KAN

Simplicial and semisimplicial complexes

M. G. Barratt , 20 xii 1956.

1. Introduction.

The realization $|B|$ of a CSS (= complete semisimplicial) complex B ([1] ; notation as in [7]) has been defined in [3,4,6]. The definition in [6], where degeneracies are taken into account, is followed here.

Theorem 1. *$|B|$ has a simplicial subdivision.*

If degeneracies are not taken into account, $|Sd^2 B|$ is obviously simplicial (Sd is defined in (2.3)). Such realizations have disadvantages, e.g. $|A| \times |B| \neq |A \times B|$.

Let R: ($RB = |B|$, $Rf = |f|$) ($|B|$, $|f|$ as in [6]) be the realization functor.

Definition. A *division functor* on a (sub)category of CSS complexes and maps to the category of CSS complexes and maps is a functor D admitting natural transformations[1]

$$\lambda : D \to 1, \qquad \alpha : RD \to R,$$

---

[1] That is, if $f: B \to C$, then there are commutative diagrams

$$\begin{array}{ccc} DB & \xrightarrow{\lambda_B} & B \\ \downarrow{\scriptstyle Df} & & \downarrow{\scriptstyle f} \\ DC & \xrightarrow{\lambda_C} & C \end{array} \quad ; \quad \begin{array}{ccc} |DB| & \xrightarrow{\alpha_B} & |B| \\ \downarrow{\scriptstyle |Df|} & & \downarrow{\scriptstyle |f|} \\ |DC| & \xrightarrow{\alpha_C} & |C| \end{array} .$$

---

[1] The author was partially supported by Air Force Contract AF 18(600)-1494 during the period when the work on this paper was being done.

such that $\alpha_B\colon |DB| \to |B|$ is a homeomorphism, and $\alpha_B \simeq |\lambda_B|$, there being a homotopy Λ such that $\Lambda(|\bar{\sigma}| \times I) \subset |\bar{\tau}|$ if $\alpha_B(|\bar{\sigma}|) \subset |\bar{\tau}|$, $(\sigma \in DB,\ \tau \in B;\ |\bar{\zeta}| =$ closure of the realization of ζ). Such functors include the barycentric subdivision functor Sd ((2.3) below) and the reverse nerve functor N^{Δ} (§2) restricted to the image of Sd. Therefore Theorem 1 is contained in the more explicit

Theorem 2 N^{Δ}Sd is a division functor from the category of CSS complexes to the subcategory of simplicial complexes.

The proof that Sd is a division functor is (fairly)straightforward. The key to Theorem 1 lies in showing that the closed cells of $|SdB|$ are elements. N^{Δ} SdB is isomorphic to the complex obtained by starring the cells of $|SdB|$ from internal points, in order of increasing dimension.

The author is indebted to helpful discussions with John Milnor, who discovered Theorem 1 independently.

2. Nerve and Star Functors.

The nerve functors apply the formal process of subdivision of simplicial complexes (see, for example, [5]) to CSS complexes; the star functors originate in Alexander's starring operation.

Definition. The nerve functor N (reverse nerve functor N^{Δ}) assigns a simplicial complex NB (N^{Δ}B) to a CSS complex B; the vertices are the nondegenerate simplices of B, and the p-simplices

$(p = 0,1,2,\ldots)$ are sequences $(\sigma_{(o)},\ldots,\sigma_{(p)})$ such that

$$\sigma_{(o)} \leq \cdots \leq \sigma_{(p)}, \qquad (\sigma_{(o)} \geq \cdots \geq \sigma_{(p)}),$$

where $\sigma < \tau$ means that σ is a face of τ. A map $f: B \to C$ induces a transformation $\overline{f}$ of the nondegenerate simplices of B to the nondegenerate simplices of C such that $f\sigma = \overline{f}\sigma$ or a degeneracy of $\overline{f}\sigma$. Then Nf $(N^{\triangle}f)$ is the map such that

$$(\sigma_{(o)},\ldots,\sigma_{(p)}) \longrightarrow (\overline{f}\sigma_{(o)},\ldots,\overline{f}\sigma_{(p)}) \ ((\sigma_{(o)},\ldots,\sigma_{(p)}) \in NB \text{ or } N^{\triangle}B).$$

Clearly, if B is a simplicial complex, NB is isomorphic to the usual barycentric subdivision of B (c.f. [5]).

