~~Then $(V \otimes W)^{\otimes h} = \bigoplus_{i+j=h} P_{ij}$,~~

~~where $P_{ij} = \bigoplus_g g(V^{\otimes i} \otimes W^{\otimes j})$~~

We will split $(V \otimes W)^{\otimes h}$ as the

sum $\bigoplus_{i+j=h} P_{ij}$ of a number of subspaces; then $(V \otimes W)^{\otimes h}/Y = \sum_{i+j=h} P_{ij}/Y$

More precisely, for any $Y \in R(\Sigma_h)$.

$$P_{ij} = \bigoplus_g g(V^{\otimes i} \otimes W^{\otimes j})$$

where ~~the sum~~ g runs over representatives for the cosets of $\Sigma_i \times \Sigma_j$ in Σ_h.

~~Let X_j with Y be representatives of Σ_i, Σ_j;~~

~~it follows that~~

~~$Hom_{\Sigma_h}($~~ $\qquad)$

It follows that if Y is a representation of Σ_h,

then $Hom_{\Sigma_h}(P_{ij}, Y)$

~~(which is the dual of P_{ij}/Y)~~

is equal to $Hom_{\Sigma_i \times \Sigma_j}(V^{\otimes i} \otimes W^{\otimes j}, Y)$

~~(which is the dual of $V^{\otimes i} \otimes W^{\otimes j}/\ldots Y$~~

$$1_d = 1_{r_1} 1_{r_2} 1_{r_3} \cdots \in A_n$$

$$\& \quad \mathcal{E}_d = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \mathcal{E}_{c_3} \cdots \in A_n$$

Lemma (i) $(\mathcal{E}_d, 1_d) = 1$

(ii) $(\mathcal{E}_d, 1_\delta) = 0$ unless $d \geq \delta$

Proof. Immediate from a Hopf algebra calculation [illegible].

From our point of view, this just confirms that the inner product between the base $\{\mathcal{E}_d\}$ and the base $\{1_\delta\}$ is unimodular, because its matrix is unitriangular with respect to a suitable ordering. But since $\mathcal{E}_d$ and 1_δ are actual representations and not just virtual ones, we get information about the irreducible representations which occur in them.

Corollary (i) There is just one irreducible representation i_d which occurs both in $\mathcal{E}_d$ and in 1_d; it occurs with multiplicity 1 in each.

(ii) The irreducible representations i_d, i_δ are inequivalent unless $d = \delta$.

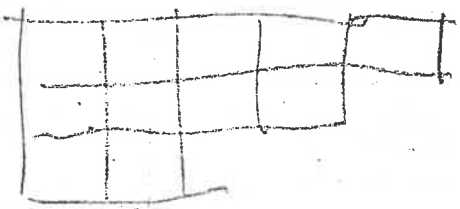
(iii) The representations [illegible] all the irreducible reps of Σ_n.

(iv) The irreducible reps which occur in 1_d are i_δ with $\delta \geq d$. The irreducible reps which occur in $\mathcal{E}_d$ are i_δ with $\delta \leq$

175

If we suppose [illegible] suff. [illegible] III
calculate with [illegible] we need [illegible] and [illegible]
on the irreducible [illegible] is [illegible].
So this would say the inner product
between $H^*(BU)$ and $H_*(BU)$ has
determinant ab. We know it has
determinant 1, so $ab = 1$,
and θ^* and θ_* are iso.

Classical combinatorialists will
undoubtedly find this insufficiently
explicit. Let us then recall that
a Young diagram is a way of depicting
2 conjugate partitions of ...



It is an arrangement of n boxes
into rows of decreasing lengths
$r_1 \geq r_2 \geq r_3 \ldots$, so that $\sum r_i = n$
so that the rows are justified
to the left; at the same time
it arranges the n boxes into columns
of decreasing length, $c_1 \geq c_2 \geq c_3 \geq \ldots$,
so that $\sum c_i = n$ and the columns are justified
from the top. We order
the diagrams, setting $d \geq d'$

if $r_1 \geq r'_1$ & $r_1 + r_2 \geq r'_1 + r'_2$ & ...,

it would be equivalent to write

$c_1 \leq c'_1$ & $c_1 + c_2 \leq c'_1 + c'_2$ & ...

To such a diagram we associate
monomials

are worked out in A. It is clear we do get a homogeneous polynomial; each product in the determinant contains one term from each row and one from each column, so its degree is

$$\sum_i \alpha_i - \sum_j Y_j + \sum_j j$$

$$= \sum_i \alpha_i - \sum_j Y_j + \ldots$$

Suffices to assume $\alpha_1 > \alpha_2 > \ldots$

Theorem. If d is a Young diagram with rows of lengths $r_1 \geq r_2 \geq r_3 \ldots \geq r_n$, then

$$i_d = \Delta(r_1 - 1, r_2 - 2, \ldots, r_n - n \,;\, 1, 2, 3 \ldots n)$$

In order to write the proof conveniently, we introduce in A the scalar product $\langle a, b \rangle$. For this take

$$A \xrightarrow{\;\psi\;} A \otimes A \xrightarrow{\;1 \otimes \langle\;, b\rangle\;} A$$

and apply it to a.

Lemma. If $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, then

$$\Delta(\alpha; Y) / 1_r = \sum \Delta(\beta; Y)$$

where the sum runs over β such that

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \alpha_3 \ldots \geq \alpha_n \geq \beta_n$$

and

$$\sum \alpha_i = r + \sum \beta_i .$$

(i) Immediate from lemma, (i).

(ii) i_d occurs in $\mathcal{E}_d$ & 1_d
i_δ occurs in $\mathcal{E}_\delta$ & 1_δ.
If they are equivalent, lemma (ii) gives $d \geq \delta$, $d \leq \delta$ whence $d = \delta$.

(iii) The irred. reps ξ_d are inequivalent by (ii), & there is the correct number of them, viz. the no. of partitions of n.

