B}$ be the category of commutative Z_p-algebras without unit. Then the functor $SP^{+}: \mathcal{A}_{Z_p} \to \mathcal{B}$ is left adjoint to the forgetful functor $\mathcal{B} \to \mathcal{A}_{Z_p}$. Now 11.2 gives the usual maps $m: SP^{+} \circ SP^{+} \to SP^{+}$ and $i: I \to SP^{+}$.

Example 11.4. The mod-p truncated symmetric algebra functor (5.5)

$$SP^{+}_{p} = \sum_{r=1}^{\infty} SP^{r}_{p}: \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

has the structure of a functor algebra. Let $\mathcal{B}$ be the category whose objects $M \in \mathcal{B}$ are commutative Z_p-algebras without unit and with $m^p = 0$ for $m \in M$. Then the functor $SP^{+}_{p}: \mathcal{A}_{Z_p} \to \mathcal{B}$ is left adjoint to the forgetful functor $\mathcal{B} \to \mathcal{A}_{Z_p}$.

Now 11.2 gives the usual maps $m: SP^+_p \circ SP^+_p \to SP^+_p$ and $i: I \longrightarrow SP^+_p$.

Example 11.5 The mod-p gamma functor (5.5)

$$\Gamma^+_p = \sum_{r=1}^{\infty} \Gamma^r_p : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$$

has the structure of a functor algebra. Let $\mathcal{B}$ be the category whose objects $M \in \mathcal{B}$ are commutative Z_p-algebras without unit and with divided powers, i.e., for $m \in M$ and $r \geq 1$ there are defined elements $\gamma_r(m) \in M$ such that:

(i) $\gamma_1(m) = m$ for $m \in M$

(ii) $\gamma_r(m) \cdot \gamma_s(m) = (r,s)\ \gamma_{r+s}(m)$ for $m \in M$ and $r,s \geq 1$

(iii) $\gamma_t(m+n) = \gamma_t(m) + \gamma_t(n) + \sum_{r=1}^{t-1} \gamma_r(m) \cdot \gamma_{t-r}(n)$ for $m,n \in M$ and $t \geq 1$.

(iv) $\gamma_r(m \cdot n) = r!\ \gamma_r(m) \cdot \gamma_r(n)$ for $m,n \in M$ and $r \geq 1$

(v) $\gamma_s(\gamma_r(m)) = k(s,r)\ \gamma_{rs}(m)$ for $m \in M$ and $r,s \geq 1$

where (r,s) denotes the binomial coefficient $\binom{r+s}{r}$ and $k(s,r) = (r,r-1)(2r,r-1) \ldots ((s-1)r, r-1)$. Then the functor $\Gamma^+_p : \mathcal{A}_{Z_p} \to \mathcal{B}$ is left adjoint to the forgetful functor $\mathcal{B} \to \mathcal{A}_{Z_p}$. Now 11.2 gives a functor algebra structure to $\Gamma^+_p : \mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$.

11.6. Stable algebras of mod-p functor algebras. Let (T,m,i) be a functor algebra over $\mathcal{A}_{Z_p}$ with $T(0) = 0$. Then $\sum_{n=0}^{\infty} L^s_n T(Z_p)$ is an algebra over the Hopf algebra $A_*(p)$, and is called the stable algebra of (T,m,i). Multiplication is given by the associative pairing

$$\left(\sum_{n=0}^{\infty} L^s_n T(Z_p)\right) \otimes_{Z_p} \left(\sum_{n=0}^{\infty} L^s_n T(Z_p)\right) \xrightarrow{c} \sum_{n=0}^{\infty} L^s_n (T \circ T)(Z_p) \xrightarrow{m_*} \sum_{n=0}^{\infty} L^s_n T(Z_p)$$

where c is the $A_*(p)$-module map of §10. The identity functor I: $\mathcal{A}_{Z_p} \to \mathcal{A}_{Z_p}$ satisfies

$$L^s{}_n I(Z_p) = \begin{cases} Z_p & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

Let $1 \in L^s{}_0 T(Z_p)$ be the image of the canonical generator $1 \in L^s{}_0 I(Z_p)$ under the map

$$i_*: L^s{}_0 I(Z_p) \to L^s{}_0 T(Z_p).$$

Then $1 \in L^s{}_0 T(Z_p)$ is the identity in the algebra $\sum_{n=0}^{\infty} L^s{}_n T(Z_p)$.