<u>Remark.</u> There are (in general, not very useful) maps

$$(2.1) \qquad \lambda: NB \to B, \qquad \lambda^{\triangle}: N^{\triangle}B \to B$$

defined on the vertices (σ) by $\lambda(\sigma)$ = last vertex of σ; $\lambda^{\triangle}(\sigma)$ = first vertex of σ, and such that $\lambda(\sigma_{(o)},\ldots,\sigma_{(p)})$, $\lambda^{\triangle}(\sigma_{(o)},\ldots,\sigma_{(p)})$ are in the least subcomplexes of B containing $\sigma_{(p)}$, $\sigma_{(o)}$, respectively.

The face and degeneracy operators in CSS complexes form a multiplicative system (abstract category) Φ of operators ϕ such that that certain values of q (depending on ϕ), $\phi\,\sigma_q$ is defined and is (for some p) a p-simplex of a CSS complex K for any q-simplex

σ_q of K_q.

<u>Definition</u> $\Phi = \{\phi\}$ is a multiplicative system generated by objects $1, \partial_i, s_i (i \geq 0)$, subject to the relations

$$\partial_i\partial_j = \partial_j\partial_{i+1}, i \geq j; \quad s_i s_j = s_{j+1}s_i, i \leq j; \quad 1\phi = \phi = \phi 1 ;$$

$$\partial_i s_i = \partial_{i+1}s_i = 1 \quad ; \quad \partial_i s_j = s_{j-1}\partial_i \text{ if } i \leq j, = s_j\partial_{i-1} \text{ if } i > j+1 .$$

Let $\Phi_{n,p}$ be the set of ϕ such that $\phi\sigma_n$ is defined and $\phi\,\sigma_n \in B_p$ for any $\sigma_n \in B_n$. Let Φ^* be an anti-isomorphic copy of Φ, and let ϕ^* correspond to ϕ in the anti-isomorphism (so that $(\phi\psi)^* = \psi^*\phi^*$). Also let $\Phi^*_{n,p}$ be the image of $\Phi_{n,p}$ in the anti-isomorphism.

Suppose $M = \{M^n\}$ is a collection of CSS complexes M^n, $n \geq 0$, (with M^0 usually consisting of one vertex and its degeneracies), and suppose that Φ^* is made to operate on M as a collection of CSS maps in such a way that if $\phi^* \in \Phi^*_{n,p}$, then $\phi^*: M^p \to M^n$. (For example, $\partial_i^*: M^{n-1} \to M^n$ for each $n \geq i$). Then define the <u>star-functor</u> M* by

<u>Definition</u> M*B is a CSS complex whose q-simplices are classes of pairs (τ_q, σ_n) $(\tau_q \in M^n,\ \sigma_n \in B_n)$ subject to the identifications

(2.21) $$(\phi^*\tau_q, \sigma_n) = (\tau_q, \phi\sigma_n) \qquad (\tau_q \in M^p,\ \phi \in \Phi_{n,p}).$$

The face and degeneracy operations in M*B are given by

(2.22) $$\phi(\tau_q, \sigma_n) = (\phi\,\tau_q,\ \sigma_n).$$

If $f: B \to C$, then $M*f: M*B \to M*C$ is given by

$$(2.23) \qquad M*f(\tau_q, \sigma_n) = (\tau_q, f\sigma_n).$$

[Remark. (2.22),(2.23) are consistent with (2.21) since ϕ^* is a CSS map, for any $\phi^* \in \Phi^*$.]

(The realization functor can be similarly defined; M^n is there a geometric n-simplex, ϕ^* a simplicial map, (2.31) a set of identifications).

If $g: M \to \overline{M}$ is a collection of CSS maps $g^n: M^n \to \overline{M}^n$ which commute with the maps of $\Phi^*(\phi^* g^p = g^n \phi^*$ if $\phi^* \in \Phi^*_{n,p})$, there is a natural transformation $\overline{g}: M* \to \overline{M}*$, such that

$$(2.24) \qquad \overline{g}_B(\tau_q, \sigma_n) = (g^n \tau_q, \sigma_n) \qquad (\tau_q, \sigma_n) \in M*B.$$

Examples of star functors. Conical functor: M^n = join of the standard n-simplex Δ^n with a fixed vertex; suspension functor: M^n = join of Δ^n with two fixed vertices; barycentric division functor: defined below.