(iv) i_δ does occur in $\mathcal{E}_\delta$, so by lemma (ii) it can only occur in 1_d if $\delta \geq d$. Similarly i_δ does occur in 1_δ, so by lemma (ii) it can only occur in $\mathcal{E}_d$ if $d \geq \delta$.

~~At this~~ The classical combinatorialist still has a challenge for us. All right, he says, write the irreducible reph i_d as an explicit Z-linear combination in the permutation reps 1_δ.

To oblige him, we will discuss a class of formulae slightly wider than he needs. Suppose given 2 sequences of numbers

$\alpha_1, \alpha_2, \ldots, \alpha_n$
$\gamma_1, \gamma_2, \ldots, \gamma_n$;

we form a matrix A whose $(i,j)^{th}$ element is

$$a_{ij} = 1_{(\alpha_i - i + \gamma_j)}$$

and set

$$\Delta(\alpha; \gamma) = \det A = \left| 1_{(\alpha_i - i + \gamma_j)} \right|$$

where the products and Z-linear combinations

$\alpha_1 \geq \beta_1 \geq \alpha_2 \cdots \geq \alpha_{n-1} \geq \beta_{n-1}, \quad \alpha_n \geq \beta_n$

$\sum \alpha_i = r + \sum \beta_i.$

Well, now, that's a few more terms than we want. Observe, however, that

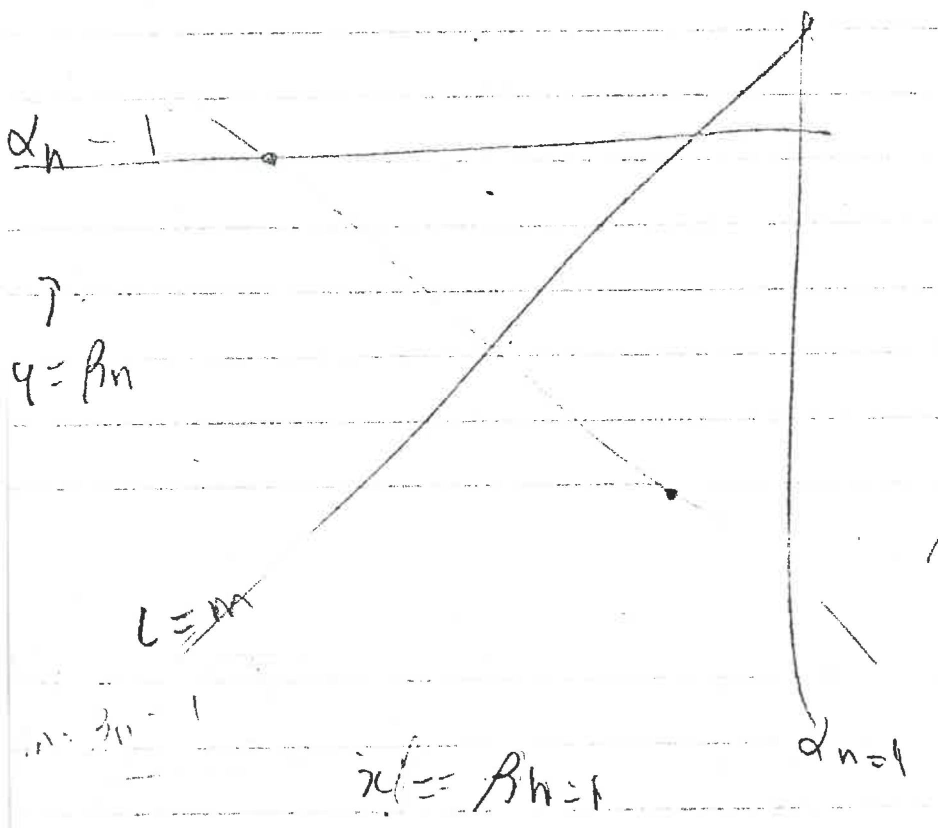
$$\Delta(\beta_1, \ldots, \beta_{n-2}, x, |x| ; y) = 0$$

because it is a determinant with 2 equal rows, and

$$\Delta(\beta_1, \ldots, \beta_{n-2}, x, |y| ; y)$$

$$= -\Delta(\beta_1, \ldots, \beta_{n-2}, y, |x| ; y)$$

by interchanging 2 rows. Well, let us consider the available values of β_{n-1} and β_n.



They are restricted by

$\beta_{n-1} \leq \alpha_{n-1}$

$\beta_n \leq \alpha_n$

and a linear relation

$$\beta_{n-1} + \beta_n = \sum_1^n \alpha_i - \sum_1^{n-2} \beta_i - r.$$

Proof. By induction over n. For $n = 1$, we just get

$$1_{\alpha_1 - \gamma_1} / 1_r = 1_{(\alpha_1 - r) - \gamma_1},$$

which is the required answer with $\beta_1 = \alpha_1 - r$.

As our inductive hypothesis, we assume the result for $n-1$. Expand the determinant by (say) the last row, ~~although it doesn't matter very much which row you use.~~ We get

$$\Delta(\alpha;\gamma) = M_{n,n}\, 1_{\alpha_n - \gamma_n} - M_{n,n-1} \cdot 1_{\alpha_n - \gamma_{n-1}} \cdots$$

Applying $/1_r$, we get

$$\Delta(\alpha;\gamma)/1_r = \sum_{s+t=r} (M_{nn}/1_s)(1_{\alpha_n - \gamma_n}/1_t) - \cdots$$

— Here $1_{\alpha_n - \gamma_n}/1_t = 1_{\alpha_n - \gamma_n - t}$, [illegible]

Each minor M_{ij} is a determinant to [illegible]

which the inductive hypothesis applies, in fact, the only reason we have the γ's here is to get the required generality into the inductive hypothesis. Applying the inductive hypothesis and collecting terms, we get

$$\Delta(\alpha;\gamma)/1_r = \sum \Delta(\beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n; \gamma)$$

where $\beta_n = \alpha_n - t$, and the sum ranges over

We have $\alpha_{n-1} > \alpha_n$, i.e.

so the line $x = \ldots$ comes through with this. All there is a point on the line it goes zero, and points symmetrically disposed about the line give results which cancel; so the [illegible] portion cancels and it is sufficient $\alpha \le \alpha_n - 1$

to run the sum over [illegible]

$\alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n$.

