DIVIDED SEQUENCES AND BIALGEBRASOF HOMOLOGY OPERATIONS

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ABSTRACT

In order to classify and construct relations in the Steenrod algebra and Dyer-Lashof algebra, we consider some general notions concerning divided sequences and divided systems in cocommutative coalgebras. We develop the idea of a test map on a coalgebra C and use it to classify divided sequences in C. It is convenient to use families of power series for this purpose. If C is cofree with test map we can construct divided sequences corresponding to every possible family of power series; thus we have a complete description of $\mathcal{B}(C)$, the set of divided sequences in C. We say C is component cofree if it is a direct sum of cofree coalgebras. If C is component cofree and has a multiplication, then $\mathcal{B}(C)$ is a monoid and the group law for C describes the multiplication on $\mathcal{B}(C)$ in terms of the corresponding power series. If C is graded it is natural to consider $\mathcal{B}^{gr}(C)$ the monoid of graded divided sequences in C.

We apply these methods to coalgebras of operations on a monoidal functor, and give general conditions for a coalgebra of operations to generate a bialgebra of operations. We consider in detail the Steenrod algebra $\mathcal{A}$, the Dyer-Lashof algebra $\mathcal{R}$, and the algebra generated by $\mathcal{R}$ and $\mathcal{A}_{opp}$ which we call the Nishida algebra $\mathcal{N}$. $\mathcal{A}$ is cofree, and $\mathcal{R}$ is isomorphic to $\sum_{\ell=0}^{\infty} \mathcal{R}[\ell]$ with each $\mathcal{R}[\ell]$ cofree. Then $\mathcal{N}$ is isomorphic to $\sum_{\ell=0}^{\infty} \mathcal{N}[\ell]$ and we define a coalgebra basis $\{ \mathfrak{A}^{(S,R)} : S \in \mathbb{N}^{\ell}\ R \in \mathbb{N}^{(\mathbb{P})}\}$ for $\mathcal{N}[\ell]$ in terms of the natural coalgebra basis for $\mathcal{R}$ and $\mathcal{A}$. Thus $\mathcal{N}$ is also component cofree. Since $\mathcal{N}[\ell]$ is not of finite type, and is positively and negatively graded, we introduce a topology and consider two-way divided sequences in the completion $\tilde{\mathcal{N}}$.

Among our results, we show that the group $\mathcal{D}^{gr}(\mathcal{A})$ is isomorphic to $\{f(w) \in Z_p[[w]] : f(w) = 1\}$ under multiplication. Similarly $\mathcal{D}^{gr}(\mathcal{R}[\ell]) \approx \{f(w) \in Z_p[w] : f(0) = 1$ and f has degree $\leq \ell\}$. Thus graded divided sequences in the two bialgebras multiply in much the same way. For example $\sum_{i,j} (-1)^j P^i \cdot P^j = \sum_n (-1)^n P^{(0,n)}$ and $\sum_{i,j} (-1)^j Q^i Q^j = \sum_n (-1)^n Q^{(0,n)}$, both corresponding to $(1 + w)(1 - w) = 1 - w^2$. We show how to express the Adem relations and the Nishida relations in terms of divided sequences with power series as generating functions for the usual binomial coefficients.

Terrence Bisson

Divided Sequences of Homology Operations

INTRODUCTION

The Steenrod algebra $\mathcal{A}$ and the Dyer-Lashof algebra $\mathcal{R}$ are actually bialgebras whose structure as cocommutative coalgebras is fairly simple. In order to make use of this fact in classifying and constructing relations in $\mathcal{A}$ and $\mathcal{R}$, we consider the set of graded divided sequences (and more generally, of graded divided systems) in $\mathcal{A}$ and $\mathcal{R}$. We also form the bialgebra $\mathcal{N}$ generated by $\mathcal{R}$ and $\mathcal{A}_{opp}$, and apply similar ideas to study it.

A sequence $(c_n)_{n=0}^{\infty}$ in a coalgeba C is called a divided sequence if $\varepsilon(c_0) = 1$ and $\psi(c_n) = \sum_{i+j=n} c_i \otimes c_j$. This notion arises naturally in the theory of cocommutative coalgebras over a field k. In a connected bialgebra for instance, if k has characteristic 0 we can construct a divided sequence over any primitive element; the fact that this method of construction fails if k has characteristic $p \neq 0$ is a major distinguishing feature of the characteristic p case (see [Sweedler, 1967] and [Dieudonné, 1973, Foreword]). Divided sequences are closely related to the notion of curves in an formal group (see [Lazard, 1975]).

In algebraic topology divided sequences appear in several ways. For instance multiplicative characteristic classes are precisely divided sequences; for the algebraic ideas behind this see [Husemoller, 1971]. In a quite different setting, Steenrod's reduced power construction [Steenrod, 1957] produces a sequence of operations $(P^n)_{n=0}^{\infty}$ on the Z_p cohomology of any space. By the Cartan formula the P^n naturally form a divided sequence; and this determines the cocommutative coalgebra structure of the Steenrod algebra $\mathcal{A}^e$ generated by the P^n. [Kudo and Araki, 1956] (for $p = 2$) and [Dyer and Lashof, 1962] (for p any prime) used an analog of the reduced power construction to produce a sequence of operations

$(Q^n)_{n=0}^{\infty}$ on the Z_p homology of any infinite loop space. The Q^n also satisfy a Cartan formula and so naturally form a divided sequence; this determines the cocommutative coalgebra structure of the Dyer-Lashof algebra $\mathcal{R}^e$ generated by the Q^n. Note that for $p \neq 2$ we omit the Bockstein operations in forming $\mathcal{A}^e$ and $\mathcal{R}^e$, which are thus evenly graded subbialgebras of the standard Steenrod and Dyer-Lashof algebras. Since no divided sequence in $\mathcal{A}$ or $\mathcal{R}$ can involve Bocksteins, we are justified in this simplification.

Milnor [1958] showed that $\mathcal{A}^e$ has a basis $\{P^{(R)} : R \in N^{(\mathbb{P})}\}$ such that $\psi P^{(R)} = \sum_{R'+R''=R} P^{(R')} \otimes P^{(R'')}$. We call such a basis a coalgebra basis; and if a coalgebra has such a basis we say it is cofree. May [1976] and Madsen [1975] pointed out that as a coalgebra $\mathcal{R}^e$ is isomorphic to the direct sum $\sum_{\ell=0}^{\infty} \mathcal{R}^e[\ell]$ where $\mathcal{R}^e[\ell]$ is spanned by the monomials of length ℓ in $\mathcal{R}^e$. In coalgebra terminology, each $\mathcal{R}^e[\ell]$ is a connected component. Each $\mathcal{R}^e[\ell]$ is a cofree coalgebra, so we say that $\mathcal{R}^e$ is component cofree. We define a preferred coalgebra basis $\{Q^{(S)} : S \in N^{\ell}\}$ for each $\mathcal{R}^e[\ell]$.

In the Z_p homology of any infinite loop spaces we have both the operations Q^n and the transposed operations P^n_t which we consider as elements of $\mathcal{A}^e_{opp}$. We form the bialgebra generated by the Q^n and P^n_t, and call it the Nishida algebra $\mathcal{N}^e$, since Nishida [1968] found the commutation relation expressing $P^n_t \cdot Q^m$ in terms of operations $Q^a \cdot P^b_t$. We show that $\mathcal{N}^e$ is again a component cofree coalgebra. $\mathcal{N}^e$ is isomorphic to $\sum_{\ell=0}^{\infty} \mathcal{N}^e[\ell]$ as coalgebra and we define the natural coalgebra basis $\{\mathcal{A}^{(S,R)} : S \in N^{\ell}, R \in N^{(\mathbb{P})}\}$ for each $\mathcal{N}^e[\ell]$. Unlike $\mathcal{A}^e$ and $\mathcal{R}^e$,

$\mathcal{N}^e$ is not of finite type, and is positively and negatively graded. Therefore we introduce a natural topology for homogeneous sums and consider two-way divided sequences in the completion $\tilde{\mathcal{N}}^e$.

Certain basic ideas are suggested by the algebraic topology and we develop them in the general setting of cocommutative coalgebras. We define the idea of a test map on a coalgebra C and use it to classify divided sequences in C. It is convenient to use families of power series for this purpose. If C is cofree with test map we can construct divided sequences corresponding to every possible family of power series; thus we have a complete description of $\mathcal{D}(C)$ the set of divided sequences in C. If C is a component cofree bialgebra then $\mathcal{D}(C)$ is a monoid and the group law for C describes the multiplication on $\mathcal{D}(C)$ in terms of the corresponding power series. If C is graded, $\mathcal{D}^{gr}(C)$ denotes the monoid of graded divided sequences in C. If C is a bialgebra with antipode map, i.e. a Hopf algebra, then $\mathcal{D}(C)$ is a group. For instance, $\mathcal{A}$ is a Hopf algebra but $\mathcal{R}$ is not, and $\mathcal{D}^{gr}(\mathcal{R})$ does not have inverses.

In Chapter 1 we develop the algebraic preliminaries. Section 1.1. contains the basic background definitions and results. Many of the ideas come from [Heyneman and Sweedler, 1969]. A divided system is a divided sequence in several parameters. We define the set $\mathcal{D}_I(C)$ of divided systems in a coalgebra C for any index set I. If C has a test map, to each element of $\mathcal{D}_I(C)$ we associate a family of power series in $k[\![t_i]\!]_I$. We prove the following classifying result: if two divided systems have the same associated family of power series, then the divided systems are equal, term by term.

In section 1.2 we describe the cofree construction in the category of connected coalgebras. The approach is based on that in [Husemoller, 1971].

Given a coalgebra C and a map α, we give necessary and sufficient conditions for C to be cofree with test map α.

In Section 2.3 we show how to construct all divided systems in a cofree coalgebra C. We then prove that there is a bijective correspondence between $\mathcal{D}_I(C)$ and families of power series. If C is a component cofree bialgebra, the group law for the multiplication in C describes the multiplication on $\mathcal{D}_I(C)$ in terms of the corresponding power series.

In Chapter 2 we apply these methods to algebraic topology. In Section 2.1 we develop the idea of a coalgebra of operations on a monoidal functor. We give general conditions for such a coalgebra to generate a bialgebra of operations on the functor. In Section 2.2 we use our condition of Section 1.2 to show that $\mathcal{A}^e$ is cofree. One major result is Theorem 2.2.50: $\mathcal{D}_I^{gr}(\mathcal{A}^e)$ is isomorphic as a group to $U^{gr}(\mathbb{Z}_p[\![\tilde{w},t_i]\!]_I)$, where $\mathbb{Z}_p[\![\tilde{w},t_i]\!]_I$ has a skew multiplication. As a special case we have Theorem 2.2.52: $\mathcal{D}^{gr}(\mathcal{A}^e) \approx \{f(w) \in Z_p[\![w]\!] : f(0) = 1\}$ as groups. The correspondence can be expressed as follows: for $(\theta_n)_{n=0}^{\infty} \in \mathcal{D}^{gr}(\mathcal{A}^e)$, $\sum_{n=0}^{\infty} \theta_n(x) = f(w)(x)$ where x is the polynomial generator in $H^*(P^{\infty};Z_p)$ (real or complex projective space according as $p = 2$ or $p \neq 2$) and $w^k(x) = x^{p^k}$. For example, it follows easily that $\sum_{i,j} (-1)^j P^i \cdot P^j = \sum_{n=0}^{\infty} (-1)^n P^{(0,n)}$, since both sides correspond to $(1 + w)(1 - w) = 1 - w^2$. We show how to express the Adem relations in terms of divided sequences, with power series considered as generating functions.

In Section 2.3 we develop the coalgebra basis for $\mathcal{R}^e$. The group law is more complicated, so $\mathcal{D}_I^{gr}(\mathcal{R}^e)$ is not so easy to describe. But the complications disappear when we consider only the divided sequences. Then we have Theorem 2.3.54: $\mathcal{D}^{gr}(\mathcal{R}^e[\ell]) \approx \{f(w) \in Z_p[w] : f(0) = 1$ and degree $f(w) \leq \ell\}$. Thus graded divided sequences multiply much the same way in $\mathcal{A}$ and in $\mathcal{R}$. We again have the relation $\sum_{i,j}(-1)^j Q^i \cdot Q^j = \sum_{n=0}^{\infty}(-1)^n Q^{(0,n)}$.