Let Δ^n be the (abstract) n-simplex (K [n] in [2]) with vertices $0,\ldots,n$. The order-preserving inclusion $[0,\ldots,n-1] \subset [0,\ldots,n]$ (image omitting i), and the order-preserving projection $[0,\ldots,n+1] \to [0,\ldots,n]$ $(i, i+1 \to i)$ induce, respectively, simplicial maps

$$\partial_i^*: \Delta^{n-1} \to \Delta^n \; ; \; s_i^*: \Delta^{n+1} \to \Delta^n .$$

This induces an operation of Φ^* on $\Delta = \{\Delta^n\}$. Let $Sd\Delta = \{N\Delta^n\}$ (N = nerve functor), and let Φ^* operate on $Sd\Delta$ by

$$\phi^* = (N\phi^*) : N\Delta^p \to N\Delta^n \qquad (\phi^* \in \Phi^*_{n,p}).$$

Similarly $Sd^r\Delta$ can be defined for any r, as $\{N^r\Delta\}$ with operators $N^r\phi^*$.

(2.3) <u>Definition</u> $Sd^rB = Sd^r\Delta * B$; $Sd^rf = Sd^r\Delta * f; r \geq 0$.
Here $Sd^o\Delta = \Delta$.

<u>Lemma</u>. <u>There are natural isomorphisms</u> $B \approx Sd^oB$, $Sd^{r+1}B \approx Sd(Sd^rB)$.
Proof. In the first case, let $e_n \in \Delta^n$ be the nondegenerate n-simplex. Then any $\tau_q \in \Delta^n$ can be expressed uniquely as $\phi^* e_p$ for some p and some $\phi^* \in \Phi^*_{n,p}$. Hence $(\tau_q, \sigma_n) = (\phi^* e_p, \sigma_n) = (e_p, \phi\sigma_n) \in Sd^oB$, and the map $(e_p, \phi\sigma_n) \to \phi\sigma_n$ is the desired isomorphism. In the second case, notice that $N\Delta^n = Sd\Delta^n$, and that $M'*M''* = M*$ if $M^n = M'*M''^n$, for any star functors $M'*$, $M''*$. Identify B, Sd^oB; and $Sd(Sd^rB)$, $Sd^{r+1}B$ by these isomorphisms.

The maps $\lambda^n: N\Delta^n \to \Delta^n$ (2.1) (such that $\lambda^n(\sigma)$ = last vertex of σ_g and, being a simplicial map, is uniquely described by this) induce $\lambda: Sd\Delta \to \Delta$ and so ((2.34)) induce

$$\text{(2.4)} \qquad \overline{\lambda}_B: Sd\,B \to B, \quad \overline{\lambda} : Sd \to Sd^o ,$$

since $Sd\,B = Sd\Delta * B$, $B = \Delta * B = Sd^o B$.

3. Sd as a division functor.

Theorem 3 Sd is a division functor

The transformation $\overline{\lambda} : \mathrm{Sd} \to 1$ is defined by (2.4), and is clearly natural. The definition of $\overline{\alpha}_B : |\mathrm{Sd}\,B| \to |B|$ requires a little care, since the obvious homeomorphism $|\mathrm{Sd}\,\Delta^n| \to |\Delta^n|$ is not compatible with identification maps $|\mathrm{Sd}\,s_i^*|$, $|s_i^*|$.

In each face e_t^q of $|\Delta^n|$, select a point A_t^q, at present arbitrary. Cover $|\Delta^n|$ by a simplicial complex isomorphic to $\mathrm{Sd}\Delta^n$ $(=N\Delta^n)$ by starring each cell e_t^q from A_t^q, in order of increasing dimension q. [That is to say, when all cells of dimension less than q have been triangulated, join each simplex on the boundary of e_t^q to A_t^q, $t = 0,1,\ldots\binom{n}{q}$]. Then there is a natural barycentric map $\beta : |\mathrm{Sd}\,\Delta^n| \to |\Delta^n|$ defined by means of this triangulation, in which $|(e_t^p)| \to A_t^p$.