~~Corollary. $(\Delta(\alpha, 0), 1_{r_1} 1_{r_2} \cdots 1_{r_n}) = 0$~~

Must ...

Corollary. Assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $r_1 \geq r_2 \geq \cdots \geq r_n$

Then $(\Delta(\alpha), 1_{r_1} 1_{r_2} \cdots 1_{r_n}) = 0$

if $(\alpha_1, \alpha_2, \ldots, \alpha_n) < (r_1, r_2, \ldots, r_n)$

in the ordering on Young diagrams.

Notice that we may have $n' > n$ — this is the only reason the proof requires care.

Proof. By induction on n. It is true for $n = 0$ and we propose to write the inductive step so that it works from 0 to 1, so we assume the result true for $n-1$.

We have $\alpha_{n-1} > \alpha_{n}$ so the line $x = y$ cuts through the lines. If there is a point on the line it is zero, and points symmetrically disposed about the line give results which cancel, so the points $x \leq \alpha_{n} - 1$ cancels and it is sufficient to make sum over

$$\alpha_{n-1} \geq \beta_{n-1} > \alpha_{n}.$$

Now set

$$\Delta(\alpha) = \Delta(\alpha_1 - 1, \alpha_2 - 2, \ldots, \alpha_n - n;\ 1, 2, 3, \ldots, n)$$

The lemma says: if $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, then

$$\Delta(\alpha) / 1_r = \sum \Delta(\beta)$$

where the sum is over β s.t.

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \alpha_3 \geq \cdots \geq \alpha_n \geq \beta_n$$

and

$$\sum \alpha_i = r + \sum \beta_i.$$

$$(\beta_1, \beta_2, \ldots, \beta_n) = (r_1, \ldots, r_{n-1}, 0).$$

But in that case, for $j < n$ we get

$$\sum_1^j r_i = \sum_1^j \beta_i \le \sum_1^j \alpha_i \le \sum_1^j r_i$$

which implies $\alpha_i = r_i$ for $i < n$.

The equations $\beta_n = 0$, $\beta_{n+1} = 0$, ...
imply $\alpha_i = 0$ for $i > n$.

Finally, we get $\alpha_n = r_n$ by subtraction.
This shows

$$(\alpha_1, \ldots, \alpha_n) = (r_1, r_2 \ldots r_n, 0 \ldots 0),$$

which contradicts our assumption. This proves the result.

Let d be a Young diagram with rows of lengths $r_1 \geq r_2 \geq \ldots \geq r_n$.
(It is easy to check

Proof of Theorem.

that
$\Delta = \Delta(r_1, r_2, \ldots, r_n)$ ~~ann~~

is a Z-linear combination of monomials $1_{s_1} 1_{s_2} \ldots 1_{s_n}$ $= 1_\delta$
with $(s = \delta \geq d)$ in the ordering of Young diagrams. Each such $1_{s_1} 1_{s_2} \ldots 1_{s_n}$ is a Z-linear combination

of irreducible $i_{\delta'}$ with $\delta' \geq \delta \geq d$.

Therefore, $\Delta = \Delta(r_1, r_2 \ldots r_n)$ ~~(d)~~ is a

Z-linear combination of $i_{\delta'}$ with $\delta' \geq d$.

183

Then $(\Delta(\alpha)\mathfrak{A}, 1_{r_1}, 1_{r_2} \dots 1_{r_n})$

$= (\Delta(\alpha)\mathfrak{A}/1_{r_n}, 1_{r_1}, 1_{r_2} \dots 1_{r_{n-1}})$

$= \sum (\Delta(\beta)\mathfrak{A}, 1_{r_1}, 1_{r_2} \dots 1_{r_{n-1}})$

where the sum runs over

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \dots \geq \alpha_n \geq \beta_n$$

$$\sum_{\alpha} \alpha_i = r_n + \sum \beta_i .$$

I claim that all our sequences β have

$$(\beta_1, \beta_2, \dots, \beta_n) \leq (r_1, \dots, r_{n-1}, 0).$$

In fact, for $j < n$ we certainly have

$$\sum_1^j \beta_i \leq \sum_1^j \alpha_i \leq \sum_1^j r_i ,$$

while for $j \geq n$ we have

$$\sum_1^j \beta_i \leq \left(\sum \alpha_i\right) - r_n = \sum_1^{n-1} r_i .$$

So the induction hypothesis asserts

$$(\Delta(\beta)\mathfrak{A}, 1_{r_1}, 1_{r_2} \dots 1_{r_{n-1}}) = 0$$

except for the possible case

Theorem $\dim i_d = \dfrac{n!}{\prod h_{ij}}$.

Proof The idea is that we know the dimension of the permutation representations:

$$\dim\left(1_{r_1} 1_{r_2} \cdots 1_{r_k}\right) = \frac{n!}{r_1!\, r_2! \cdots r_k!}.$$

Therefore we know the dimension of everything via the formula

$$i_d = \Delta(r_1, r_2, \ldots, r_k)$$

we have

$$\dim i_d = n! \begin{vmatrix} \frac{1}{r_1!} & \frac{1}{(r_1+1)!} & \cdots & \frac{1}{(r_1+k-1)!} \\ \frac{1}{(r_2-1)!} & \frac{1}{r_2!} & \cdots & \frac{1}{(r_2+k-2)!} \\ \vdots & & & \\ \frac{1}{(r_k-k+1)!} & & & \frac{1}{r_k!} \end{vmatrix}$$

Notice that r_1+k-1, r_2+k-2, $\ldots$ r_k are the hook-lengths for the first column, h_{11}, $h_{21}, \ldots$ h_{k1}.