In Section 2.4 we develop the proper notion of excess in the Nishida algebra; each $\mathcal{A} \in \mathcal{N}$ has an associated excess interval $\overline{e}(\mathcal{A})$. We show that excess and the Nishida relations generate all the relations in the Nishida algebra. We define the natural coalgebra basis $\{\mathcal{A}^{(S,R)}$: $S \in \mathbb{N}^{\ell}\ R \in \mathbb{N}^{(\mathbb{P})}\}$ for $\mathcal{N}^e[\ell]$, where $\mathcal{A}^{(S,R)} = Q^{(S+|R|\cdot\Delta\ell)}\cdot P_t^{(R)}$. We give some examples to show the necessity of introducing infinite homogeneous sums and topological ideas. We define $\mathcal{D}_{\pm}(\tilde{\mathcal{N}})$ the set of graded two way divided sequences. Each element $\tilde{\mathcal{A}}_*$ of $\mathcal{D}_{\pm}(\tilde{\mathcal{N}})$ is associated with a unique family of constants (μ_i, ν_k) in $\mathbb{Z}_p$ (see 2.4.48). We consider the way in which properties of $\tilde{\mathcal{A}}_*$ depend on some properties of the associated constants. We show how to express the Nishida relations in terms of graded divided sequences, and give a simple consequence: for any $n \geq 0$

$$\sum_{i=0}^{\infty} P_t^i Q^{n+i} = \begin{cases} 0 & \text{if } n > 0 \\ \sum_{j=0}^{\infty} Q^j P_t^j & \text{if } n = 0 \end{cases}$$

Section 2.4. The Nishida Algebra

Recall that the Steenrod algebra $\mathcal{A}^e$ is the bialgebra of operations on the functor $H^* : \mathbb{T}op_* \to \mathbb{L}in$ generated by the set of operations $\{P^n : n \geq 0\}$.

Definition 2.4.1: For any linear natural transformation θ on the functor $H^* : \mathbb{T}op_* \to \mathbb{L}in$, let θ_t denote the transpose linear natural transformation on the functor $H_* : \mathbb{T}op_* \to \mathbb{L}in$; i.e. for every $X \in \mathbb{T}op_*$, $\theta : H^*(X) \to H^*(X)$, and $\theta_t : H_*(X) \to H_*(X)$ is the transpose of θ. In particular, consider the operations $\{P^n_t : n \geq 0\}$. Then each P^n_t lowers degree by $d\cdot(p-1)\cdot n$, and satisfies the following excess condition.

Proposition 2.4.2: Let $X \in \mathbb{T}op_*$ and let V denote the shift map on the coalgebra $H_*(X)$; then $P^n_t(x) = \begin{cases} V(x) & \text{if degree } x = d\cdot pn \\ 0 & \text{if degree } x < d\cdot pn \end{cases}$.

Proof: Let m = degree of x and consider $P^n : H^{m-d(p-1)n}(X) \to H^m(X)$. if $m = dpn$ then $m - d(p-1)n = dn$, and on H^{dn} P^n is the $p\underline{^{th}}$ power map so that on $H^{dpn}P^n_t = V$, since for any coalgebra V is transpose to the $p\underline{^{th}}$ power map on the dual algebra (see 1.1). If $m < dpn$ then $m - d(p-1)n < d\cdot n$ so on $H^{m-d(p-1)n}$ P^n is the zero map, and therefore P^n_t is the zero map on $H_m(X)$. □

The suspension map $s : H_*(X) \to H_{*-1}(X)$; and the P^n_t obey the Cartan formula: For $x \in H_*(X)$, $y \in H_*(Y)$, and $x \otimes y \in H_*(X \otimes Y)$, we have $P^n_t(x \otimes y) = \sum_{i+j=n} P^i_t(x) \otimes P^j_t(y)$. Therefore, as for $\mathcal{A}^e$, the P^n_t generate a bialgebra of operations on $H_* : \mathbb{T}op_* \to \mathbb{L}in$.

Definition 2.4.3: Let $\mathcal{A}^e_{opp}$ denote this bialgebra. Then:

Proposition 2.4.4: $\mathcal{A}^e \to \mathcal{A}^e_{opp} : \theta \longmapsto \theta_t$ is an isomorphism of coalgebras and an antiisomorphism of algebras. □

In particular, we can consider each P^n_t as an operation on the functor H_*: Loop → Lin.

Proposition 2.4.5: $\mathcal{A}^e_{opp}$ is the bialgebra of operations on the functor H_* : Loop → Lin generated by the P^n_t.

Proof: We need only show that if θ_t is the zero operation on every $\underline{X} \in$ Loop, then $\theta_t = 0$ in $\mathcal{A}^e_{opp}$. For any m, the sequence $\{K_{m+n} : n \geq 0\} = \underline{K_m}$ is an object in Loop. Suppose $\theta_t(z) = 0$ for every $m \geq 0$, every $z \in H_*(\underline{K_m})$; then $\langle\theta(\iota_m), z\rangle = \langle\iota_m, \theta_t(z)\rangle = 0$, so $\theta(\iota_m) = 0$ for every m. Therefore $\theta = 0$ in $\mathcal{A}^e$, so $\theta_t = 0$ in $\mathcal{A}^e_{opp}$. □

Recall that the Dyer-Lashof algebra $\mathcal{R}^e$ is the bialgebra of operations on the functor H_* : Loop → Lin generated by the set of operations $\{Q^n : n \geq 0\}$.

Definition 2.4.6: Let $\mathcal{N}^e$ denote the bialgebra of operations on the functor H_* : Loop → Lin generated by the operations $\{P^n_t : n \geq 0\}$ and $\{Q^m : m \geq 0\}$. We call $\mathcal{N}^e$ the Nishida algebra. Nishida [1968] showed how to compute $P^n_t \cdot Q^m$; it can be expressed as a linear combination of terms $Q^i \cdot P^j_t$. We will discuss the precise formula, called the Nishida relations, later.

Notation 2.4.7: We will write $\mathcal{A}, \mathcal{A}_{opp}, \mathcal{R}, \mathcal{N}$ in place of $\mathcal{A}^e, \mathcal{A}^e_{opp}, \mathcal{R}^e, \mathcal{N}^e$. For $\mathfrak{A} \in \mathcal{N}$ we say $[\mathfrak{A}] = n$ if $\mathfrak{A}$ raises degree by $d \cdot n$, and similarly for $\theta \in \mathcal{A}$ and $Q \in \mathcal{R}$.

Since $\mathcal{A}_{opp}$ and $\mathcal{R}$ act faithfully on the functor H_* : Loop → Lin we have $\mathcal{A}_{opp} \hookrightarrow \mathcal{N}$ and $\mathcal{R} \hookrightarrow \mathcal{N}$ inclusions of bialgebras. Then the multiplication (composition of operations) in $\mathcal{N}$ gives $\mathcal{R} \otimes \mathcal{A}_{opp} \to \mathcal{N} \otimes \mathcal{N} \xrightarrow{m} \mathcal{N}$; since m is a coalgebra map, $\mathcal{R} \otimes \mathcal{A}_{opp} \to \mathcal{N}$: $Q \otimes \theta_t \longmapsto Q \circ \theta_t$ is a coalgebra map.

Proposition 2.4.8: $\mathcal{R} \otimes \mathcal{A}_{opp} \to \mathcal{N}$ is a surjective coalgebra map.

Proof: $\mathcal{N}$ is spanned by words in the symbols P_t^n and Q^m. But each occurance of $P_t^n \cdot Q^m$ can be replaced by a linear combination of words with all Q^m's preceding all P_t^n's. □

By Proposition 2.3.14, $\mathcal{R}$ can be described as the collection of linear operations on the functor $M : \mathbb{L}in \to \mathbb{L}in$ that commute with suspension. In general an operation in $\mathcal{N}$ does not operate on the functor $M : \mathbb{L}in \to \mathbb{L}in$.

Proposition 2.4.9: Each operation in $\mathcal{N}$ can be considered as an operation on the functor $M \circ \overline{H} : \mathbb{T}op_* \to \mathbb{L}in$.

Proof: For $X \in \mathbb{T}op_*$, consider $\overline{H}_*(X) \subset H_*(\underline{Q}X)$. $\mathcal{N}$ operates on $H_*(\underline{Q}X)$, and we need only show that $\mathcal{N} \cdot \overline{H}_*(X) \subset M(\overline{H}_*(X)) = \mathcal{R} \cdot \overline{H}_*(X)$. But by Proposition 2.4.7, we need only note that $Q \cdot \theta_t \cdot \overline{H}_*(X) \subset Q \cdot \overline{H}_*(X) \subset \mathcal{R} \cdot \overline{H}_*(X)$ for every $Q \in \mathcal{R}$ and every $\theta_t \in \mathcal{A}_{opp}$. □

Recall that $\mathcal{R}$ is a component coalgebra; $\mathcal{R} \approx \sum_{\ell \geq 0} \mathcal{R}[\ell]$ as coalgebras where $\mathcal{R}[\ell]$ is the connected component of the grouplike element Q^{0_ℓ}. $\mathcal{A}_{opp}$ is a connected coalgebra, with 1 its unique grouplike element.

Proposition 2.4.10: $\mathcal{N}$ is a component coalgebra with $\{Q^{0_\ell} : \ell \geq 0\}$ as its set of grouplike elements.

Proof: In degree 0, $[\mathcal{N}]_0$ has basis $\{Q^{0_\ell} : \ell \geq 0\}$; 2.3.3 and 1.1.16. □

Definition 2.4.11: Let $\mathcal{N}[\ell]$ denote the connected component of Q^{0_ℓ} in $\mathcal{N}$. Then $\mathcal{N} \approx \sum_{\ell\geq 0} \mathcal{N}[\ell]$ as coalgebras.

Corollary 2.4.12: 1) For $\ell \geq 0$, $\mathcal{R}[\ell]$ is a subcoalgebra of $\mathcal{N}[\ell]$.

2) $\mathcal{N}[0]$ is a subbialgebra of $\mathcal{N}$ and $\mathcal{A}_{opp} \hookrightarrow \mathcal{N}[0]$ as bialgebras.

Proof: 1) The coalgebra map $\mathcal{R} \hookrightarrow \mathcal{N}$ must preserve components; see 1.1.

2) $Q^{0_0} = 1$ and the component of 1 is always a subbialgebra. The coalgebra map $\mathcal{A}_{opp} \hookrightarrow \mathcal{N}$ preserves components. □

Multiplication in $\mathcal{N}$ is a coalgebra map taking $\mathcal{N}[\ell_1] \otimes \mathcal{N}[\ell_2] \xrightarrow{m} \mathcal{N}[\ell_1 + \ell_2]$. Consider $\mathcal{R}[\ell] \otimes \mathcal{A}_{opp} \to \mathcal{N}[\ell] \otimes \mathcal{N}[0] \xrightarrow{m} \mathcal{N}[\ell]$: $Q \otimes \theta_t \longmapsto Q\cdot\theta_t$.

Proposition 2.4.13: $\mathcal{R}[\ell] \otimes \mathcal{A}_{opp} \to \mathcal{N}[\ell]$ is a surjective coalgebra map.

Proof: $\mathcal{R} \otimes \mathcal{A}_{opp}$ is the component coalgebra $\sum_{\ell\geq 0} \mathcal{R}[\ell] \otimes \mathcal{A}_{opp}$, and $\sum_{\ell\geq 0} \mathcal{R}[\ell] \otimes \mathcal{A}_{opp} \to \sum_{\ell\geq 0} \mathcal{N}[\ell]$ is a surjection with $\mathcal{R}[\ell] \otimes \mathcal{A}_{opp} \to \mathcal{N}[\ell]$. □

We will describe the kernel of this map in terms of excess.

Recall the definition of excess in $\mathcal{A}$ and in $\mathcal{R}$.

Definition 2.4.14: For $\theta \in \mathcal{A}$, we say excess $\theta = e(\theta) = d\cdot n$ if $\theta\cdot H^m(X) = 0$ for every $m < d\cdot n$ every $X \in \mathfrak{Top}_*$ but $\theta(\iota_{dn}) \neq 0$ in $H^*(K_{dn})$. For any admissible sequence $I = (i_1,\ldots,i_k)$, $e(P^I) = e(I) = d\cdot(p\cdot i_1 - [I])$, where $[I] = (p-1)\cdot \sum_{j=1}^{k} i_j$ and for any multiindex $R \in \mathbb{N}^{(\mathbb{P})}$, $e(P^{(R)}) = d\cdot|R|$ where $|R| = \sum_{\mathbb{P}} r_k$. Recall that $\{\theta \in \mathcal{A} : e(\theta) \geq d\cdot n\} = \{\theta \in \mathcal{A} : \theta\cdot H^m = 0$ when $m < d\cdot n\}$ is the left ideal $\{\theta \in \mathcal{A} : \theta(\iota_{d\cdot(n-1)}) = 0\}$.

Definition 2.4.15: For $Q \in \mathcal{R}$, we say excess $Q = e(Q) = d\cdot n$ if $Q\cdot H^m(\underline{X}) = 0$ for every $m > d\cdot n$ every $\underline{X} \in \mathfrak{Loop}$, but $\theta(y_n) \neq 0$ in M_n;

then for any $y \in \overline{H}_{d\cdot n}(Y)$, $Y \in \mathbb{T}op_*$, $Q(y) \neq 0$ in $H_*(QY)$. For any allowable sequence I, $e(Q^I) = e(I)$; and for any multiindex $R \in \mathbb{N}^{\ell}$, $e(Q^{(R)}) = d\cdot\pi(R)$ where $\pi(R)$ is the last entry, i.e. $\pi(R) = r_\ell$ for $R \in \mathbb{N}^{\ell}$. Recall that $\{Q \in \mathcal{R} : e(Q) \leq d\cdot n\} = \{Q \in \mathcal{R} : Q\cdot H_m(\mathbb{L}oop) = 0$ when $m > d\cdot n\}$ is the two-sided ideal and subcoalgebra $\{Q \in \mathcal{R} : Q(y_{n+1}) = 0\}$.