For each $\sigma_n \in B_n$, select $\{A_t^p\}$ as follows. If the face $\sigma_{p,t}$ of σ_n corresponding to e_t^p (i.e. $\phi\, e_n = e_t^p$, $\phi\, \sigma_n = \sigma_{p,t}$ for some $\phi \in \Phi$) is non-degenerate, then A_t^p is the barycentre (= centroid). If for some ϕ, $\sigma_{p,t} = \phi\, \sigma_{q,u}$, then $\phi^* : e_t^p \to e_u^q$ is a simplicial map. For each vertex V_i of e_u^q, let G_i be the centroid of the vertices of e_t^p in $\phi^{*-1}(V_i)$, and let A_t^p be the centroid of the G_i's $(i = 0,1,\ldots,q)$. This set of choices defines a triangulation of $|\Delta^n|$ and so a map

$$\beta(\sigma_n) : |\mathrm{Sd}\,\Delta^n| \to |\Delta^n|.$$

These maps, for all simplices of B, are easily shown to have the property

<u>Lemma 3.1</u> <u>If for some</u> $\phi \in \Phi_{n,m}$, $\sigma_m = \phi\, \sigma_n$, <u>then the diagram</u>

$$\begin{array}{ccc} |Sd\,\Delta^n| & \xrightarrow{\beta(\sigma_n)} & |\Delta^n| \\ \Big\downarrow |Sd\,\phi^*| & & \Big\downarrow |\phi^*| \\ |Sd\,\Delta^m| & \xrightarrow{\beta(\sigma_m)} & |\Delta^m| \end{array}$$

<u>Commutes.</u>

(It suffices to consider $\phi = s_i$, $\phi = \partial_i$ only).

The (unique) CSS map $[\sigma_n] : \Delta^n \to B$ which maps e_n on σ_n has a realization

$$f(\sigma_n) : |\Delta^n| \to |B|;$$

while $Sd\,[\sigma_n] : Sd\,\Delta^n \to Sd\,B$ similarly defines

$$g(\sigma_n) : |Sd\Delta^n| \to |Sd\,B| .$$

It follows from the definition of $|B|$, $|SdB|$ as identification spaces that the map

$$\bar{\alpha}_B \,|\, (g(\sigma_n)|Sd\,\Delta^n|) = f(\sigma_n) \circ \beta(\sigma_n) \circ g(\sigma_n)^{-1}$$

is single valued and so continuous ([8]), 1-1 and so a homeomorphism ($|\bar{B}|$ is Hausdorff, $g(\sigma_n)|Sd\,\Delta^n|$ is compact). Also, under the hypotheses of (3.1), $g(\sigma_m) \circ |Sd\,\phi^*| = g(\sigma_n)$; $f(\sigma_m) \circ |\phi^*| = f(\sigma_n)$; and hence by (3.1), $\bar{\alpha}_B$, defined piecewise over $|Sd\,B|$, is a homeomorphism. ($\overline{\alpha}_B$ is continuous, since $|Sd\,B|$ is a CW complex, and it has a continuous inverse similarly constructed).

Now $\overline{\alpha}_B \simeq |\overline{\lambda}_B|$; for, with the triangulation of $|\Delta^n|$ under $\beta(\sigma_n)$, define $h_t(\sigma_n): |\Delta^n| \to |\Delta^n|$ such that $h_o(\sigma_n) = 1$, $h_t(\sigma_n)|A^p_s$ is a linear map of $A^p_s \times I$ on the segment joining A^p_s to the last vertex of e^p_t, and $h_t(\sigma_n)$ is a linear extension of this. These homotopies define $H_t: |B| \to |B|$ by

$$H_t|(f(\sigma_n)|\Delta^n|) = f(\sigma_n) \circ h_t(\sigma_n) \circ f(\sigma_n)^{-1}.$$

Then it is clear that $H_o\overline{\alpha}_B = \overline{\alpha}_B$; $H_1\overline{\alpha}_B = |\overline{\lambda}_B|$. This completes the proof of Theorem 3.

4. Regular Complexes

Definition A CSS complex is regular if each non-degenerate σ_n has a vertex such that no face of σ_n containing this vertex is degenerate.