§12. Examples of Stable Algebras

Stable algebras of the functor algebras (§11) SP^+, SP^+_p, and Γ^+_p will now be determined.

The natural maps (5.5)

$$f: SP^+ \to SP^+_p$$

$$g: SP^+_p \to \Gamma^+_p$$

are maps of functor algebras. Thus

$$f_*: L^s{}_*SP^+(Z_p) \to L^s{}_*SP^+_p(Z_p)$$

$$g_*: L^s{}_*SP^+_p(Z_p) \to L^s{}_*\Gamma^+_p(Z_p)$$

are homomorphisms of algebras over $A_*(p)$. We will show that f_* and g_* are monomorphisms and will compute the images of $L^s{}_*SP^+(Z_p)$ and $L^s{}_*SP^+_p(Z_p)$ in $L^s{}_*\Gamma^+_p(Z_p)$.

__Lemma 12.1.__ _If_ $r \geq 1$ _and_ $r \neq p^j$ _for any_ j, _then_ $L^s_*\, SP^r(Z_p) = 0$, $L^s_*\, SP^r{}_p(Z_p) = 0$, _and_ $L^s_*\, \Gamma^r{}_p(Z_p) = 0$.

__Proof.__ Since SP^r, $SP^r{}_p$, and $\Gamma^r{}_p$ are r-homogeneous [2], this follows by the mod-p version of [2, 5.11]. It can also be proved by constructing maps (see 8.14)

$$T^r \to T^t \otimes T^{r-t} \to T^r$$

whose composition is an isomorphism, where $T^i = SP^i$, $SP^i{}_p$, or $\Gamma^i{}_p$. Since $L^s{}_*(T^t \otimes T^{r-t}) = 0$ by 9.1 (5), the lemma follows.

By 12.1 we may view

$$L^s{}_*\Gamma^+{}_p(Z_p) = \sum_{n,j\geq 0} L^s{}_n\Gamma^{p^j}{}_p(Z_p)$$

as a bigraded algebra.

Now let $p = 2$.

For $i \geq 0$ define

$$\alpha_i \in L^s{}_i\Gamma^2{}_2(Z_2)$$

by $\alpha_i = (\xi_1)^i\alpha_0$, where $\alpha_0 \in L^s{}_0\Gamma^2{}_2(Z_2) \approx Z_2$ is the non-zero element.

Lemma 12.2. *The elements α_i, $i \geq 0$, are a Z_2-basis for $L^s{}_*\Gamma^2{}_2(Z_2)$. If $i \geq 0$ then $\xi_1\alpha_i = \alpha_{i+1}$ and $\xi_k\alpha_i = 0$ for $k \geq 2$.*

Proof. This is a stable consequence of 5.6 and 5.7.

Theorem 12.3. *The algebra $L^s{}_*\Gamma^+{}_2(Z_2)$ is given by the generators α_i for $i \geq 0$ and the relations:*

(i) $$0 = \sum_{i+j=n} \binom{n}{i} \alpha_{2m+i}\alpha_{1+m+j} \text{ for } m,n \geq 0$$

(ii) $$0 = \sum_{i+j=n} \binom{n}{i} \alpha_{1+2m+i}\alpha_{1+m+j} \text{ for } m,n \geq 0.$$

Furthermore the image of the monomorphism

$$g_*: L^s{}_*SP^+{}_2(Z_2) \to L^s{}_*\Gamma^+{}_2(Z_2)$$

is the subalgebra generated by α_i for $i \geq 1$; and the image of the monomorphism

$$g_*\circ f_*: L^s{}_*SP^+(Z_2) \to L^s{}_*\Gamma^+{}_2(Z_2)$$

is the subalgebra generated by α_i for $i \geq 2$.