So multiplying up we get

$$\dim i_d = \frac{n!}{h_{11}!\, h_{21}! \cdots h_{k1}!} \begin{vmatrix} h_{11}(h_{11}-1) & \cdots & h_{11}(h_{11}-1) & h_{11} \\ & \cdots & h_{12}(h_{12}-1) & h_{12} \\ & & & \\ & & & h_{k1} \end{vmatrix}$$

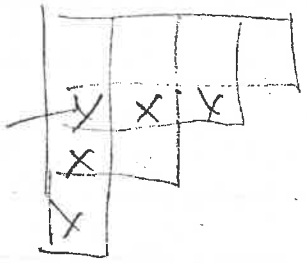
The irreducible i_d occurs with multiplicity 1 because it occurs once in 1_d, which occurs once in Δ. We show that the other irreducibles i_δ, $\delta > d$ occur in Δ with multiplicity zero, and this is done by induction downwards on δ. In fact, by the corollary

$$(\Delta, 1_\delta) = 0,$$

and 1_δ contains i_δ with multiplicity 1, plus other irreducibles which annihilate Δ by induction hypothesis. This completes the proof.

From this theorem I shall derive a formula which I actually need in case [illegible], and this gives the dimension of the irreducible reps i_d.

Suppose given a Young diagram, say



Take the box in the i^{th} row & j^{th} column. Take the number of boxes in the same row to the right of the box where you started, plus the number of boxes in the same column below the box where you started, counting the original box just once. That gives you the hook length h_{ij}. In our case,

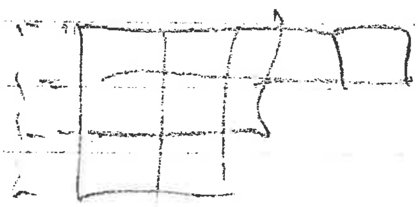
$$h_{21} = 5.$$

That multiplies the LHS by

$$\frac{(h_{11}+1)(h_{21}+1)\cdots(h_{n1}+1)}{h_{11}\;h_{21}\;\cdots\;h_{n1}}.$$

That's exactly what happens on the RHS, so if the formula was true for the old diagram it's true for the new one.

Now suppose the formula is true for diagram



and I form a new one by adding a new first column of the same length. That multiplies the LHS by

$$(h_{11}+1)(h_{21}+1)\cdots(h_{n1}+1).$$

Again that's exactly what happens on the RHS, so if the formula was true for the old diagram it's true for the new one.

Any diagram can be obtained from □ by iterating these steps, so the result is always true.

Using elementary column operations from the right, this gives

$$\dim i_\alpha = \frac{n!}{h_{11}!\,h_{21}!\cdots} \begin{vmatrix} h_{11}^{n-1} & \cdots & h_{11}^2 & h_{11} & 1 \\ h_{21}^{n-1} & \cdots & h_{21}^2 & h_{21} & 1 \\ \vdots & & & & \vdots \\ \cdots & \cdots & h_{n1}^2 & h_{n1} & 1 \end{vmatrix}$$

This is a Vandermonde det., & it gives

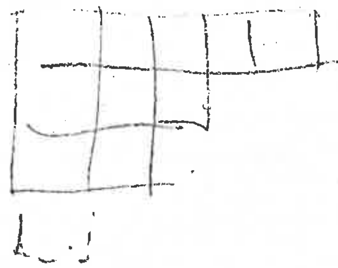
$$\dim i_\alpha = \frac{n!}{h_{11}!\,h_{21}!\cdots} \prod_{i<j} (h_{i1} - h_{j1}).$$

Of course this is written entirely in terms of hook lengths from the first column; we still need to prove

$$\frac{h_{11}!\,h_{21}!\cdots h_{n1}!}{\prod_{i<j}(h_{i1} - h_{j1})} = \prod_{i,j} h_{ij}.$$

Well, we do this by induction over the diagrams. For the diagram $\square$ both sides are 1.

Suppose the formula is true for a given diagram, and consider a new diagram with a new box added at the bottom of the first column



we can find such transposition v, h for every q except the q in one double coset VqH, which has the (irregular) number $|V|\,|H|$ of members.

Of course, if the equation $vq = qh$ is soluble for one q in a double coset VqH, then it is soluble for all the others.

<u>Proof</u> First suppose $\exists\, i, j$ in the same row of T s.t. qi, qj are in the same column of t. In this case we can take $h = (i\,j)$, $v = (qi\,qj)$ and we have solved the equation $vq = qh$.

The contrary case, then, is that in which any 2 distinct elements i, j in the same row of T map to qi, qj in distinct columns of t. Thus in particular, all the ρ_1 cells of the first row of T go to distinct columns of t. So postmultiplying q by some $v \in V$, we can suppose they all go to the first row of t; in particular, $\nu_1 \geq \rho_1$.

Now consider the elements in the second row of T. Again they all go into distinct columns, so by postmultiplying the q by some $v \in V$ we can suppose they go into the highest places we haven't used yet, in order [illegible].

In particular, $\nu_1 + \nu_2 \geq \rho_1 + \rho_2$.

Continuing in this way, we see that in the contrary case we have $d \geq \delta$.

Hence, if $d = \delta$, we have shown that

At this point we want a really splendid gadget which we owe to the Rev. Alfred Young, and that is the Young idempotent. Let me build up gently.

First suppose I have two Young tableaux, that is, diagrams filled up with numbers like this:

d δ

t

4	5	8
3	6	
2	7	
1		

τ

2	3	6	7
1	4	5	8

The no. of boxes is supposed to be n in both cases.