The appropriate idea of excess in the Nishida algebra is that of an excess interval.

Definition 2.4.16: For any interval $[a,b] \subseteq \mathbb{N}$, we say $\mathcal{A} \in \mathcal{N}$ has excess interval $\overline{e}(\mathcal{A}) \subseteq [a,b]$ if $\mathcal{A}\cdot H_m(\underline{X}) = 0$, for $m \notin [a,b]$ and $\underline{X} \in \mathbb{L}oop$. If $[a,b]$ is the minimal interval with this property, we say $\overline{e}(\mathcal{A}) = [a,b]$.

Proposition 2.4.17: For $\theta \neq 0$ in $\mathcal{A}$ consider $\theta_t \in \mathcal{A}_{opp} \subset \mathcal{N}$; then $\overline{e}(\theta_t) = [e(\theta) + d\cdot[\theta],\infty) = [e(\theta) - d\cdot[\theta_t],\infty)$.

Proof: $\theta_t : H_n\underline{X} \to H_{n-d\cdot[\theta]}\underline{X}$ is transpose to $\theta : H^{n-d\cdot[\theta]}\underline{X} \to H^n\underline{X}$ which is 0 if $e(\theta) > n - d[\theta]$. Therefore $\theta_t\cdot H_n = 0$ for $n < e(\theta) + d\cdot[\theta]$, i.e. $\overline{e}(\theta_t) \subseteq [e(\theta) + d[\theta],\infty)$. Let $m \geq e(\theta)$, so $\theta(\iota_m) \neq 0$ in $H^*(K_m)$; then $\theta_t(z) \neq 0$ for some $z \in H_{m+d\cdot[\theta]}\underline{K}_m$ since otherwise $\langle\theta(\iota_m),z\rangle = \langle\iota_m,\theta_t(z)\rangle = 0$, so that $\theta(\iota_m) = 0$ a contradiction. Therefore $\overline{e}(\theta_t) = [e(\theta) + d\cdot[\theta],\infty)$. $\square$

Proposition 2.4.18: For $Q \neq 1$ in $\mathcal{R} \subset \mathcal{N}$ $\overline{e}(Q) = [0,e(Q)]$.

Proof: $Q\cdot H_m(\mathbb{L}oop) = 0$ for $m > e(Q)$, so $\overline{e}(Q) \subseteq [0,e(Q)]$. Let $0 \leq m \leq e(Q)$; then for any $y \in \overline{H}_m(Y)$ $Y \in \mathbb{T}op_*$, $Q(y) \neq 0$ in $H_*(\underline{Q}Y)$. So $\overline{e}(Q) = [0,e(Q)]$. $\square$

Proposition 2.4.19: For $\mathcal{A}_1$ and $\mathcal{A}_2 \in \mathcal{N}$, $\overline{e}(\mathcal{A}_1 \cdot \mathcal{A}_2) \subseteq \overline{e}(\mathcal{A}_2) \cap (\overline{e}(\mathcal{A}_1) - d\cdot[\mathcal{A}_2])$.

Proof: Let $x \in H_n(\underline{X})$. If $n \in \overline{e}(\mathcal{A}_2) \cap (\overline{e}(\mathcal{A}_1) - d\cdot[\mathcal{A}_2])$ then either $n \notin \overline{e}(\mathcal{A}_2)$ whence $\mathcal{A}_2(X) = 0$; or $n \notin \overline{e}(\mathcal{A}_1) - d\cdot[\mathcal{A}_2]$, i.e. $n + d\cdot[\mathcal{A}_2] \notin \overline{e}(\mathcal{A}_1)$ whence $\mathcal{A}_1(\mathcal{A}_2(x)) = 0$ since $\mathcal{A}_2(x) \in H_{n+d\cdot[\mathcal{A}_2]}(\underline{X})$. □

Since for $\mathcal{A} \in \mathcal{N}$, $\mathcal{A} = 0$ in $\mathcal{N}$ iff $\overline{e}(\mathcal{A}) = \emptyset$, we have:

Proposition 2.4.20: For $Q \in \mathcal{R}$ and $\theta \in \mathcal{Q}$, $Q\cdot\theta_t = 0$ in $\mathcal{N}$ if $e(\theta) > e(Q)$.

Proof: By Proposition 2.4.18 $\overline{e}(Q\cdot\theta_t) \subseteq \overline{e}(\theta_t) \cap (\overline{e}(Q) - d\cdot[\theta_t]) = [e(\theta) - d\cdot[\theta_t],\infty) \cap ([0,e(Q)] - d\cdot[\theta_t])$ by Propositions 2.4.16 and 2.4.17. Therefore, $e(\theta) > e(Q)$ implies $\overline{e}(Q\cdot\theta_t) = \emptyset$. □

Thus for each $\ell \geq 0$, $\mathrm{span}\{Q \otimes \theta_t : e(\theta) > e(Q),\ Q \in \mathcal{R}[\ell]\} \subseteq \mathrm{Ker}(\mathcal{R}[\ell] \otimes \mathcal{Q}_{opp} \to \mathcal{N}[\ell])$. We will show they are equal.

Definition 2.4.21: For $n \neq 0$ we define elements $\xi^R \in \overline{H}_*(K_{d\cdot n})$ for $|R| \leq n$ as follows: recall that $\mathcal{Q}^e\cdot\iota_{dn} \subseteq H^*(K_{d\cdot n})$ has basis $\{P^{(R)}(\iota_{dn}) : |R| \leq n\}$. Since $H^*(K_{dn}) \approx V(\mathcal{Q}^e\cdot\iota_{dn}) \otimes E$ as algebras, where V denotes the restricted enveloping algebra and E is an exterior algebra present for $p \neq 2$, we can extend the set $\{P^{(R)}(\iota_{dn}) : |R| \leq n\}$ naturally to a basis of $V(\mathcal{Q}^e\cdot\iota_{dn})$. Then there are naturally defined elements $\{\xi^R : |R| \leq n\}$ in $H_*(K_{dn})$ orthogonal to E and dual to the elements $\{P^{(R)}(\iota_{dn}) : |R| \leq n\}$.

Definition 2.4.22: Consider $\iota_{d\cdot n} \in H^*(K_{dn})$ as the Z_p-linear map $H^*(K_{dn}) \xrightarrow{\iota_{dn}} Z_p : z \longmapsto \langle\iota_{dn}, z\rangle$.

Proposition 2.4.23: For $R, R' \in \mathbb{N}^{(\mathbb{P})}$ with $|R| \leq n$, consider $P_t^{(R')}(\xi^R) \in \overline{H}_*(K_{dn})$. Then $H_*(K_{dn}) \xrightarrow{\iota_{dn}} \mathbb{Z}_p$ takes $P_t^{(R')}(\xi^R)$ to $\delta_{R,R'}$.

Proof: $\langle \iota_{dn}, P_t^{(R')}(\xi^R)\rangle = \langle P^{(R')}(\iota_{dn}), \xi^R\rangle = \delta_{R,R'}$. □

Definition 2.4.24: Consider $M(\overline{H}_*(K_{dn})) = \mathcal{R}\cdot\overline{H}_*(K_{dn}) \subset H_*(QK_{dn})$. By Proposition 2.4.9, $M(\overline{H}_*K_{dn})$ is an $\mathcal{N}$-module, and we consider the ξ^R as elements of it for $|R| \leq n$. Let $\eta : M(\overline{H}_*K_{dn}) \to M_n$ denote the $\mathcal{R}$-linear map induced by $\overline{H}_*K_{dn} \xrightarrow{\iota_{dn}} \mathbb{Z}_p \approx \mathbb{Z}_p\cdot\{y_n\}$.

Proposition 2.4.25: For any $Q \in \mathcal{R}$ and $R, R' \in \mathbb{N}^{(\mathbb{P})}$ with $|R| \leq n$, consider $QP_t^{(R')} \in \mathcal{N}$. Then $QP_t^{(R')}(\xi^R) \in M(\overline{H}_*K_{dn})$ and $\eta(QP_t^{(R')}(\xi^R)) = \delta_{R,R'}\cdot Q(y_n) \in M_n$. □

Theorem 2.4.26: For $Q \in \mathcal{R}$ and $\theta \in \mathcal{Q}$, $\overline{e}(Q\cdot\theta_t) = [e(\theta), e(Q)] - d\cdot[\theta_t] = [e(\theta), e(Q)] + d\cdot[\theta]$.

Proof: We have containment by Proposition 2.4.18. We will show that for any $d\cdot n \in [e(\theta), e(Q)] + d\cdot[\theta]$ there is an element of degree $d\cdot n$ on which $Q\cdot\theta_t$ is nonzero. Let $\theta = \nu_R P^{(R)} + \sum_{R'} \nu_{R'} P^{(R')}$ where $\nu_R \neq 0$ and $e(\theta) = d\cdot|R| \leq d\cdot|R'|$. Then $d\cdot n \in [d\cdot|R|, e(Q)] + d\cdot[R]$; let $m = n - [R]$, then $d\cdot m \in [d\cdot|R|, e(Q)]$. Therefore we can consider $\xi^R \in \overline{H}_*(K_{dm}) \subset M(\overline{H}_*K_{dm})$ since $|R| \leq m$, and ξ^R has degree $d\cdot m + d\cdot[R] = d\cdot n$. Consider $\eta(Q\theta_t(\xi^R)) = \nu_R\cdot Q(y_m)$ in M_m; since $d\cdot m \leq e(Q)$ this is nonzero, so $Q\theta_t(\xi^R) \neq 0$. □

Recall that $\mathcal{Q}$ has coalgebra basis $\{P^{(R)} : R \in \mathbb{N}^{(\mathbb{P})}\}$, and that for $\ell \geq 0$, $\mathcal{R}[\ell]$ has coalgebra basis $\{Q^{(S)} : S \in \mathbb{N}^{\ell}\}$. Also, by 2.4.19 we have $Q^{(S)}P_t^{(R)} = 0$ by excess if $|R| > \pi(S)$, where $\pi(S) = s_\ell$, the last entry in S.

Theorem 2.4.27: For $\ell \geq 0$, the set $\{Q^{(S)} \cdot P_t^{(R)} : S \in \mathbb{N}^{\ell}, R \in \mathbb{N}^{(\mathbb{P})}, |R| \leq \pi(S)\}$ is linearly independent in $\mathcal{N}[\ell]$.

Proof: Suppose $\mathcal{A} = \sum \mu_{SR} Q^{(S)} P_t^{(R)} = 0$. We will show that $\mu_{SR} = 0$ if $|R| \leq \pi(S)$. Pick an R, let $n = |R|$. Consider $\xi^R \in \overline{H}_* K_{dn}$; then $\eta(\mathcal{A}(\xi^R)) = \sum_S \mu_{SR} Q^{(S)}(y_n)$ in M_n. But $\{Q^{(S)}(y_n) : \pi(S) \geq n\}$ is linearly independent in M_n. Therefore $\mu_{SR} = 0$ for every S such that $\pi(S) \geq |R|$. □

Corollary 2.4.28: For $\ell \geq 0$, $\operatorname{span}\{Q \otimes \theta_t : e(\theta) > e(Q)\}$ is the kernel of $\mathcal{R}[\ell] \otimes \mathcal{A}_{opp} \to \mathcal{N}[\ell]$.

Proof: $\mathcal{R}[\ell] \otimes \mathcal{A}_{opp} = \operatorname{span}\{Q^{(S)} \otimes P_t^{(R)} : \pi(S) \geq |R|\} \oplus \operatorname{span}\{Q \otimes \theta_t : e(\theta) > e(Q)\}$ and the map is injective on the first summand by 2.4.27 and is zero on the second summand. □

We can now see that $\mathcal{N}$ is a component cofree coalgebra.

Definition 2.4.29: For $S,R \in \mathbb{N}^{\ell} \times \mathbb{N}^{(\mathbb{P})}$, let $\mathcal{A}^{(S,R)}$ denote $Q^{(S+|R|\cdot\Delta_\ell)} \cdot P_t^{(R)}$. Then by 2.4.25, $\overline{e}(\mathcal{A}^{(S,R)}) = d\cdot[|R|, \pi(S) + |R|] + d\cdot[R]_{\mathcal{a}} = d\cdot(|R| + [R]_{\mathcal{a}} + [0,\pi(S)])$, where $|R| = \sum_{\mathbb{P}} r_k$ and $[R]_{\mathcal{a}} = \sum_{\mathbb{P}} (p^k-1) r_k$, so that $|R| + [R]_{\mathcal{a}} = \sum_{\mathbb{P}} p^k r_k$.