Remark 12.4. The $A_*(2)$-module structure of $L^s{}_*\Gamma^+{}_2(Z_2)$ is determined by 12.2. In particular the action of $A_*(2)$ on the relations $\alpha_0\alpha_1 = 0$ and $\alpha_1\alpha_1 = 0$

gives all the other relations. A consequence of 12.3 is that $L^s{}_*\Gamma^+{}_2(Z_2)$ has as a Z_2-basis the element $1 \in L^s{}_0\Gamma^1{}_2(Z_2)$ together with products $\alpha_{i_1}\cdots\alpha_{i_k} \in L^s{}_*\Gamma^{2^k}{}_2(Z_2)$ where $k \geq 1$, $i_1,\ldots,i_k \geq 0$, and $i_{j-1} \geq 2i_j$ for all j. We shall prove 12.3 in 12.8.

Now let p be an odd prime.

Let $\nu_0 \in L^s{}_0\Gamma^p{}_p(Z_p)$ be the stabilization of ν (see 6.7). For $i \geq 0$ define

$$\nu_i \in L^s{}_{2i(p-1)}\Gamma^p{}_p(Z_p)$$

by $\nu_i = (\xi_1)^i\nu_0$, and for $i \geq 0$ define

$$\eta_i \in L^s{}_{2i(p-1)+1}\Gamma^p{}_p(Z_p)$$

by $\eta_i = \tau_0\nu_i$.

<u>Lemma 12.5</u>. <u>The elements</u> ν_i <u>for</u> $i \geq 0$ <u>and</u> η_i <u>for</u> $i \geq 0$ <u>are a</u> Z_p<u>-basis for</u> $L^s{}_*\Gamma^p{}_p(Z_p)$. <u>If</u> $i \geq 0$ <u>then</u> $\xi_1\nu_i = \nu_{i+1}$, $\xi_1\eta_i = \eta_{i+1}$, $\xi_k\nu_i = 0$ <u>for</u> $k \geq 2$, $\xi_k\eta_i = 0$ <u>for</u> $k \geq 2$, $\tau_0\nu_i = \eta_i$, $\tau_k\nu_i = 0$ <u>for</u> $k \geq 1$, <u>and</u> $\tau_k\eta_i = 0$ <u>for</u> $k \geq 0$.

<u>Proof</u>. This is a stable consequence of 6.7 and 6.8.

<u>Theorem 12.6</u>. <u>The algebra</u> $L^s{}_*\Gamma^+{}_p(Z_p)$ <u>is given by the generators</u> ν_i <u>and</u> η_i <u>for</u> $i \geq 0$ <u>and by the relations</u>:

(i) $\quad 0 = \sum_{i+j=n} \binom{n}{i} \nu_{r+pm+i}\,\nu_{1+m+j}$ <u>for</u> $0 \leq r < p$ <u>and</u> $m,n \geq 0$.

(ii) $\quad 0 = \sum_{i+j=n} \binom{n}{i} \nu_i\eta_j$ <u>for</u> $n \geq 0$.

(iii) $\quad 0 = \sum_{i+j=n} \binom{n}{i} \nu_{r+pm+i}\,\eta_{1+m+j}$ <u>for</u> $1 \leq r \leq p$ <u>and</u> $m,n \geq 0$.

(iv) $0 = \sum_{i+j=n} \binom{n}{i} (\eta_{pm+i}\, \nu_{1+m+j} + \nu_{pm+i}\, \eta_{1+m+j})$ *for* $m,n \geq 0$.

(v) $0 = \sum_{i+j=n} \binom{n}{i} \eta_{r+pm+i}\, \nu_{1+m+j}$ *for* $1 \leq r < p$ *and* $m,n \geq 0$.

(vi) $0 = \sum_{i+j=n} \binom{n}{i} \eta_i \eta_j$ *for* $n \geq 0$.