It is time then to introduce 2 subgroups, as follows:

V, the vertical subgroup, consists of those permutations which take any number to a number in the same column of t:

$$V = \Sigma(1,2,3,4) \times \Sigma(5,6,7) \times \Sigma(8)$$

H, the horizontal subgroup, consisting of those permutations which take any number to a number in the same row of τ:

$$H = \Sigma(2,3,6,7) \times \Sigma(1,4,5,8).$$

<u>Lemma</u>. Unless $d \geq \delta$, we can find for every $g \in G \equiv \Sigma_n$ transpositions $v \in V$, $h \in H$ s.t. $vg = gh$.

In the limiting case $d = \delta$

192

So if we can solve the eqn $vg = gh$ the transpositions $v \in V$, $h \in H$ we get
$-L(g) = L(vg) = L(gh) = L(g)$
and $L(g) = 0$.

① Unless $d \geq \delta$, this applies to all $g \in \Sigma_n$ and we have $L = 0$.
In the limiting case $d = \delta$ it applies to all g except the g in one double coset VgH; of course all these g give the same outcome up to sign, and it is non-zero because the elements of the double coset are all distinct, and they each come either with a + sign or a − sign.

<u>Corollary</u>. Unless $d \geq \delta$ we have

$(\mathrm{ind}_V^G \varepsilon_V, \mathrm{ind}_H^G 1_H) = 0$,

in the limiting case $d = \delta$ it is 1.

This repeats something we already proved.

<u>Proof</u> (from this point of view). For any complex representation U of $G = \Sigma_n$ the representation gives a map of algebras

$$\mathbb{C}G \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, U).$$

With respect to this we have

we can postmultiply q by $\varepsilon(v)$ $v \in V$ until q maps row 1 to row 1, row 2 to row 2 etc. Then we can premultiply by some $h \in H$ to make sure that q gains each number in τ to the number in the corresponding place of t; i.e. we have shown that the bad q's lie in only one double coset.

As further value of the double coset, we may as well reduce to the case in which t and τ are the same tableau (i.e. the same Young diagram with the numbers in the same places), and ask about $V \cap H$, i.e. the set of permutations which keep every number in the same row and in the same column. Clearly such a permutation leaves each number fixed, so $V \cap H = 1$.

Now we consider the linear map

$$L : \mathbb{C}G \longrightarrow \mathbb{C}G \qquad (G = S_n)$$

given by

$$L(a) = \sum_{v \in V,\, h \in H} \frac{\varepsilon(v)}{|V||H|}\, v a h,$$

That is, we antisymmetrise the left action of V and symmetrise the right action of H.

Lemma. Unless $d \geq \delta$, we have $L = 0$; in the limiting case $d = \delta$, L has rank 1.

Proof. Observe that $L(va) = \varepsilon(v) L(a)$, $L(ah) = L(a)$.

194a

Thus we have $t = \tau$.

__Corollary__. The element $e = \frac{1}{\lambda} \sum_{v \in V, h \in H} \varepsilon(v) vh$ is an idempotent in $\mathbb{C}G$ provided we have

$$\lambda = \frac{n!}{\dim_{\mathbb{C}} W} = \prod_{ij} h_{ij}.$$

__Proof__. Set $E = \sum_{v \in V, h \in H} \varepsilon(v) vh$.

Then clearly $E^2 \in \operatorname{Im} l$, so by the lemma, $E^2 = \lambda E$. To compute λ

$$\mathbb{C}G \xrightarrow{\cong} \prod_U \mathrm{Hom}_{\mathbb{C}}(U, U),$$

where the product runs over one rep for each iso class of irred. reps. The fundamental theorem of rep. theory says that this map is iso. This isom. carries the left action of G on $\mathbb{C}G$ onto the action on Hom which comes from the action of G on the target; and it carries the right action of G on $\mathbb{C}G$ to the action on Hom which comes from the action of G on the source. Thus the summand

$m(l) = \tilde{V}\, \mathbb{C}(G)\, \bar{H}$ corresponds to

$$\prod_U \mathrm{Hom}_{\mathbb{C}}(\tilde{V} U, \bar{H} U).$$

If this has dimension 0 then there are ~~exactly~~ no U with $\tilde{V} U \neq 0$ and $\bar{H} U \neq 0$;

if this has dimension 1 then there is exactly one such rep, say W, and it has

$\dim_{\mathbb{C}} \tilde{V} W = 1$, $\dim_{\mathbb{C}} \bar{H} W = 1$.

But of course

~~$\dim \bar{H} W = (1_H, \mathrm{res}^G_H W) = (\mathrm{ind}$~~

$\dim \tilde{V} W = (\varepsilon_V, \mathrm{res}^G_V W) = (\mathrm{ind}^G_V \varepsilon_V, W)$

$\dim \bar{H} W = (1_H, \mathrm{res}^G_H W) = (\mathrm{ind}^G_H 1_H, W)$.

Secondly we use the representation

$$C(G) \xrightarrow{\cong} \prod_W \mathrm{Hom}_{\mathbb{C}}(W, W)$$

For other representations W' either $\bar{V}W' = 0$ or $\bar{H}W' = 0$, and E acts as 0 on W'.

On W, E acts as a rank-1 idempotent up to the scalar factor λ, so its trace is λ.

On $\mathrm{Hom}_{\mathbb{C}}(W, W)$, with E acting on the left, the trace is $(\dim_{\mathbb{C}} W)\lambda$.

Thus $(\dim_{\mathbb{C}} W)\lambda = n!$,

and $\lambda = \dfrac{n!}{\dim_{\mathbb{C}} W}$.

194 b

$\prod_W \mathrm{Hom}_C(\tilde{\nabla} W, \overline{H} W)$. If this has dimension 1 then there is exactly one W such that $\tilde{\nabla} W$ and $\overline{H} W$ are both non-zero, and then they both have dim 1. But of course

(1) $\dim \overline{H} W = (1_H, \mathrm{res}^G_H W) = (\mathrm{ind}^G_H 1_H, W)$

& (2) $\dim \tilde{\nabla} W = (\varepsilon_V, \mathrm{res}^G_V W) = (\mathrm{ind}^G_V \varepsilon_V, W)$.