Definition A CW complex is regular if the closure of each n-cell is an n-element, and if e_1, e_2 are two cells, then for some cell e_3, $\overline{e}_1 \cap \overline{e}_2 = \overline{e}_3$ (or $= \phi$).

Theorem 4. The realization of a regular CSS complex is a regular CW complex, and so can be triangulated by starring each cell in order of increasing dimension.

Notice that

Lemma 4.1 $Sd\,B$ is regular, for any CSS complex B.

(In fact, no face of a nondegenerate $\sigma_n \in Sd\,B$, containing the last vertex of σ_n, can be degenerate).

It is clear that if B is regular, and $|\bar{\sigma}_n|$ is an n-element for each nondegenerate $\sigma_n \in B$, then $|B|$ is regular. (For any two cells meet in $\emptyset$ or a common face). Also, if σ_n is nondegenerate and has degenerate faces, then there is a unique degenerate face of maximal dimension. Hence, by an inductive argument, there is a sequence of faces

$$\sigma_n > \sigma_{p_1} > \sigma_{q_1} > \cdots > \sigma_{p_s} > \sigma_{q_s}$$

such that σ_{p_i} is a degeneracy of σ_{q_i}, and is the maximal degenerate face of nondegenerate $\sigma_{q_{i-1}}$, where $q_o = n$ and σ_{q_s} is nondegenerate. All degeneracies of faces of σ_n are consequences of these degeneracies.

Then $|\bar{\sigma}_n|$ is the image of $|\Delta^n|$ under an identification map in which certain faces $|\Delta^{p_i}|$ are identified with $|\Delta^{q_i}|$ by simplicial retractions $|\phi_i^*|$. Split these retractions, for convenience, into sequences

$$|\Delta^{p_i}| \to |\Delta^{p_i-1}| \to \cdots \to |\Delta^{q_i+1}| \to |\Delta^{q_i}|,$$

and renumber the faces so that there is a sequence

$$n > p_1 > q_1 \geq p_2 > q_2 \geq \cdots \geq p_t > q_t$$

where $q_i = p_i - 1$, and, for each i, there is specified a simplicial retraction $f_i : |\Delta^{p_i}| \to |\Delta^{q_i}|$. Perform the identifications by stages, beginning with f_1, let $F_i|\Delta^n|$ be the identification space defined by $(f_1, \ldots, f_i)$, and suppose

<u>Lemma</u> 4.2 If $f: |\Delta^p| \to |\Delta^{p-1}|$ is a simplicial retraction, there is an extension $f: |\Delta^n| \to |\Delta^n|$ which maps $|\Delta^n| - |\Delta^p|$ homeomorphically on $|\Delta^n| - |\Delta^{p-1}|$.

The proof of this is given in §6.

Then, with $p = p_1$, $\tilde{f}_1 F_1^{-1} : F_1|\Delta^n| \to |\Delta^n|$ is single valued and so continuous, 1-1, and so a homeomorphism, and, since f_1 is a retraction, maps $|\Delta^{p_1 - 1}| = F_1|\Delta^{p_1-1}|$ identically on itself. Hence, by iteration, there is a homeomorphism of $F_i|\Delta^n|$ on $|\Delta^n|$ for every i; in particular, $F_t|\Delta^n|$ is an n-element, and Theorem 4 follows.

5. <u>N^Δ as a division functor.</u>

<u>Theorem</u> 5 N^Δ <u>is a division functor on the image of Sd.</u>

Triangulate $|\mathrm{Sd}\,B|$ as in Theorem 4; $|\mathrm{Sd}\,B|$ is now covered by a simplicial complex isomorphic to $N^\Delta \mathrm{Sd}\,B$, for a suitable ordering of the vertices. Therefore there is a simplicial map inducing a homeomorphism $|N^\Delta \mathrm{Sd}\,B| \to |\mathrm{Sd}\,B|$. In order that this should be α^Δ_B (which has to be natural), the triangulation of $|\mathrm{Sd}\,B|$ has to be achieved canonically.

$\lambda^\Delta_B : N^\Delta \mathrm{Sd}\,B \to \mathrm{Sd}\,B$ is the map referred to in (2.1), and sends (σ) to the first vertex of σ; it is uniquely described by this and the condition that $\lambda_B^\Delta(\sigma_{(o)},\ldots,\sigma_{(p)})$ is to be $\sigma_{(o)}$ or a face of $\sigma_{(o)}$ if $(\sigma_{(o)},\ldots,\sigma_{(p)})$ is nondegenerate. This is clearly natural, and $|\lambda_B^\Delta|$ is shown to be homotopic to α_B^Δ (for any construction as in the previous paragraph) by the method at the end of §3.