(vii) $0 = \sum_{i+j=n} \binom{n}{i} \eta_{r+pm+i}\, \eta_{1+m+j}$ *for* $1 \leq r \leq p$ *and* $m,n \geq 0$

Furthermore the image of the monomorphism

$$g_*: L^s{}_*SP^+{}_p(Z_p) \to L^s{}_*\Gamma^+{}_p(Z_p)$$

is the subalgebra generated by ν_i *for* $i \geq 1$ *and* η_j *for* $j \geq 0$; *and the image of the monomorphism*

$$g_* \circ f_*: L^s{}_*SP^+(Z_p) \to L^s{}_*\Gamma^+{}_p(Z_p)$$

is the subalgebra generated by ν_i *for* $i \geq 1$ *and* η_j *for* $j \geq 1$.

Remark 12.7. The $A_*(p)$-module structure of $L^s{}_*\Gamma^+{}_p(Z_p)$ is determined by 12.5. In particular the action of $A_*(p)$ on the relations $\nu_0\eta_0 = 0$, $\nu_r\nu_1 = 0$ for $0 \leq r < p$, and $\eta_r\nu_1 = 0$ for $1 \leq r < p$, gives all the other relations. For $i,j \geq 0$ call $\nu_i\nu_j$ and $\eta_i\nu_j$ admissible if $i \geq pj$; for $i,j \geq 0$ call $\nu_i\eta_j$ and $\eta_i\eta_j$ admissible if $i \geq pj+1$. Call a product $\omega_1 \ldots \omega_k$ admissible if $k \geq 1$, each factor ω_h equals some ν_i on η_i for $i \geq 0$, and each pair of successive factors $\omega_h\omega_{h+1}$ is admissible. By 12.6, $L^s{}_*\Gamma^+{}_p(Z_p)$ has as a Z_p-basis the elements $1 \in \Gamma^1{}_p(Z_p)$ together with all admissible products. The proof of 12.6 is discussed in 12.9.

12.8. Proof of 12.3. An argument similar to 8.14 shows the algebra $L^s{}_*\Gamma^+{}_2(Z_2)$ is generated by α_i for $i \geq 0$. The relation $\alpha_0\alpha_1 = 0$ is proved in the same way as 8.15 (2). The relation $\alpha_1\alpha_1 = 0$ follows from the corresponding relation in $L^s{}_*SP^+{}_2(Z_2)$, which is proved using $L_3SP^4{}_2(Z_2,1) = 0$. Under the action of $A_*(2)$, these two relations give all the desired relations in 12.3. But 4.1 implies for $r \geq 1$

$$L_q\Gamma^r{}_2(Z_2,n) \approx \pi_q(Z_2 \boxtimes \Gamma^r K(Z,n)) \approx \pi_{q+2r}(Z_2 \boxtimes SP^r K(Z,n+2)) \approx L_{q+2r}SP^r(Z_2,n+2)$$

so

$$L^s{}_i\Gamma^r{}_2(Z_2) \approx L^s{}_{i+2r-2}SP^r(Z_2)$$

for $i \geq 0$, $r \geq 1$. A counting argument using 8.6 implies that $L^s{}_*\Gamma^+{}_2(Z_2)$ is as claimed.

The argument of 8.14 shows that the algebras $L^s{}_*SP^+(Z_2)$ and $L^s{}_*SP^+{}_2(Z_2)$ are generated respectively by $L^s{}_*SP^2(Z_2)$ and $L^s{}_*SP^2{}_2(Z_2)$. Thus 5.6 implies that the images of g_* and $g_* \circ f_*$ are as stated in 12.3. The maps g_* and $g_* \circ f_*$ are monomorphisms by a counting argument using the known dimension (see §8) of $L^s{}_qSP^+(Z_2)$ and the fact that

$$L_qSP^+{}_2(Z_2,n) \approx \tilde{H}_q(Z_2,n;Z_2)$$

so $L^s{}_*SP^+{}_2(Z_2) \approx A_*(2)$ as Z_2-modules.

12.9. Proof of 12.6. The proof is essentially the same as 12.8, using mod-p results. We note that the needed relations (see 12.7) in $L^s{}_*\Gamma^+{}_p(Z_p)$ are stable consequences of 8.15 (1)-(4).

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