Theorem 2.4.30: $(\mathcal{A}^{(S,R)} : S,R \in \mathbb{N}^{\ell} \times \mathbb{N}^{(\mathbb{P})})$ is a divided system in $\mathcal{N}[\ell]$.

Proof: $\psi(Q^{(S+|R|\Delta_\ell)} \cdot P_t^{(R)}) = \sum Q^{(T')} P_t^{(R')} \otimes Q^{(T'')} P_t^{(R'')}$ summed over $R' + R'' = R''$ and $T' + T'' = S + |R|\cdot\Delta_\ell$. If $\pi(T') < |R'|$ or $\pi(T'') < |R''|$ the term is zero and can be ignored. For each nonzero term let $S' = T' - |R'|\cdot\Delta_\ell$ and $S'' = T'' - |R''|\cdot\Delta_\ell$; then $S' + S'' = S$, and the sum can be written

$$\sum_{\substack{S'+S''=S \\ R'+R''=R}} Q^{(S'+|R'|\cdot\Delta_\ell)} \cdot P_t^{(R')} \otimes Q^{(S''+|R''|\Delta_\ell)} P_t^{(R'')}. \ \square$$

Since $\{\mathcal{A}^{(S,R)} : S,R \in \mathbb{N}^{\ell} \times \mathbb{N}^{(P)}\} = \{Q^{(T)} \cdot P_t^{(R)} : \pi(T) \geq |R|\}$ is a basis for $\mathcal{N}[\ell]$ by Theorem 2.4.26, we have:

Theorem 2.4.31: For $\ell \geq 0$ $\mathcal{N}[\ell]$ is a cofree coalgebra with coalgebra basis $\{ \mathcal{A}^{(S,R)} : S,R \in \mathbb{N}^{\ell} \times \mathbb{N}^{(\mathbb{P})} \}$. □

For each ℓ, $\mathcal{N}[\ell]$ is a graded cofree coalgebra, indexed by $\ell \cup \mathbb{P}$ with weighting $d\cdot(d_i, e_k : i \in \ell, k \in \mathbb{P})$ where $d_i = p^{\ell-i}(p^i-1)$ and $e_k = (p^{\ell}-1) - (p^k-1) = p^{\ell} - p^k$; thus $e_{\ell} = 0$ and $e_k < 0$ for $k > \ell$. We have $[\mathcal{A}^{(\Delta_i\ell,\Delta_0)}] = d_i$ and $[\mathcal{A}^{(\Delta_0\ell,\Delta_k)}] = e_k$; and for $\ell = 0$, $[\mathcal{A}^{(S,R)}] = [\mathcal{A}^{(S,R+n\cdot\Delta_{\ell})}]$ for every n, so that there are infinitely many basis elements of the same degree.

Example 2.4.32: Consider $Q^n P_t^n = \mathcal{A}^{(0,n)}$ in $\mathcal{N}[1]$. Each $Q^n P_t^n$ has degree 0. $\bar{e}(\mathcal{A}^{(0,n)}) = [d\cdot n, d\cdot n] + d\cdot(p-1)n = \{d\cdot p\cdot n\} \in \mathbb{N}$; so for any $\underline{X} \in \mathbb{L}\mathrm{oop}$ and any $m \in \mathbb{N}$, $Q^n P_t^n(H_m(\underline{X})) = 0$ unless $m = dpn$. By Proposition 2.4.2, on $H_{dpn}(\underline{X})$ $Q^n P_t^n = w\circ V$, where V is the shift map and w the restriction map on the bicommutative bialgebra $H_*(\underline{X})$, since $P_t^n(x) = V(x) \in H_{d\cdot n}$ so that $Q^n(P_t^n(x)) = w(V(x)) = V(x)^P$. Thus for each $x \in H_*(\underline{X})$, $\sum_{n=0}^{\infty} Q^n P_t^n(x)$ has only one non-zero term, which is equal to $w\circ V(x)$, so that $\sum_{n=0}^{\infty} Q^n P_t^n$ is a well defined operation on $H_*(\underline{X})$, taking x to $w\circ V(x)$.

We define a filtration on $\mathcal{N}[\ell]$ in each degree so as to be able to consider summable homogeneous series.

Definition 2.4.33: Let $\mathcal{F}_N = \mathcal{E}_{d\cdot[N,\infty)} = \{ \mathcal{A} \in \mathcal{N} : \mathcal{A}\cdot H_{d\cdot[0,N)}(\underline{X})) = 0$ for every $\underline{X} \in \mathbb{L}\mathrm{oop}\}$. Then $\{\mathcal{F}_N\}_{N=0}^{\infty}$ is a decreasing filtration with $\mathcal{F}_0 = \mathcal{N}$ and $\bigcap_{N=0}^{\infty} \mathcal{F}_N = 0$. For each ℓ and each degree $d\cdot n$ we consider the restriction of this filtration to $[\mathcal{N}[\ell]]_n$ and form the completion $[\tilde{\mathcal{N}}[\ell]]_n$. For each ℓ, $\mathcal{N}[\ell] \hookrightarrow \tilde{\mathcal{N}}[\ell]$ as a graded subspace. In particular

$\mathcal{N}[0] = \tilde{\mathcal{N}}[0]$ since $\mathcal{N}[0]$ is finite dimensional in each degree. Let $\tilde{\mathcal{N}} = \sum \tilde{\mathcal{N}}[\ell]$, so that we have $\mathcal{N} \hookrightarrow \tilde{\mathcal{N}}$. A homogeneous family $\{\mathcal{A}_i\}$ in $\mathcal{N}[\ell]$ is summable in $\tilde{\mathcal{N}}[\ell]$ iff for every N $\mathcal{A}_i \in \mathcal{F}_N = \mathcal{E}_{d\cdot[N,\infty)}$ for all but finitely many i.

$\mathcal{N}[\ell]$ has coalgebra basis $\{\mathcal{A}^{(S,R)} : S,R \in \mathbb{N}^\ell \times \mathbb{N}^{(\mathbb{P})}\}$.

Proposition 2.4.34: $\mathcal{F}_N = \mathcal{E}_{d\cdot[N,\infty)}$ has basis $\{\mathcal{A}^{(S,R)} : S,R \in \mathbb{N}^\ell \times \mathbb{N}^{(\mathbb{P})}$ and $|R| + [R]_{\mathcal{a}} = \sum_{\mathbb{P}} p^k r_k \geq N\}$.

Proof: $\bar{e}(\mathcal{A}^{(S,R)}) = d\cdot([0,\pi(S)] + |R| + [R]_{\mathcal{a}})$, so $\mathcal{A}^{(S,R)} \in \mathcal{E}_{d\cdot[N,\infty)}$ iff $N \leq |R| + [R]_{\mathcal{a}} = \sum_{\mathbb{P}} p^k r_k$. □

Theorem 2.4.35: For each ℓ and n every homogeneous family $\{\mu_{SR}\mathcal{A}^{(S,R)}\}$ in $[\mathcal{N}[\ell]]_n$ is summable.

Proof: For any N, $\sum_{\mathbb{P}} p^k r_k < N$ for only finitely many $R \in \mathbb{N}^{(\mathbb{P})}$. For each such R there are only finitely many $S \in \mathbb{N}^\ell$ such that $[\mathcal{A}^{(S,R)}] = n$, since $[\mathcal{A}^{(S,R)}] = [S]_{\mathcal{R}} + |R|\cdot(p^\ell-1) - [R]_{\mathcal{a}}$ so that $[S]_{\mathcal{R}}$ is bounded above by $n + [R]_{\mathcal{a}} - |R|(p^\ell-1)$. Thus for any N, all but finitely many $\mu_{SR}\mathcal{A}^{(S,R)}$ are in $\mathcal{F}_N$. □

Corollary 2.4.36: Every element in $\tilde{\mathcal{N}}[\ell]$ can be expressed uniquely as a homogeneous infinite sum $\sum \mu_{SR}\mathcal{A}^{(S,R)}$.

We have a similar result for the allowable/admissible basis.

Theorem 2.4.37: $\mathcal{N}[\ell]$ has a basis $\{Q^J\cdot P_t^I$: J allowable, $\ell(J) = \ell$, I admissible, and $e(J) \geq e(I)\}$. Every homogeneous family $\{\mu_{IJ}Q^J P_t^I\}$ in $[\mathcal{N}[\ell]]_n$ is summable.

Proof: $\bar{e}(Q^J P_t^I) = [e(I), e(J)] + d\cdot[I] \subseteq [e(I) + d\cdot[I],\infty)$, and $e(I) + d\cdot[I] = d\cdot p\cdot \mathrm{lead}(I)$. For any N there are only finitely many admissible sequences I

with $d\cdot p\cdot \mathrm{lead}(I) < N$. For each such I there are only finitely many allowable sequences J such that $[Q^J P^I_t] = n$, since $[Q^J P^I_t] = [J] - [I]$ so that $[J]$ is bounded above by $n + [I]$. Thus for any N, all but finitely many $\mu_{IJ} Q^J P^I_t$ are in $\mathcal{F}_N$. $\square$

Not every naturally occurring homogeneous family is summable.

Example 2.4.38: We will show (2.4.78) that for any n the homogeneous family $\{\chi(P^j)_t \cdot P^i_t \cdot Q^{n+i+j} : i,j \geq 0\}$ is not summable where χ is the antipode map in $\mathcal{A}$, by using the fact that a summable family can be evaluated on any $x \in H_*(\underline{X})$ to give a finite sum. If we apply 2.4.19, we get $\bar{e}((\chi P^j)_t P^i_t Q^{n+i+j}) = (P^i \cdot \chi P^j)_t Q^{n+i+j}) = [e(P^i \cdot \chi P^j) - d(p-1)n, d(n+i+j)]$. But $e(P^i \chi(P^j)) \not\to \infty$; in fact given N we can find $i,j > N$ such that $e(P^i \chi P^j) = 1$. Take $i = p^k$ and $(p-1)\cdot j = p^k - 1$ for k sufficiently large; then $\chi(P^j) = \sum_{[R]=(p-1)j} \pm P^{(R)}$ has $P^{(\Delta_k)} = P^{I_k}$ as a term, so $P^{p^k} \cdot \chi(P^j)$ has $P^{p^k} \cdot P^{I_k} = P^{I_{k+1}}$ as a term and $e(P^{I_{k+1}}) = 1$. Thus $e(P^i \cdot \chi(P^j)) = 1$.

Definition 2.4.39: Give $\mathcal{N} \otimes \mathcal{N}$ the tensor product filtration: $\mathcal{F}_N(\mathcal{N} \otimes \mathcal{N}) = \sum_{i+j=N} \mathcal{F}_i(\mathcal{N}) \otimes \mathcal{F}_j(\mathcal{N})$. For each degree n and each ℓ, let $[\tilde{\mathcal{N}}[\ell] \hat{\otimes} \tilde{\mathcal{N}}[\ell]]_n$ denote the completion of $[\mathcal{N}[\ell] \otimes \mathcal{N}[\ell]]_n$ with respect to this filtration. Let $\tilde{\mathcal{N}} \hat{\otimes} \tilde{\mathcal{N}} = \sum_\ell \tilde{\mathcal{N}}[\ell] \hat{\otimes} \tilde{\mathcal{N}}[\ell]$.

Proposition 2.4.40: $\psi : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ carries $\mathcal{F}_N(\mathcal{N})$ into $\mathcal{F}_N(\mathcal{N} \otimes \mathcal{N})$ and therefore induces $\tilde{\mathcal{N}} \to \tilde{\mathcal{N}} \hat{\otimes} \tilde{\mathcal{N}}$.

Proof: $\{\mathcal{A}^{(S,R)} : |R| + [R] \geq N\}$ is a basis for $\mathcal{F}_N(\mathcal{N})$, and $\psi \mathcal{A}^{(S,R)} = \sum_{S'R'+S''R''=SR} \mathcal{A}^{(S',R')} \otimes \mathcal{A}^{(S'',R'')}$ where $(|R'| + [R']) + (|R''| + [R'']) = |R| + [R] \geq N$. Thus there exists i and j such that $i + j = N$ and $\mathcal{A}^{(S',R')} \in \mathcal{F}_i(\mathcal{N})$ and $\mathcal{A}^{(S'',R'')} \in \mathcal{F}_j(\mathcal{N})$. $\square$

Proposition 2.4.41: For each ℓ and n, every homogeneous family $\{\mu_{R'S'R''S''}\,\mathcal{A}^{(S',R')} \otimes \mathcal{A}^{(S'',R'')}\}$ in $[\mathcal{N}[\ell] \otimes \mathcal{N}[\ell]]_n$ is summable.

Proof: The same as the proof of Theorem 2.4.35. □

Example 2.4.42: For any i consider $\{Q^{n+i}P^n_t = \mathcal{A}^{(i,n)}\}^\infty_{n=0}$. This is a homogenous summable family in $\tilde{\mathcal{N}}[1]$, of degree $(p-1)-i$; let $\tilde{\mathcal{A}}_i = \sum_{n=0}^{\infty} Q^{n+i}P^n_t \in \tilde{\mathcal{N}}[1]$. Then $\psi\tilde{\mathcal{A}}_i = \sum_{i_1+i_2=i} \tilde{\mathcal{A}}_{i_1} \otimes \tilde{\mathcal{A}}_{i_2}$, i.e. $(\tilde{\mathcal{A}}_i)^\infty_{i=0}$ is a graded divided sequence of weight $d\cdot(p-1)$ in $\tilde{\mathcal{N}}[1]$.