The construction of $\beta(\sigma_n)$ in §3, but for $\sigma_n \in \mathrm{Sd}\,B$, is now used to triangulate $|\mathrm{Sd}\,B|$ canonically. Let A_t^p be as before,

$$f(\sigma_n): |\Delta^n| \to |\mathrm{Sd}\,B|$$

the identification map. $|\bar{\sigma}_n|$ will now be starred from the points $X_t^p = f(\sigma_n)A_t^p = f(\sigma_{p,t})A^p$.

The 0-section of $|\mathrm{Sd}\,B|$ is triangulated. Suppose the (n-1)-section is triangulated, using the points constructed above. It is necessary to describe how X^n is joined to a simplex ζ on the boundary of $|\bar{\sigma}_n|$. Let

$$C_t^p = \text{centroid } f(\sigma_n)^{-1} X_t^p ;$$

if ζ has vertices $X_t^p, X_u^q, \ldots, X_w^s$, take the rectilinear simplex in $|\Delta^n|$ with vertices $A^n, C_t^p, C_u^q, \ldots, C_w^s$ and map this into $|\bar{\sigma}_n|$ by $f|(\sigma_n)$. Since $f(\sigma_n)\, C_t^p = X_t^p$, etc., this defines a cone on ζ with vertex X^n, and the construction, carried out for all ζ on $|\bar{\sigma}_n|^o$, clearly extends the triangulation over $|\bar{\sigma}_n|$. This construction, repeated on all cells of $|\mathrm{Sd}\,B|$ in order of increasing dimension, defines the canonical triangulation of $|\mathrm{Sd}\,B|$; the α_B^Δ defined with respect to this is then natural.

Theorem 2, and hence Theorem 1, follow from Theorems 3,5, since the composite of a division functor is again a division functor. (Remark. The reverse nerve functor N^{Δ} is required to construct a CSS map λ^{Δ}_{B}. Alternatively, a reverse barycentric functor Sd^{Δ} using $\{N^{\Delta}\Delta^{n}\}$ followed by N can be used).

§6. Proof of (4.2)

E^n is Euclidean space of points $(x_1,\ldots,x_n)$; E^n_+ the half-space $x_n \geq 0$; E^q the subspace $0 = x_{q+1} = \cdots = x_n$; $\overline{E}^n = E^n \cup (\infty)$ the compactification. Let Δ^k denote a geometric k-simplex throughout this section. Choose a face $\Delta^{n-1} \supset \Delta^p$ in Δ^n, and embed Δ^{n-1} in E^{n-1} so that $\Delta^p \subset E^p$, $\Delta^{p-1} \subset E^{p-1}$, and so that the edge of Δ^p collapsed to a vertex in Δ^{p-1} by f lies on a line orthogonal to E^{p-1}. Then $f\colon \Delta^p \to \Delta^{p-1}$ extends to the orthogonal projection $E^n \to E^{p-1}$. Extend the embedding $\Delta^{n-1} \subset E^{n-1}$ to a homeomorphism $h\colon \Delta^n \to \overline{E}^n_+ \subset \overline{E}^n$.

Let $g\colon E^n \to \Delta^{p-1}$ send $x \in E^n$ to the nearest point of Δ^{p-1}. This is single valued and continuous (Δ^{p-1} is compact and convex). Also g can be factored through the orthogonal projection $E^n \to E^{p-1}$ followed by $g|E^{p-1}$; hence g extends f. In the usual metric, let $d(x,y) = \|x - y\|$, and define

$$\delta(x) = d(x,\Delta^p)/d(x,g(x)) \qquad \text{all } x \in E^n - \Delta^{p-1}$$

$$(6.1) \qquad \overline{f}(x) = g(x) + \delta(x)(x - g(x))$$

$$\overline{f}(x) = x \qquad x \in (\overline{E}^n - E^n) \cup \Delta^{p-1}.$$

Then $\overline{f}$ will be shown to have the properties

(6.2)
(i) $\overline{f}$ is continuous and extends f ;
(ii) $\overline{f}$ is a homeomorphism of $\overline{E}^n - \Delta^p$ on $\overline{E}^n - \Delta^{p-1}$;
(iii) $\overline{f}$ maps $\overline{E}^n_+$ and its complement on themselves.