Corollary. The element $e = \frac{1}{d} \sum_{v \in V, h \in H} \varepsilon(v) vh$ is an idempotent in CG provided that

$d = \frac{n!}{\dim_C W}$.

Proof. Set $E = \sum_{v \in V, h \in H} \varepsilon(v) vh$

Then clearly $E^2 \in \mathrm{Im}\, L$, so by Lemma 2, $E^2 = \lambda E$. To compute λ the easiest way is to compute the trace of

$$CG \xrightarrow{a \mapsto Ea} CG$$

in two different ways.

First we use the basis of elts of G. Since E contains 1 with coeff 1, we see

$$\mathrm{Tr}(a \mapsto Ea) = n!$$

to act the factors arranged in the order down the columns. Then our element becomes

$$(b_1 \otimes b_2 \otimes \cdots \otimes b_{c_1}) \otimes (b_1 \otimes b_2 \otimes \cdots \otimes b_{c_2}) \otimes \cdots$$

Now the operator $\sum_{v \in V} \varepsilon(v) v$ acts as

$$a_{c_1} \otimes a_{c_2} \otimes \cdots$$

where a_c is the obvious non-zero alternating tensor in $b_1, b_2, \ldots, b_c$.

Now we suppose that $\lambda = \prod_{i,j} h_{ij}$ is prime to p, so in this case the Young idempotent

$$e = \frac{1}{\lambda} \sum_{v \in V, h \in H} \varepsilon(v) v h$$

acts on $V^{\otimes n}$ when V is a vspace over $\mathbb{F}_p$.

Prop. It is still true that e acts on $V^{\otimes n}$ as $\begin{cases} \text{zero if } \dim V < c_1 \\ \text{nonzero if } \dim V \geq c_1. \end{cases}$

Proof. Take a free module M over $\mathbb{Z}_{(p)}$ with rank equal to $\dim V$. Then e acts on $M^{\otimes n}$, and it is an idempotent, so $M^{\otimes n} = I \oplus K$ where I, K are free $\mathbb{Z}_{(p)}$-modules on which e acts as $1, 0$. We can determine the ranks of I, K by applying $\otimes \mathbb{Q}$, which reduces us to the previous result; so $I = 0$ if $\operatorname{rank} M < c_1$, We now consider the action of Σ_n on $V^{\otimes n}$ by permuting the factors. For example, when Σ_2 acts on $V^{\otimes 2}$ the +1 eigenspace is $\sigma^2(V)$, the −1 eigenspace is $\lambda^2(V)$. I assume V is a vector space over $\mathbb{Q}$.

<u>Remark</u>. The Young idempotent

$$e = \frac{1}{\lambda} \sum_{v \in V,\, h \in H} \varepsilon(v)\, v\, h$$

acts on $V^{\otimes n}$ as $\begin{cases} \text{zero if } \dim V < c_1 \\ \text{non-zero if } \dim V \geq c_1 \end{cases}$

<u>Proof</u>. i) If $\dim V < c_1$ then the operator $\sum_{v' \in \Sigma_{c_1}} \varepsilon(v')\, v'$ annihilates $V^{\otimes c_1}$; a fortiori, $\sum_{v' \in \Sigma_{c_1},\, v'' \in \Sigma_{c_2} \times \cdots \times} \varepsilon(v')\varepsilon(v'')\, v' \otimes v''$ annihilates $V^{\otimes c_1} \otimes V^{\otimes c_2} \cdots$.

ii) Suppose V has a base $b_1, b_2, \ldots b_N$, $N \geq c_1$; and suppose our Young tableau is

1	2	3 …	r_1
r_1+1			

Consider the vector $\underbrace{b_1 \otimes \cdots \otimes b_1}_{r_1} \otimes \underbrace{b_2 \otimes \cdots \otimes b_2}_{r_2} \cdots \otimes b_{c_1}$

This vector is fixed under H, so the averaging operator $\sum_{h \in H} h$ multiplies it by $|H|$.

Now let us permute the factors of $V^{\otimes n}$ so as

§9. Exploitation of Idempotents

First I'll give a general explanation of how you exploit idempotents.

Let X be any CW-complex and $e: X_{p,\infty} \longrightarrow X_{p,\infty}$ a map which is idempotent up to htpy, $e^2 \simeq e$.

<u>Lemma</u>. Then we can factorise e

as $X \xrightarrow{j} Y \xrightarrow{i} X$, $X \xrightarrow{e} X$

where ~~(?)~~ $ji \simeq 1$ so that Y is in effect a summand of ~~X~~.

Notice that if $ji \simeq 1$ then certainly ~~(?)~~ $(ij)(ij) \simeq ij$.

Proof. Construct

$Y = \mathrm{Tel}\,(X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \cdots$

Then certainly $\pi_*(Y) \cong e\,\pi_*(X)$

The diagram $Y \xrightarrow{e} Y \xrightarrow{e} X$

$e \searrow \quad \swarrow e$

X

gives a map $Y \xrightarrow{i} X$ which induces iso

$\pi_*(Y) \xrightarrow{\cong} e\,\pi_*(X)$.

rank $I > 0$ if rank $M \geq c_1$.