We want to consider graded divided sequences in $\mathcal{N}[\ell]$ (or $\tilde{\mathcal{N}}[\ell]$) for each ℓ; since $\mathcal{N}$ has weight $d\cdot(p-1)$, we consider a collection of elements $(\tilde{\mathcal{A}}_n)_{n\in\mathbb{Z}}$ to be graded if it has weight $d\cdot(p-1)$, i.e. $[\tilde{\mathcal{A}}_n] = d\cdot(p-1)\cdot n$ for every n. The graded coalgebra inclusion $\mathcal{R}[\ell] \hookrightarrow \mathcal{N}[\ell]$ gives $\mathcal{D}^{gr}(\mathcal{R}[\ell]) \hookrightarrow \mathcal{D}^{gr}(\mathcal{N}[\ell])$. We also have the coalgebra inclusion $\mathcal{A}_{opp} \hookrightarrow \mathcal{N}[0]$ giving $\mathcal{D}^{gr}(\mathcal{A}_{opp}) \hookrightarrow \mathcal{D}^{gr}(\mathcal{N}[0])$. $\mathcal{A}$ is antiisomorphic to $\mathcal{A}_{opp}$ as Hopf algebras, so $\mathcal{D}^{gr}(\mathcal{A}_{opp})$ is antiisomorphic to $\mathcal{D}^{gr}(\mathcal{A})$ as groups, but an element of $\mathcal{D}^{gr}(\mathcal{A}_{opp})$ is negatively graded when considered in $\mathcal{N}[0]$; if $[\theta_n] = n$ in $\mathcal{A}$, then $[(\theta_n)_t] = -n$ in $\mathcal{N}[0]$. Thus in $\mathcal{N}$, some simple examples are positively graded and some are negatively graded. We want to consider both positively and negatively graded divided sequences in $\mathcal{N}$, and to retain the convolution product so far as is possible.

Suppose $(\mathcal{A}'_n)^\infty_{n=0}$ is a positively graded divided sequence in $\mathcal{N}$ and $(\mathcal{A}''_m)^\infty_{m=0}$ is a negatively graded divided sequence in $\mathcal{N}$. For each $n \in \mathbb{Z}$ let $\tilde{\mathcal{A}}_n$ be the homogeneous sum of the family $\{\mathcal{A}'_i\cdot\mathcal{A}''_j : i,j \geq 0,$ $[\mathcal{A}'_i\cdot\mathcal{A}''_j] = (p-1)n\}$. This is defined since we have:

Proposition 2.4.43: For each n this infinite family is summable.

Proof: Consider $\bar{e}(\mathcal{A}'_{n+i}\cdot\mathcal{A}''_i) \subseteq \bar{e}(\mathcal{A}''_i) \cap (\bar{e}(\mathcal{A}'_{n+i}) - d\cdot[\mathcal{A}''_i]) \subseteq d\cdot(p-1)[i,\infty)$ since $-[\mathcal{A}''_i] = +(p-1)i$. Therefore, for fixed n, $\mathcal{A}'_{n+i}\cdot\mathcal{A}''_i \in \mathcal{E}_{d[N,\infty)}$ for all but finitely many i. □

Proposition 2.4.44: For fixed $n \in \mathbb{Z}$, the family $\{(\mathcal{A}'_*{*}\mathcal{A}''_*)_{n_1} \otimes (\mathcal{A}'_*{*}\mathcal{A}''_*)_{n_2} : n_1 + n_2 = n,\ n_1, n_2 \in \mathbb{Z}\}$ is summable in $\tilde{\mathcal{N}} \hat{\otimes} \tilde{\mathcal{N}}$, and $\psi((\mathcal{A}'_*{*}\mathcal{A}''_*)_n) = \sum_{\substack{n_1+n_2=n \\ n_1,n_2\ \mathbb{Z}}} (\mathcal{A}'_*{*}\mathcal{A}''_*)_{n_1} \otimes (\mathcal{A}'_*{*}\mathcal{A}''_*)_{n_2}$.

Proof: By the proof of 2.4.43, the family is summable. We will show in Proposition 2.4.52 below that the comultiplication acts as stated. □

Example 2.4.45: Consider $\{Q^n \cdot P_t^{(0,m)} : n,m \geq 0\}$. Collecting the terms of like degree we have a collection of summable families in $\tilde{\mathcal{N}}[1]$. Let $\tilde{\mathcal{A}}_i = \sum_{i=n-(p+1)m} Q^n P_t^{(0,m)}$ in $[\mathcal{N}[\ell]]_{(p-1)i}$ for $i \in \mathbb{Z}$. Then $\psi\tilde{\mathcal{A}}_i = \sum_{i_1+i_2=i} \tilde{\mathcal{A}}_{i_1} \otimes \tilde{\mathcal{A}}_{i_2}$ where i_1 and i_2 range over $\mathbb{Z}$.

Definition 2.4.46: We say a collection of elements $(\tilde{\mathcal{A}}_n)_{n\in\mathbb{Z}}$ in $\tilde{\mathcal{N}}$ is a graded two-way divided sequence if $[\tilde{\mathcal{A}}_n] = (p-1)\cdot n$ and $\psi\tilde{\mathcal{A}}_n = \sum_{\substack{n_1+n_2=n \\ n_1,n_2\in\mathbb{Z}}} \tilde{\mathcal{A}}_{n_1} \otimes \tilde{\mathcal{A}}_{n_2}$, where we think of $\tilde{\mathcal{N}}$ as a completed coalgebra with completed comultiplication $\psi : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}} \hat{\otimes} \tilde{\mathcal{N}}$. Then 2.4.44 states that a two-way divided sequence $(\tilde{\mathcal{A}}_n)_{\mathbb{Z}}$ results from the "convolution" of a positively and a negatively graded divided sequence. Let $\mathcal{D}_{\pm}(\tilde{\mathcal{N}})$ denote the set of all (graded) two-way divided sequence. Let $\mathcal{D}_+(\tilde{\mathcal{N}})$ and $\mathcal{D}_-(\tilde{\mathcal{N}})$ denote the sets of positively and negatively graded divided sequences. We can consider $\mathcal{D}_+(\tilde{\mathcal{N}}) \hookrightarrow \mathcal{D}_{\pm}(\tilde{\mathcal{N}})$ as those collections $(\tilde{\mathcal{A}}_n)_{\mathbb{Z}}$ such that $\tilde{\mathcal{A}}_n = 0$ for $n < 0$. Similarly $\mathcal{D}_-(\tilde{\mathcal{N}}) \hookrightarrow \mathcal{D}_{\pm}(\tilde{\mathcal{N}})$ as those collection $(\tilde{\mathcal{A}}_n)_{\mathbb{Z}}$ such that

$\tilde{\mathcal{A}}_n = 0$ for $n > 0$. Let $G(\tilde{\mathcal{N}})$ denote $\{\tilde{\mathcal{A}} \in [\tilde{\mathcal{N}}]_0 : \psi\tilde{\mathcal{A}} = \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}\}$; then we can consider $G(\tilde{\mathcal{N}}) \hookrightarrow \mathfrak{D}_{\pm}(\tilde{\mathcal{N}})$ as those collections $(\tilde{\mathcal{A}}_n)_{\mathbb{Z}}$ with $\tilde{\mathcal{A}}_n = 0$ for $n \neq 0$. Note that $G(\tilde{\mathcal{N}})$ properly contains $G(\mathcal{N})$.

Example 2.4.47: Consider $\tilde{\mathcal{A}} = \sum_{n=0}^{\infty} Q^n P_t^n$. Then $\psi\tilde{\mathcal{A}} = \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}$, so $\tilde{\mathcal{A}} \in G(\tilde{\mathcal{N}})$.

Rather than extend our theory for coalgebras and divided sequences to "completed coalgebras" and two-way divided sequences, we handle the situation directly.

Theorem 2.4.48: Any $\tilde{\mathcal{A}}_* \in \mathfrak{D}_{\pm}(\tilde{\mathcal{N}}[\ell])$ can be written as $\tilde{\mathcal{A}}_n = \sum_{[\mathcal{A}^{(S,R)}]=(p-1)\cdot n} \mu^S \nu^R \mathcal{A}^{(S,R)}$, summing over $S,R \in \mathbb{N}^{\ell} \times \mathbb{N}^{(\mathbb{P})}$, for a unique family of constants $\mu_1,\ldots,\mu_\ell$ and $(\nu_k)_{\mathbb{P}}$ in $\mathbb{Z}_p$.

Proof: For each n, $\tilde{\mathcal{A}}_n \in [\tilde{\mathcal{N}}[\ell]]_n$ so $\tilde{\mathcal{A}}_n = \sum \mu_{S\,R}\, \mathcal{A}^{(S,R)}$ for unique $\mu_{SR} \in \mathbb{Z}_p$ such that $[\mathcal{A}^{(S,R)}] = n$. Then $\sum_{\mathbb{Z}} \tilde{\mathcal{A}}_n = \sum_{S,R} \mu_{SR}\, \mathcal{A}^{(S,R)}$ in each degree. So $\psi\sum_{\mathbb{Z}} \tilde{\mathcal{A}}_n = \sum_{S,R} \mu_{SR}\psi\, \mathcal{A}^{(S,R)} = \sum \mu_{S'+S'',R''+R''}\, \mathcal{A}^{(S',R')} \otimes \mathcal{A}^{(S'',R'')}$ in $\tilde{\mathcal{N}}[\ell] \hat{\otimes} \tilde{\mathcal{N}}[\ell]$. But $\psi\sum_{\mathbb{Z}} \tilde{\mathcal{A}}_n = \sum_{\mathbb{Z}} \tilde{\mathcal{A}}_i \otimes \sum_{\mathbb{Z}} \tilde{\mathcal{A}}_j = \sum \mu_{S'R'}\, \mathcal{A}^{(S',R')} \otimes \sum \mu_{S''R''}\, \mathcal{A}^{(S'',R'')}$ $= \sum \mu_{R'S'} \cdot \mu_{R''S''}\, \mathcal{A}^{(S',R'')} \otimes \mathcal{A}^{(S'',R'')}$ in $\tilde{\mathcal{N}}[\ell] \hat{\otimes} \tilde{\mathcal{N}}[\ell]$. Therefore $\mu_{S'+S'',R'+R''} = \mu_{S'R'} \cdot \mu_{S''R''}$ for every S',S'',R',R''. Let $\mu_i = \mu_{\Delta_i\ell,\Delta_0}$ and $\nu_k = \mu_{\Delta_0\ell,\Delta_k}$. Then $\mu_{SR} = \mu^S \cdot \nu^R$. □

Corollary 2.4.49: There is a bijective correspondence between $\mathfrak{D}_{\pm}(\tilde{\mathcal{N}}[\ell])$ and families $(\mu_i, \nu_k : i \in \ell,\ k \in \mathbb{P})$ in $\mathbb{Z}_p$. □

Definition 2.4.50: Consider $\tilde{\mathcal{A}}'_* \in \mathfrak{D}_{\pm}(\tilde{\mathcal{N}}[\ell_1])$ and $\tilde{\mathcal{A}}''_* \in \mathfrak{D}_{\pm}(\tilde{\mathcal{N}}[\ell_2])$, and consider the family $\{\tilde{\mathcal{A}}'_i \cdot \tilde{\mathcal{A}}''_j : i + j = n,\ i,j \in \mathbb{Z}\}$ in $\tilde{\mathcal{N}}[\ell_1 + \ell_2]$. If this family is summable we say that the convolution of $\tilde{\mathcal{A}}'_*$ and $\tilde{\mathcal{A}}''_*$ is defined and that $\tilde{\mathcal{A}}'_* * \tilde{\mathcal{A}}''_* = \tilde{\mathcal{A}}_*$ where $\tilde{\mathcal{A}}_n = \sum_{i+j=n} \tilde{\mathcal{A}}_i \cdot \tilde{\mathcal{A}}_j$.

Examples 2.4.51: If $\tilde{\mathcal{A}}'_*$ and $\tilde{\mathcal{A}}''_* \in \mathcal{D}_+$ then the convolution is defined since the sums involved are finite. Similarly for $\tilde{\mathcal{A}}'_*$ and $\tilde{\mathcal{A}}''_* \in \mathcal{D}_-$. By Proposition 2.4.43, if $\tilde{\mathcal{A}}'_* \in \mathcal{D}_+$ and $\tilde{\mathcal{A}}''_* \in \mathcal{D}_-$ the convolution is defined.

Proposition 2.4.52: If the convolution $\tilde{\mathcal{A}}_*$ is defined, then $\tilde{\mathcal{A}}_* \in \mathcal{D}_\pm(\tilde{\mathcal{N}}[\ell_1 + \ell_2])$.