Therefore $\overline{f}h(\Delta^n) = \overline{E}^n_+ = h(\Delta^n)$, and $h^{-1}\overline{f}\circ h = \tilde{f}$ is the required map.

Now $g(x) \in \Delta^{p-1} \subset \Delta^p$ and is the nearest point of Δ^{p-1} to x. Hence

$$0 \leq d(x,\Delta^p)/d(x,g(x)) \leq 1$$
$$0 \leq d(x,g(x)) - d(x,\Delta^p) \leq \text{diam } \Delta^p.$$

and so

$$\|\overline{f}(x) - \overline{f}(y)\| \leq \|g(x) - g(y)\| + \|x-y\|$$

for all $x,y \in E^n - \Delta^{p-1}$. This is also satisfied if one, or both, $x,y \in \Delta^{p-1}$. Therefore $\overline{f}$ is continuous on E^n, since g is. Also

$$\|x - \overline{f}(x)\| \leq \text{diam } \Delta^p$$

so that $\overline{f}$ is continuous on a neighborhood of (∞). This proves (6.2(i)).

$g(x)$ has the property that if $x,y,g(x)$ or $y,x,g(x)$ are colinear and in order, then $g(x) = g(y)$. For, if y separates $x,g(x)$, then

$$d(x,g(y)) \leq d(x,y) + d(y,g(y)) \leq d(x,y) + d(y,g(x)) = d(x,g(x));$$

hence $g(y) = g(x)$. If x separates $y, g(x)$, let $T: E^n \to E^n$ be the similarity transformation

$$T(z) = g(x) + \tau(z - g(x))$$

where $\tau > 1$ is chosen so that $Tx = y$. Since $\Delta^{p-1} \subset T\Delta^{p-1}$,

$$d(x,g(x)) = d(Tx,Tg(x)) = d(Tx,T\Delta^{p-1}) \leq d(Tx,\Delta^{p-1}) = d(y,\Delta^{p-1}) = d(y,g(y))$$

so $g(x) = g(y)$ again.

Let L be the half line ending at $g(x)$, containing x. Then $g(y) = g(x)$ for all y on L. Then $d(y,\Delta^p)$ is strictly monotone for $y \in L - \Delta^p$, for, since $g(y) = g(x)$, we may suppose that $y, x, g(x)$ are in order, and take T as before: since $\tau > 1$,

$$d(y,\Delta^p) \geq d(y,T\Delta^p) = d(Tx,T\Delta^p) = \tau d(x,\Delta^p) > d(x,\Delta^p),$$

provided $x \notin \Delta^p$. But then, if $x \in E^n - \Delta^{p-1}$,

$$\|\bar{F}(x) - g(x)\| = \|x - g(x)\| \, \delta(x) = d(x,\Delta^p),$$

and hence g maps $L - \Delta^p$ homeomorphically on $L - g(x)$, and (6.2(ii)) follows at once.

Princeton University

References

1. S. Eilenberg and J. A. Zilber, Semisimplical complexes and singular homology, Ann. Math. 51 (1950), p.499-512.

2. ———— On Products of Complexes, Am. J. Math.75(1953), p. 200-4.

3. J. B. Giever, On the equivalence of two singular homology theories, Ann. Math. 51 (1950), p. 178-191.

4. S. T. Hu, On the realizability of homotopy groups and their operations, Pacific J. Math. 1 (1951),p.583-602.

5. S. Letschetz, Introduction to Topology, Princeton.

6. J. Milnor, The geometric realization of a semisimplicial complex Lecture notes, Princeton University, 1955-6.

7. J. Moore, Semisimplicial complexes ibid.

8. J. H. C. Whitehead, Note on a theorem due to Borsuk. Bull. Am. Math. Soc. 54 (1948), p. 1125-32.

9. ———— Combinatorial Homotopy I, ibid, 55(1949), p. 213-45.

10. ———— A Certain Exact Sequence, Ann. Math. 52 (1950) p. 51-110.