Now M/pM is a v'space over F_p of the required dim, and

$$(M/pM)^{\otimes n} = M^{\otimes n} / pM^{\otimes n}.$$

Evidently e acts as 1 on I/pI, 0 on K/pK, where

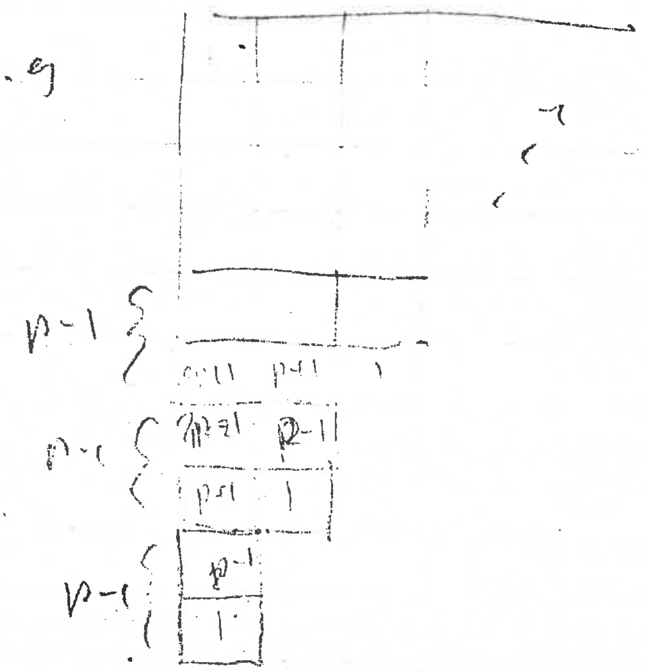
I/pI is 0 if $\dim V < c_1$

& non-zero if $\dim V \geq c_1$.

Of course, one might worry whether there is a sufficient supply of idempotents with $\lambda = \prod_{ij} h_{ij}$ prime to p.

Prop For any assigned p and c_1 there is an n and a Young idempotent with the assigned c_1 and $\lambda = \prod_{ij} h_{ij}$ prime to p.

Proof. e.g.



Then we form $\bigwedge_1^m X$, and Σ_m acts on it by permuting coordinates. In order to make sense of linear combinations of group elements we pass to $S^2 \bigwedge_1^m X$; then we have $\mathbb{Z}[\Sigma_m]$ acting. Now if our space is not p-local already then we localise it and get $\mathbb{Z}_{(p)}[\Sigma_m]$ acting on $(S^2 \bigwedge_1^m X)_{(p)}$. In our application

$X = \mathbb{R}P^{2^{n+1}} / \mathbb{R}P^1$ is not 2-local ($H_2 X = \mathbb{Z}$) so we localise at 2.
For any idempotent e in $\mathbb{Z}_{(p)}(\Sigma_m)$ we can apply the process above, and we get a p-local CW-complex Y which is a summand of $(S^2 \wedge \bigwedge_1^m X)_{(p)}$.

In particular, in our case, we have

$$\dim_{F_2}(\tilde{H}_*(X; F_2)) = N = 2^{n+1} - 2.$$

Using the last prop. of the last section, we find a group Σ_m and a Young idempotent ~~[illegible]~~ with $c_1 = N$ and $\lambda = \prod_{i,j} h_{ij}$ prime to 2. It comes out with $m = \frac{1}{2} N(N+1)$, but who cares; we get a Young idempotent $e \in \mathbb{Z}_{(2)}(\Sigma_m)$ and perform the construction above to get Y. Clearly Y is 2-local, and

Conversely the injection

$$\begin{array}{c} X \\ \downarrow \\ X \xrightarrow{e} X \xrightarrow{e} X \to \cdots \end{array}$$

gives a map $X \xrightarrow{j} Y$ which induces

$$\pi_*(X) \xrightarrow{e} e\pi_*(X) = \pi_*(Y)$$

Clearly we have $ij \simeq e$.

Consider the composite

$$\begin{array}{ccc} Y = \mathrm{Tel}(X \xrightarrow{e} X \cdots) & & \\ {\scriptstyle ij}\downarrow & & X \\ X & & \\ {\scriptstyle j}\downarrow & & \\ Y = \mathrm{Tel}(X \xrightarrow{e} X \cdots) & & \end{array}$$

This Tel covers the identity map of $\pi_*(Y) = e\,\pi_*(X)$, so by the Theorem of Whitehead it is an equivalence. Since $ij \simeq e$, we get

$$\text{ei jj } ijij \simeq e^2 \simeq e \simeq ij$$

whence

$$jijiji \simeq ji1ji.$$

Since ji is an equivalence, we deduce $ji \simeq 1$.

For any generalized homol. or cohomol. theory, say E_*, we have $E_*Y = e\,E_*X$.

In practice we begin with some space X. For example, in Jeff Smith's construction, with $p = 2$, we choose in [illegible] Fake

$$X = \mathbb{R}P^{2^{n+1}} / \mathbb{R}P^1.$$

So these SS collapse and $\widehat{H}(i)_*(X)$, $\hat{h}(i)_*(X)$ are free over $F_2[t,t^{-1}]$, $F_2[t]$ on the same no. of generators as $\widehat{H}_*(X)$, viz N.

For $i<n$ the first bdry in the AHSS is given by an operation $Sq^{0\ldots01}$ of degree $\le 2^n-1$ which is not identically zero in X. Thus $\widehat{H}(i)_*(X)$ is of dim $< N$; we don't discuss what happens to $\hat{h}(i)_*(X)$, because it develops torsion.

We can now apply the Künneth formula to find

$$\widehat{H}(i)_*(S^2 \wedge \bigwedge_1^n X)_{(2)} \cong F_2[t,t^{-1}] \otimes \bigotimes_1^n V$$

$$\hat{h}(i)_*(S^2 \wedge \bigwedge_1^n X)_{(2)} \cong F_2[t] \otimes \bigotimes_1^n V \quad (i \ge n)$$

where $\dim_{F_2} V = N$ or $< N$ acc. as $i \ge n$ or $i < n$.

So $\widehat{H}(i)_*(Y) \cong F_2[t,t^{-1}] \otimes e \bigotimes_1^m V$

$\hat{h}(i)_*(Y) \cong F_2[t] \otimes e \bigotimes_1^m V \quad (i \ge n)$.