Proof: Consider $\sum_{\mathbb{Z}} \tilde{\mathcal{A}}_n = (\sum \tilde{\mathcal{A}}_i)(\sum \tilde{\mathcal{A}}_j)$. Then $\psi(\sum \tilde{\mathcal{A}}_*) = \psi(\sum \mathcal{A}_i) \cdot \psi(\sum \mathcal{A}_j) =$ $(\sum \mathcal{A}_{i_1} \otimes \sum \mathcal{A}_{i_2})(\sum \mathcal{A}_{j_1} \otimes \sum \mathcal{A}_{j_2}) = (\sum \mathcal{A}_{i_1} \cdot \mathcal{A}_{j_1}) \otimes (\sum \mathcal{A}_{i_2} \cdot \mathcal{A}_{j_2}) = \sum \mathcal{A}_{n_1} \otimes \mathcal{A}_{n_2}$. □

Example 2.4.53: Consider $(\chi(P^j)_t)_{j=0}^{\infty} \in \mathcal{D}_-(\mathcal{N}[0])$ and $(\sum_{i=0}^{\infty} P_t^i Q^{n+i})_{n=0}^{\infty} \in \mathcal{D}_+(\tilde{\mathcal{N}}[1])$. Since $\{\chi(P^j) \cdot P^i \cdot Q^{n+i+j}\}$ is not summable (2.4.38), the convolution is not defined.

Theorem 2.4.54: If $\tilde{\mathcal{A}}_* \in \mathcal{D}_\pm(\tilde{\mathcal{N}}[\ell])$ corresponds to $(\mu_i, \nu_k)_{\ell, \mathbb{P}}$ where $(\nu_k)_{\mathbb{P}}$ has the property $\{k \in \mathbb{P} : \nu_k \neq 0\}$ is bounded, then the convolution of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ is defined for any $\tilde{\mathcal{A}}' \in \mathcal{D}_\pm(\tilde{\mathcal{N}})$.

Proof: Let m be large enough that $\nu_k = 0$ for $k > m$. By 2.4.48 we have $\sum \mathcal{A}_n = \sum \mu^S \nu^R \mathcal{A}^{(S,R)}$ summing over $S \in \mathbb{N}^\ell$ and $R \in \mathbb{N}^m$. Similarly $\sum \mathcal{A}'_n = \sum \mu'^{S'} \nu'^{R'} \mathcal{A}^{(S',R')}$ summing over $S',R' \in \mathbb{N}^{\ell'} \times \mathbb{N}^{(\mathbb{P})}$. Then we need to show that $\{\mu^S \nu^R \mu'^{S'} \nu'^{R'} \mathcal{A}^{(S,R)} \mathcal{A}^{(S',R')} : [\mathcal{A}^{(S,R)} \cdot \mathcal{A}^{(S',R')}] = n\}$ is summable for each n. We will show that $\{\mathcal{A}^{(S,R)} \cdot \mathcal{A}^{(S',R')} : S,R \in \mathbb{N}^\ell \times \mathbb{N}^m,$ $S',R' \in \mathbb{N}^{\ell'} \times \mathbb{N}^{(\mathbb{P})}$ all in degree $n\}$ is summable. Let N be given, $\bar{e}(\mathcal{A}^{(S,R)} \cdot \mathcal{A}^{(S',R')}) \subseteq \bar{e}(\mathcal{A}^{(S',R')}) \cap (\bar{e}(\mathcal{A}^{(S,R)}) - d \cdot [\mathcal{A}^{(S',R')}]) \subseteq$ $d \cdot [\max\{|R'| + [R'] \;,\; |R| + [R] \;-\; [\mathcal{A}^{(S',R')}]\}, \infty)$. We have

$[\mathcal{A}^{(S',R')}] = n - [\mathcal{A}^{(S,R)}] = n - [S]_{\mathcal{R}} - |R|(p^{\ell}-1) - [R]_{\mathcal{a}}$, so $|R| + [R]_{\mathcal{a}} - [\mathcal{A}^{(S',R')}] = p^{\ell}\cdot|R| + [S]_{\mathcal{R}} - n$. There are only finitely many $(S,R) \in \mathbb{N}^{\ell} \times \mathbb{N}^{m}$ such that $p^{\ell}\cdot|R| + [S]_{\mathcal{R}} - n \le N$. Since $|R'| + [R']_{\mathcal{a}} = \sum_{\mathbb{P}} p^{k} r_{k}$ there are only finitely many $R' \in \mathbb{N}^{(\mathbb{P})}$ such that $|R'| + [R']_{\mathcal{a}} \le N$, so for any fixed (S,R) there are only finitely many (S',R') such that $|R'| + [R'] \le N$ and $[\mathcal{A}^{(S,R)}\cdot\mathcal{A}^{(S',R')}] = n$. Therefore $\mathcal{A}^{(S,R)}\cdot\mathcal{A}^{(S',R')} \in \mathcal{E}_{d\cdot[N,\infty)}$ for all but finitely many $(S,R),(S',R')$. □

Proposition 2.4.55: If $\tilde{\mathcal{A}}_{*} \in \mathcal{B}_{+}(\tilde{\mathcal{N}}[\ell])$, and $\tilde{\mathcal{A}}_{*}$ corresponds to $(\mu_i,\nu_k)_{\ell,\mathbb{P}}$, then $\mu_k = 0$ for $k > \ell$. Thus we get Proposition 2.4.43 as a corollary.

Proof: If $\mu_k \neq 0$ for some $k > \ell$ then $\sum\tilde{\mathcal{A}}_n$ has a term $\mu_k\,\mathcal{A}^{(\Delta_0\ell,\Delta_k)}$ which has negative degree. □

Theorem 2.4.56: If $\tilde{\mathcal{A}}_{*} \in \mathcal{B}_{+}(\tilde{\mathcal{N}}[\ell])$ corresponds to $(\mu_i,\nu_k)_{\ell,\mathbb{P}}$ with $\mu_{\ell} \neq 0$, then $\tilde{\mathcal{A}}_{*}$ can be expressed as $Q_{*}*(\theta_{*})_{t}$ for some $Q_{*} \in \mathcal{B}^{gr}(\mathcal{R}[\ell])$ and $\theta_{*} \in \mathcal{B}^{gr}(\mathcal{a})$. If $\mu_{\ell} = 0$, then $\tilde{\mathcal{A}}_{*}$ can only be expressed as such a convolution if $\tilde{\mathcal{A}}_{*} \in \mathcal{B}^{gr}(\mathcal{R}[\ell])$.

Proof: Assume that $\mu \neq 0$. For each S,R $\mathcal{A}^{(S,R)} = Q^{(S+|R|\Delta_{\ell})}P_t^{(R)}$. If $S' = S + |R|\Delta_{\ell}$, then $\mu^{S'} = \mu^{S}\cdot\mu_{\ell}^{|R|}$, so $\mu^{S}\nu^{R} = \mu^{S'}\prod_{\mathbb{P}}(\nu_k/\mu_{\ell})^{r_k} = \mu^{S'}\nu'^{R}$ where $\nu'_k = \nu_k/\mu$. Then $\sum\mathcal{A}_n = \sum\mu^{S}\nu^{R}Q^{(S+|R|\Delta_{\ell})}P_t^{(R)} = (\sum_{S'}\mu^{S'}Q^{(S')})\cdot(\sum_{R}\nu'^{R}P_t^{(R)})$, since $Q^{(S')}P_t^{(R)} = 0$ unless $S' = S + |R|\Delta_{\ell}$ for some S. Then $\tilde{\mathcal{A}}_{*} = Q_{*}*(\theta_{*})_{t}$. If $\tilde{\mathcal{A}}_{*} = Q_{*}*(\theta_{*})_{t}$, let $Q_{*} = \sum\mu^{S'}Q^{(S')}$ and $\theta_{*} = \sum\nu'^{R}P^{(R)}$. Then by the above argument we have $\tilde{\mathcal{A}}_{*} = \sum_{S,R}\mu^{S}\nu^{R}\,\mathcal{A}^{(S,R)}$ where $\nu_k = \nu'_k\cdot\mu_{\ell}$ and the μ_i are unchanged. Thus if $\mu_{\ell} = 0$ for $\tilde{\mathcal{A}}_{*}$, every ν_k must be zero, so $\tilde{\mathcal{A}}_{*} = \sum_{S}\mu^{S}Q^{(S)} \in \mathcal{B}(\mathcal{R}[\ell])$. □

Example 2.4.57: $\sum_{n=0}^{\infty} Q^{n}P_t^{n} = \sum_{n=0}^{\infty}\mathcal{A}^{(0,n)}$ in $\tilde{\mathcal{N}}[1]$ is not the convolution of an $\mathcal{R}$ and an $\mathcal{A}_{opp}$ divided sequence, since it corresponds to $(\mu_1 = 0, \nu_1 = 1)$.

H_* can be considered as a monoidal functor $\mathbb{L}\mathrm{oop} \to \mathcal{R}$-Mod. For any object $\underline{X} \in \mathbb{L}\mathrm{oop}$ $H_*\underline{X}$ is a bicommutative Hopf algebra over $\mathcal{R}$, i.e. the product and coproduct and antipode map χ are maps in $\mathcal{R}$-Mod. Recall the terminology in 2.3.57. Since $X[g] \hookrightarrow X$ in $\mathbb{T}\mathrm{op}$, each $H_*X[g]$ is a coalgebra over $\mathcal{A}_{opp}$. Since H_*X is a Hopf algebra, we have $H_*X[g_1] \approx H_*X[g_2]$ as coalgebras for any $g_1, g_2 \in G$, via multiplication by $g_2 - g_1 = g_2\chi(g_1)$ in G. Since this multiplication is induced by a map in $\mathbb{T}\mathrm{op}$ we have $H_*X[g_1] \approx H_*X[g_2]$ as $\mathcal{A}_{opp}$-modules.

Proposition 2.4.58: 1) If $\tilde{\mathcal{A}} \in \tilde{\mathcal{N}}[\ell]$ then $\tilde{\mathcal{A}} : H_*X[g] \to H_*X[p^{\ell}\cdot g]$.

2) Each $\tilde{\mathcal{A}}_{a} \in \mathcal{D}_{\pm}(\tilde{\mathcal{N}})$ gives a multiplicative transformation $H_*\underline{X} \to \hat{H}_*\underline{X} : x \longmapsto \sum_{\mathbb{Z}} \tilde{\mathcal{A}}_n(x)$.

3) If $(x_R)_{\mathbb{N}}(I)$ is a graded divided system in $H_*\underline{X}$, then $(x'_{m,R})_{\mathbb{N}\times\mathbb{N}}(I)$ is a divided system in $H_*\underline{X}$ where $x'_{m,R} = \tilde{\mathcal{A}}_{m-[R]}(x_R)$.

Proof: 1) $\tilde{\mathcal{A}} = \sum_{S\in\mathbb{N}^{\ell}} \mu_{SR}\, \mathcal{A}^{(S,R)}$ where $\mathcal{A}^{(S,R)} = Q^{(S'')}\cdot P_t^{(R)}$ with $S' \in \mathbb{N}^{\ell}$. Then $\mathcal{A}^{(S,R)}$ takes $H_*X[g] \to H_*X[g] \to H_*X[p^{\ell}\cdot g]$.

2) For $x \in H_m\underline{X}$, each $\tilde{\mathcal{A}}_n(x)$ is a well-defined element of $H_{m+d\cdot n}\underline{X}$. Then $\sum_{\mathbb{Z}} \tilde{\mathcal{A}}_n(x) = \sum_{d\cdot n \geq -m} \tilde{\mathcal{A}}_n(x)$ which is summable in the graded completion of $\hat{H}X$. Since $\psi \sum \tilde{\mathcal{A}}_n = \sum \tilde{\mathcal{A}}_i \otimes \sum \tilde{\mathcal{A}}_j$, $\tilde{\mathcal{A}}_*$ is multiplicative.

3) $x'_{m,R}$ is the m-component of the coefficient of t^R in $\sum_{n,R} \tilde{\mathcal{A}}_n(x_R)t^R$. Since $\psi(\sum \tilde{\mathcal{A}}_n(x_R)t^R) = \sum \tilde{\mathcal{A}}_i(x_{R'})t^{R'} \otimes \sum \tilde{\mathcal{A}}_j(x_{R''})t^{R''}$, this is a divided system. $\square$

For a positively graded divided sequence, the map $H_* \to \hat{H}_*$ always extends to $\hat{H}_* \to \hat{H}_*$ and thus composition, which equals convolution, is possible.

If negatively graded parts are present, though, the map $H_* \to \hat{H}_*$ need not extend.

Definition 2.4.59: Recall that H_* and $\hat{H}_*$ have the grading filtration $H_{[N,\infty)}$ and $\hat{H}_{[N,\infty)}$. A nonhomogeneous linear map $\tilde{\mathcal{A}}_* : H_* \to \hat{H}_*$ is said to be continuous if for every N there exists M such that $\tilde{\mathcal{A}}_*(H_{[M,\infty)}) \subseteq \hat{H}_{[N,\infty)}$.

Example 2.4.60: Consider $(\chi P^n_t)^\infty_{n=0}$ in $\mathfrak{D}_{\pm}(\mathcal{N}[0])$. We will show (Theorem 2.4.61) that it is not continuous. For example $\sum_{n=0}^{\infty} \chi(P^n)_t$ cannot be evaluated on $\sum_{\mathbb{P}} \xi^{\Delta_k}$ in $\hat{H}_*(K_d)$, since in chomology $[\sum_{n=0}^{\infty} \chi(P^n)](x = i_d) = \sum_{\mathbb{P}} P^{(\Delta_k)}(x)$.

Theorem 2.4.61: If $\tilde{\mathcal{A}}_* \in \mathfrak{D}_{\pm}(\tilde{\mathcal{N}}[\ell])$ corresponds to $(\mu_i, \nu_k)_{\ell,\mathbb{P}}$, then $\tilde{\mathcal{A}}_*$ is continuous iff $\{k \in \mathbb{P} : \nu_k \neq 0\}$ is bounded.

Proof: ($\Longleftarrow$): Suppose $\nu_k = 0$ for $k > m$. Then $\tilde{\mathcal{A}}_n = \sum \mu^S \nu^R \mathcal{A}^{(S,R)}$ over $S,R \in \mathbb{N}^\ell \times \mathbb{N}^m$. We need only consider terms of negative degree, since terms that raise degree are automatically continuous; therefore we assume $m > \ell$, since otherwise all occurring terms have positive degree. Given N we will find M such that $[\mathcal{A}^{(S,R)}] < -M$ implies $N - [\mathcal{A}^{(S,R)}] < [R]_{\mathfrak{a}} + |R|$. Since $\bar{e}(\mathcal{A}^{(S,R)}) \subseteq d\cdot[|R| + [R]_{\mathfrak{a}},\infty)$ this implies that $\mathcal{A}^{(S,R)}$ takes H_* into $H_{d\cdot[N,\infty)}$. Since $(S,R) \in \mathbb{N}^\ell \times \mathbb{N}^m$ there are only finitely many $\mathcal{A}^{(S,R)}$ with $-M < [\mathcal{A}^{(S,R)}] < 0$, so $\tilde{\mathcal{A}}_*$ is continuous. Let $M = N\cdot(p^m - p^\ell)/p^\ell$ and suppose $-M > [\mathcal{A}^{(S,R)}] = [S]_{\mathfrak{R}} + \sum_1^m (p^\ell - p^k) r_k$. Then $M + p^\ell \cdot |R| < \sum_1^m p^k r_k \leq p^m \cdot |R|$, so $N(p^m - p^\ell)/p^\ell = M < (p^m - p^\ell)|R|$, i.e. $N < p^\ell \cdot |R|$. Then $N - [\mathcal{A}^{(S,R)}] = N - [S]_{\mathfrak{R}} - p^\ell |R| + \sum p^k r_k < \sum p^k r_k = |R| + [R]_{\mathfrak{a}}$ since $[S]_{\mathfrak{R}} \geq 0$. $\square$

($\Longrightarrow$) Assume $\nu_k \neq 0$ for infinitely many k. We will show that for $N \geq d\cdot p^{\ell}$ there does not exist any M such that $[\sum \tilde{\mathcal{R}}_n]H_{[M,\infty)} \subset \hat{H}_{[N,\infty)}$. For any k consider $\xi^{\Delta_k} \in H_{d\cdot p^k}(\underline{Q}K_d)$. Let n_k be such that $[\tilde{\mathcal{R}}_{n_k}] = [\mathcal{R}^{(0,\Delta_k)}]$. Then $\tilde{\mathcal{R}}_{n_k} = \nu_k \mathcal{R}^{(0,\Delta_k)} + \sum \mu^S \nu^R \mathcal{R}^{(S,R)}$ for other S,R of the same degree. Thus $\tilde{\mathcal{R}}_{n_k}(\xi^{\Delta_k})$ has a term $\nu_k \cdot Q^{\Delta_\ell \ell}(\xi^0)$ in $H_{d\cdot p^\ell}(\underline{Q}K_d)$. Given any M we can choose k such that $d\cdot p^k \geq M$ and $\nu_k \neq 0$. Then $\xi^{\Delta_k} \in H_{[M,\infty)}(\underline{Q}K_d)$ and $[\sum \tilde{\mathcal{R}}_n](\xi^{\Delta_k})$ has a nonzero term $\nu_k Q^{(\Delta_\ell \ell)}(\xi^0) \notin H_{[N,\infty)}(\underline{Q}K_d)$. Thus $\sum_{\mathbb{Z}} \tilde{\mathcal{R}}_n$ is not continuous. $\square$

Corollary 2.4.62: If $\tilde{\mathcal{R}}_*$ is continuous, then the convolution $\tilde{\mathcal{R}}_* * \tilde{\mathcal{R}}'_*$ is defined for every $\tilde{\mathcal{R}}'_* \in \mathcal{D}_{\pm}(\tilde{\mathcal{N}})$. $\square$ by Theorem 2.4.54.

Example 2.4.63: Let $\mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$ denote the set of two way divided sequences that are continuous. $\mathcal{D}^{gr}(\mathcal{R}) \hookrightarrow \mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$ since $\mathcal{D}_{+}(\tilde{\mathcal{N}}) \hookrightarrow \mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$. By Theorem 2.4.61, $\theta_* \in \mathcal{D}^{gr}(\mathcal{Q})$ gives $\theta_{*_t} \in \mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$ iff θ_* corresponds to $f(w) \in U(\mathbb{Z}_p[\![w]\!])$ where $f(w)$ is a polynomial. From Theorem 2.3.75 we see that this is equivalent to $e(\theta_n) \underset{n\to\infty}{\to} \infty$, and to the condition that $\sum_{n=0}^{\infty} \theta_n(x)$ always give a finite sum in cohomology.

Corollary 2.4.64: For $\theta_* \in \mathcal{D}^{gr}(\mathcal{Q})$, either $\theta_{*_t} \in \mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$ or $\chi(\theta_*)_t \in \mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$. Both are in $\mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$ iff $\theta_* = 1$.

Proof: For $f(w) \in U(\mathbb{Z}_p[\![w]\!])$, either $f(w)$ or $f(w)^{-1}$ is polynomial. Both are polynomial iff $f(w) = 1$. $\square$

Theorem 2.4.65: $\mathcal{D}^{cts}_{\pm}(\tilde{\mathcal{N}})$ is a monoid of multiplicative operations on the functor $\hat{H}_*(\text{Loop})$; the convolution product agrees with the composition of operations.

Proof: $\tilde{\mathcal{A}}_* \in \mathcal{B}_{\underline{+}}^{cts}(\tilde{\mathcal{N}})$ iff $\tilde{\mathcal{A}}_* : H_*\underline{X} \to \hat{H}_*\underline{X}$ is continuous for every $\underline{X}$, i.e. iff $\tilde{\mathcal{A}}_*$ extends to $\hat{H}_*\underline{X} \to \hat{H}_*\underline{X}$ for every $\underline{X}$. Thus composition is defined, and by Proposition 2.4.62 convolution is defined. □

By Theorem 2.4.48 each $\tilde{\mathcal{A}}_* \in \mathcal{D}_{\underline{+}}(\tilde{\mathcal{N}}[\ell])$ corresponds to a unique family $(\mu_i,\nu_k)_{\ell,\mathbb{P}}$ in $\mathbb{Z}_p$ such that $\tilde{\mathcal{A}}_* = \sum_{S,R} \mu^S \nu^R \mathcal{A}^{(S,R)}$. Since each $\mathcal{N}[\ell]$ has a preferred coalgebra basis, there is a preferred test map $\mathcal{N} \to P(\mathcal{N})$. Since for each ℓ $P(\mathcal{N}[\ell])$ has finite type, this extends to a map $\tilde{\mathcal{N}} \to P(\mathcal{N})$. We want a more natural description of this test map. The primitives of $\mathcal{N}$ group into two classes, those contained in $\mathcal{R}$ and those associated with $\mathcal{a}_{opp}$. We give two partial test maps $\alpha_{\mathcal{R}}$ and $\alpha_{\mathcal{a}}$ on $\mathcal{N}$. If $\tilde{\mathcal{A}}_* \in \mathcal{D}_{\underline{+}}(\tilde{\mathcal{N}}[\ell])$ corresponds to $(\mu_i,\nu_k)_{\ell,\mathbb{P}}$, then $\alpha_{\mathcal{R}}$ determines $(\mu_i)_\ell$ and $\alpha_{\mathcal{a}}$ determines $(\nu_k)_{\mathbb{P}}$.

Consider the left $\mathcal{N}$-module M_0 with $y_0 \in M_0$. Then $\mathcal{N} \to M_0$: $\mathcal{A} \longmapsto \mathcal{A}(y_0)$ is a left $\mathcal{N}$-linear map. Recall that M_0 can be considered as $\mathcal{R}\cdot y_0 \subset H_*(\underline{QS}^0)$ with $y_0 \in \overline{H_0}(S^0)$. Then M_0 is a sub-coalgebra of $H_*(\underline{QS}^0)$, and $\mathcal{N} \to M_0$ is a map of coalgebras; restricted to $\mathcal{R}$ it is an isomorphism of coalgebras $\mathcal{R} \to M_0 : Q \longmapsto Q(y_0)$.

Proposition 2.4.66: $\operatorname{Ker}(\mathcal{N} \to M_0)$ has basis $\{\mathcal{A}^{(S,R)} : R \neq \Delta_0\}$.

Proof: If $R \neq \Delta_0$ then $\mathcal{A}^{(S,R)}(y_0) = Q^{(S')}(P_t^{(R)}(y_0) = 0)$ since $[y_0] = 0$. $\{\mathcal{A}^{(S,\Delta_0)}(y_0) = Q^{(S)}(y_0)\}$ is linearly independent. □

Definition 2.4.67: Following Madsen [1975] and May [1976], we can use the isomorphism $\mathcal{R} \approx M_0$ and the action of $\mathcal{N}$ on M_0 to define an action of $\mathcal{N}$ on $\mathcal{R}$. In particular this gives an action of $\mathcal{a}_{opp} \subset \mathcal{N}$ on $\mathcal{R}$ such that $\mathcal{R}$ is a coalgebra over $\mathcal{a}_{opp}$ satisfying the excess relations 2.4.2. We have the left $\mathcal{N}$-linear map $\mathcal{N} \to \mathcal{R} : \mathcal{A} \longmapsto \mathcal{A}(1)$; restricted to $\mathcal{R}$ this is the identity map.

Definition 2.4.68: Let $\alpha_{\mathcal{R}}$ denote the map $\mathcal{N} \to M_0 \to \overline{M}_0$, or equivalently the map $\mathcal{N} \to \mathcal{R} \xrightarrow{\alpha} \overline{M}_0$ where $\alpha : \mathcal{R} \to \overline{M}_0 : Q \longmapsto Q(b_0)$ is the test map for $\mathcal{R}$. Then $\alpha_{\mathcal{R}}$ is the composition of a left $\mathcal{N}$-linear map with a map which is only $\mathcal{R}$-linear. Thus all calculations involving elements of $\mathcal{A}_{opp}$ must be done in $\mathcal{R} \approx M_0$ before applying α.

Proposition 2.4.69: $\alpha_{\mathcal{R}}(\mathcal{A}^{(S,R)}) = \begin{cases} \rho^{\ell-i}(b_i) & \text{if } S = \Delta_i\ell \text{ and } R = \Delta_0 . \\ 0 & \text{otherwise} \end{cases}$

If $\tilde{\mathcal{A}}_*$ corresponds to $(\mu_i,\nu_k)_{\ell,P}$ then $\alpha_{\mathcal{R}}(\tilde{\mathcal{A}}_*) = \rho^{\ell}(b_0) + \sum_1^{\ell}\mu_i\rho^{\ell-i}(b_i)$. Then the coalgebra map $\mathcal{N} \to \mathcal{R}$ takes $\tilde{\mathcal{A}}_*$ to $Q_* \in \mathcal{B}^{gr}(\mathcal{R})$, where Q_* corresponds to $(\mu_1,\ldots,\mu_\ell)$.

Proof: $\mathcal{A}^{(S,R)} \in \operatorname{Ker}(\mathcal{N} \to M_0)$ unless $R = \Delta_0$. If $R = \Delta_0$, then $\mathcal{A}^{(S,R)} = Q^{(S)}$ and $\alpha_{\mathcal{R}}(Q^{(S)}) = \alpha(Q^{(S)})$. If $\tilde{\mathcal{A}}_* = \sum\mu^S\nu^R\mathcal{A}^{(S,R)}$ then $\alpha_{\mathcal{R}}(\tilde{\mathcal{A}}_*) = \sum\mu^S\nu^R\alpha_{\mathcal{R}}(\mathcal{A}^{(S,R)}) = \alpha_{\mathcal{R}}(\mathcal{A}^{(\Delta_0\ell,\Delta_0)}) + \sum_1^{\ell}\mu_i\cdot\alpha_{\mathcal{R}}(\mathcal{A}^{(\Delta_i\ell,\Delta_0)})$. Since $\alpha(Q_*) = \alpha_{\mathcal{R}}(\tilde{\mathcal{A}}_*)$, Q_* corresponds to $(\mu_1,\ldots,\mu_\ell)$. $\square$

Consider the graded divided system $(\sum P^n_t s^n)(\sum Q^m t^m) = \sum_{n,m} P^n_t\cdot Q^m s^n t^m$ in $\mathcal{N}[1]$ with $[s] = -(p-1)$, $[t] = (p-1)$. Since $P^n_t\cdot Q^m = 0$ by excess for $n > m$, this can be rewritten as $\sum_{m\geq n} P^n_t\cdot Q^m(st)^n t^{m-n} = \sum_{n,r} P^n_t Q^{n+r} u^n t^r$ where $u = st$, so $[u] = 0$. Dropping the variable t we have a graded divided system determined as the $r\cdot(p-1)$ component of the coefficient of u^n in $\sum_{n,r} P^n_t Q^{n+r} u^n$. As a special case of the Nishida relations [May, 1976] we have

Proposition 2.4.70: As a graded divided system in $\mathcal{N}[1]$, $\alpha_{\mathcal{R}}(\sum_{n,r} P^n_t Q^{n+r} u^n) = \rho^1(b_0) + b_1(1-u)^{p-1}$. The image under $\mathcal{N} \to \mathcal{R}$ is a graded divided system $(P^n_t(Q^{n+r}))_{n,r} \in \mathcal{B}_{\{u,t\}}(\mathcal{R}[1])$ which is completely determined by $\alpha(\sum P^n_t(Q^{n+r})u^n) = \rho^1(b_0) + b_1(1-u)^{p-1}$.

Proof: The action of P_t^n on elements of $\mathcal{R}[1]$ is given by the Nishida relations as $P_t^n(Q^{n+r}) = (-1)^r \binom{n(p-1)}{r} Q^n$. Thus $\alpha(\sum_{n,r} P_t^n(Q^{n+r})u^n) = \alpha(Q^0) + \alpha(Q^1)\cdot(\sum_r (-1)^r \binom{p-1}{r} u^r) = \rho^1(b_0) + b_1\cdot(1-u)^{p-1}$. □

Definition 2.4.71: Consider the left $\mathcal{N}$-module $M(\overline{H}_*(K_d))$. For $k \in \mathbb{P}$ let ξ_k denote $\xi^{\Delta_k} \in \overline{H}_{d\cdot p^k}(K_d) \subset M(\overline{H}_{d\cdot p^k}(K_d))$, and let ξ_* denote the formal sum $\sum_{\mathbb{P}} \xi_k \cdot w^k$. Then any $\mathcal{A} \in \mathcal{N}$ can be evaluated on ξ_* to give the formal sum $\mathcal{A}(\xi_*) = \sum_{\mathbb{P}} \mathcal{A}(\xi_k)\cdot w^k$. There is a left $\mathcal{R}$-linear (but not $\mathcal{a}_{opp}$-linear) map $\eta : M(\overline{H}_*(K_d)) \to \overline{M}_1 : P_t^{(R)}(\xi_k) \longmapsto \delta_{R,\Delta_k}\cdot c$ where $c = \overline{y_1}$ in $\overline{M}_1$. Recall that $\overline{M}_1$ has basis $\{\omega^\ell(c) : \ell \geq 1\}$ and $Q^{(S)}(c) = \begin{cases} \omega^\ell(c) & \text{if } S = \Delta_\ell \ell \\ 0 & \text{other} \end{cases}$. Let $\alpha_{\mathcal{a}}$ assign to each $\mathcal{A} \in \mathcal{N}$ the formal sum $\eta(\mathcal{A}(\xi_*)) = \sum_{\mathbb{P}} \eta(\mathcal{A}(\xi_k))\cdot w^k$. Thus $\alpha_{\mathcal{a}}$ is the composition of a left $\mathcal{N}$-linear map with a map which is only $\mathcal{R}$-linear, so all calculations involving elements of $\mathcal{a}_{opp}$ must be done in the $\mathcal{N}$-module $M(\overline{H}_*(K_d)) \subset H_*(QK_d)$.

Proposition 2.4.72: $\alpha_{\mathcal{a}}(\mathcal{A}^{(S,R)}) = \begin{cases} \omega^\ell(c)\cdot w^k & \text{if } S = \Delta_0 \ell \text{ and } R = \Delta_k,\ k \in \mathbb{P} \\ 0 & \text{other} \end{cases}$.

If $\tilde{\mathcal{A}}_* \in \mathcal{D}_{\pm}(\tilde{\mathcal{N}}[\ell])$ corresponds to $(\mu_i, \nu_k)_{\ell,\mathbb{P}}$ then $\alpha_{\mathcal{a}}(\tilde{\mathcal{A}}_*) = \omega^\ell(c)\cdot(\sum_{\mathbb{P}} \nu_k \cdot w^k)$.

Proof: For $k \in \mathbb{P}$ and $S \in \mathbb{N}^\ell$, $\eta(\mathcal{A}^{(S,R)}(\xi_k)) = \eta(Q^{(S+|R|\cdot\Delta_\ell \ell)} P_t^{(R)}(\xi_k)) = Q^{(S+|R|\Delta_\ell \ell)} \eta(P_t^{(R)}(\xi_k)) = \delta_{R,\Delta_k}\cdot Q^{(S+|R|\Delta_\ell \ell)}(c)$, and $Q^{(S+\Delta_\ell \ell)}(c) = \begin{cases} \omega^\ell(c) & \text{if } S = \Delta_0 \ell \\ 0 & \text{other} \end{cases}$. Then $\alpha_{\mathcal{a}}(\tilde{\mathcal{A}}_*) = \sum \mu^S \nu^R \alpha_{\mathcal{a}}(\mathcal{A}^{(S,R)}) = \sum_{\mathbb{P}} \nu_k \alpha_{\mathcal{a}}(\mathcal{A}^{(\Delta_0 \ell, \Delta_k)}) = \sum_{\mathbb{P}} \nu_k \cdot \omega^\ell(c)\cdot w^k$. □

As a special case of the Nishida relations we have $P_t^n \cdot Q^n = \begin{cases} Q^i \cdot P_t^i & \text{if } n = p \cdot i \\ 0 & \text{other} \end{cases}$. Then we can calculate $\alpha_{\mathcal{A}}$ of the divided system in 2.4.70.

Proposition 2.4.73: $\alpha_{\mathcal{A}}(P_t^n \cdot Q^{n+r}) = \begin{cases} \omega(c) \cdot w & \text{if } r = p,\ n = 0 \\ 0 & \text{other} \end{cases}$, so $\alpha_{\mathcal{A}}(\sum_{n,r} P_t^n Q^{n+r} u^n) = \omega(c) \cdot w \cdot u$.

Proof: First we show that if $r \neq 0$ then $\alpha_{\mathcal{A}}(P_t^n Q^{n+r}) = 0$. $P_t^n Q^{n+r}$ can be written as $\sum \mu_i Q^{i+r} P_t^i$, and $\alpha_{\mathcal{A}}(Q^{i+r} P_t^i = \mathcal{A}^{(r,i)}) = 0$ if $r \neq 0$. For $r = 0$, $\alpha_{\mathcal{A}}(P_t^n Q^n) = \alpha_{\mathcal{A}}(Q^i P_t^i = \mathcal{A}^{(0,i)})$ if $n = p \cdot i$. So $\alpha_{\mathcal{A}}(P_t^n Q^n) = 0$ unless $n = p \cdot 1$. □

Theorem 2.4.74: The graded divided system in $\mathcal{D}_{\{u,t\}}(\mathcal{N}[1])$ represented by $\sum P_t^n Q^{n+r} u^n$ is uniquely determined by $\alpha_{\mathcal{R}}(\sum P_t^n Q^{n+r} u^n) = \rho(b_0) + b_1 \cdot (1-u)^{p-1}$ and $\alpha_{\mathcal{A}}(\sum P_t^n Q^{n+r} u^n) = \omega \cdot (wu)$. In particular in terms of the coalgebra basis of $\mathcal{N}[1]$, we have $\sum P_t^n Q^{n+r} u^n = \sum_{a,b} \mathcal{A}^{(a,b)} (1-u)^{(p-1)a} u^b = \sum Q^{a+b} P_t^b (1-u)^{(p-1)a} u^b$.

Proof: If $\sum P_t^n Q^{n+r} u^n = \sum \mathcal{A}^{(a,R)} f(u)^a g(u)^R$ then $\alpha_{\mathcal{R}}(\sum \mathcal{A}^{(a,R)} f(u)^a g(u)^R) = \rho(b_0) + b_1 \cdot f(u)$ and $\alpha_{\mathcal{A}}(\sum \mathcal{A}^{(a,R)} f(u)^a g(u)^R) = \omega \cdot (\sum_{\mathbb{P}} w^k g_k(u))$. So $f(u) = (1-u)^{p-1}$ and $g_k(u) = \begin{cases} u & \text{if } k = 1 \\ 0 & \text{other} \end{cases}$. □

Example 2.4.75: We can recover the Nishida relations from 2.4.73 as follows. $P_t^n Q^{n+r}$ is the $r \cdot (p-1)$ component of the coefficient of u^n in $\sum_{a,b} Q^{a+b} P_t^b (1-u)^{a \cdot (p-1)} u^b$. Let $a = r$. Then $(1-u)^{r(p-1)} u^b = \sum_i (-1)^i \binom{r \cdot (p-1)}{i} u^{i+b}$, so the coefficient of u^n is $(-1)^{n-b} \binom{r \cdot (p-1)}{n-b}$ and we have $P_t^n Q^{n+r} = \sum_b (-1)^{n-b} \binom{r \cdot (p-1)}{n-b} Q^{r+b} \cdot P_t^b$.

Consider $P_t^* \in \mathcal{B}_-(\mathcal{N}[0])$ and $Q^* \in \mathcal{B}_+(\mathcal{N}[1])$. Since $P_t^* \in \mathcal{B}_{\pm}^{cts}(\mathcal{N}[0])$, by 2.4.62 the convolution $P_t^* * Q^*$ is defined, giving an element of $\mathcal{B}_{\pm}(\tilde{\mathcal{N}}[1])$.

Theorem 2.4.76: $P_t^* * Q^* = \sum\limits_{n,r} P_t^n \cdot Q^{n+r} = \sum\limits_{b=0}^{\infty} Q^b P_t^b$ in $G(\tilde{\mathcal{N}}[1])$. In particular, for any $r \neq 0$, $\sum\limits_{n=0}^{\infty} P_t^n \cdot Q^{n+r} = 0$ in $[\tilde{\mathcal{N}}[1]]_{(p-1)\cdot r}$.

Proof: If we let $u = 1$ in $\sum P_t^n Q^{n+r} u^n = \sum Q^{a+b} P_t^b (1-u)^{a(p-1)} u^b$ we get the stated result, since $(1-u)^{a(p-1)} \to 0$ unless $a = 0$. □

Note that by 2.4.32, $\sum\limits_{b=0}^{\infty} Q^b P_t^b : x \longmapsto V \circ w(x) = w \circ V(x)$ as a multiplicative operation on $H_*(\mathbb{L}oop)$. Thus

Corollary 2.4.77: For any $x \in H_*(\underline{X})$, $\sum\limits_{n=0}^{\infty} P_t^n \cdot Q^n(x) = V \circ w(x) = w \circ V(x)$. □

Theorem 2.4.78: For any n the family $\{\chi(P^i)_t \cdot P_t^j \cdot Q^{n+i+j}\}_{i,j}$ is not summable in $[\tilde{\mathcal{N}}[1]]_n$.

Proof: If it were summable, then for any $x \in H_*(\underline{X})$ we would have
$\sum\limits_i \chi(P^i)_t \ (\sum\limits_j P_t^j Q^{(n+i)+j}(x)) = \sum\limits_m (\sum\limits_{i+j=m} \chi(P^i)_t \cdot P_t^j) Q^{n+m}(x)$. But by 2.4.77, in the left side $\sum\limits_j P_t^j Q^{(n+i)+j}(x) = \begin{cases} V \circ w(x) & \text{if } n = 0,\ i = 0 \\ 0 & \text{Other} \end{cases}$; so the left side equals $\begin{cases} V \circ w(x) & \text{if } n = 0 \\ 0 & \text{other} \end{cases}$. Since $\sum\limits_{i+j=m} \chi(P^i)_t P_t^j = (\sum\limits_{i+j=m} P^j \cdot \chi(P^i))_t$ $= \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{other} \end{cases}$, the right side equals $Q^n(x)$. Even if $n = 0$, we can find $x \in H_*(\underline{X})$ such that $Q^0(x) \neq V \circ w(x)$: just take x to be of positive degree with $V \circ w(x)$ nonzero. Contradiction. □

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