This shows that $\widehat{H}(i)_*(Y) = 0$ for $i < n$, while for $i \ge n$ $\widehat{H}(i)_*(Y)$ and $\hat{h}(i)_*(Y)$ are free over $F_2[t,t^{-1}]$ and $F_2[t]$ of rank equal to the dim of $\widehat{H}_*(Y)$.

For $i \ge n$, ~~May~~ the universal coeff theorem now says that it is finite in the sense appropriate to 2-local complexes. We will now study its properties.

<u>Theorem 2.</u> (i) $\bar{K}(i)_*(Y)$ is zero torsion and non-zero for $i \geq n$. (ii) Indeed, for $i \geq n$ $\bar{K}(i)_*(Y \wedge DY)$ is non-zero and is free over $F_2[t]$ of the same rank that $H_*(Y \wedge DY; F_2)$ is over F_2. (iii) For suitable 2^e there is an element f of Adams filtration 2^e in $\pi_{2^e(2^n-1)2}(Y \wedge DY)$ whose image in $\bar{K}(i)_*(Y \wedge DY)$ is

$\begin{cases} \text{(if } i = n) & t^{2^e} \cdot 1 + \nu \text{ with } \nu \text{ nilpotent} \\ \text{(if } i > n) & \text{nilpotent.} \end{cases}$

The construction of f is going to be by the classical Adams spectral sequence. This will take us some way in proving results about Ext_A, and the whole point of Jeff Smith's construction is to obtain a space Y so that the structure of $H^*(Y; F_2)$ as an A-module is so favourable that you can get through. For the moment we prove the easy parts of the theorem and make a preliminary reduction of the problem.

<u>Proof</u> We may compute $\bar{K}(i)_*(X)$, $\bar{K}(i)_*(X)$ by the AHSS. Forgetting the differentials these E_2 are zero for dimensional reasons; they lower degree by at least $2^{n+1}-1$.

P^0_{i+1} is a derivation –

$$P^0_{i+1}(xy) = (P^0_{i+1}x)y + x(P^0_{i+1}y)$$

and we carefully arranged $P^0_{i+1} = 0$ in $\tilde{H}^*(X)$ for $i \geq n$. It follows that it is 0 in $\tilde{H}^*(S^2 \wedge \bigwedge^n_1 X)_{(2)}$, in Y, ~~so~~ in DY and in $Y \wedge DY$. In any case, we have

$$\mathrm{Ext}^{**}_{A}(H^*(h(i)), Y \wedge DY); F_2)$$

$$\cong \mathrm{Ext}^{**}_{A(P^0_{i+1})}(\tilde{H}^*(Y \wedge DY); F_2)$$

where P^0_{i+1} acts as zero on $H^*(Y \wedge DY)$. But now the E_2-term has only just the correct rank over $F_2[t]$; any non-zero differential would lower the rank, so the differentials must be zero. ⇐.

__Lemma 4__: To prove Theorem 2 (iii), it is sufficient to construct an element f of Adams filtration 2^e in $\pi_{2^e(2^n-1)2}(Y \wedge DY)$ whose image in $\mathrm{Ext}^{**}_{A(P^0_{i+1})}(\tilde{H}^*(Y \wedge DY); F_2)$ is the class of $t^{2^e}\eta$.

__Proof__. By the proof of Corollary 3,

$k(i)^*(Y)$ is free over $F_2[t]$ of rank equal to the dim of $\bar{H}^*(Y)$. This is the same as saying that $h(i)_*(DY)$ is free over $F_2[t]$ of rank equal to the dim of $\bar{H}_*(DY)$. Now the Künneth theorem again says that $h(i)_*(Y \wedge DY)$ is free over $F_2[t]$ ~~and~~ of rank equal to the dim of $\bar{H}_*(Y \wedge DY)$. This proves (i), (ii).

Corollary 3. For $i \geq n$, the Adams spectral sequence

$$\mathrm{Ext}_A^{**}(H^*(h(i) \wedge Y \wedge DY); F_2) \Rightarrow h(i)_*(Y \wedge DY)$$

collapses; all its differentials are 0.

Proof. Sooner or later we are going to need notation for Steenrod operations. The ~~Milnor base~~ Milnor elements $Sq^{h_1 h_2 \cdots h_r}$ are the dual base to the base of monomials $\xi_1^{h_1} \xi_2^{h_2} \cdots \xi_r^{h_r}$ in A_*. In particular, P^s_t is the Milnor element $Sq^{0 \cdots 0\, 2^s\, 0 \cdots 0}$ (with 2^s in place t) dual to the monomial $\xi_t^{2^s}$. Well,

$$H^*(h(i)) \cong A \otimes_{\Lambda(P^0_{i+1})} F_2.$$

Well, now we have to get to grips with

$$\mathrm{Ext}_A^{s,t}(\bar H^*(X \wedge DX); F_2).$$

Our agenda is as follows:

(i) Prove that it has a suitable vanishing line, i.e. it is zero for

$$t < -c + 1(2^{n+1}-1)s$$

(ii) Prove that in a suitable strip

$$-c + (2^{n+1} - ~~8~~)s \leq t \leq (2^{n+1}-1)s + c'$$

the map

$$\mathrm{Ext}_A^{s,t}(\bar H^*(X \wedge DX); F_2) \to \mathrm{Ext}_{A_m}^{s,t}(\bar H^*(X \wedge DX); F_2)$$

is iso provided we take m large enough. Here I recall that A_m is the finite subalgebra of the mod 2 Steenrod algebra generated by $Sq^1, Sq^2, \ldots, Sq^{2^m}$.

(iii) ~~Use the fact that~~ [illegible] Work out then to construct an element x in

$$\mathrm{Ext}_A^{s,t}(\bar H^*(X \wedge DX); F_2)$$

whose image in $E_1^{s,t}{}_{\Lambda(P^0_{n+1})}(\bar H^*(\ldots)\ldots)$ [illegible]

is the class of $t^{2^?}\eta$.

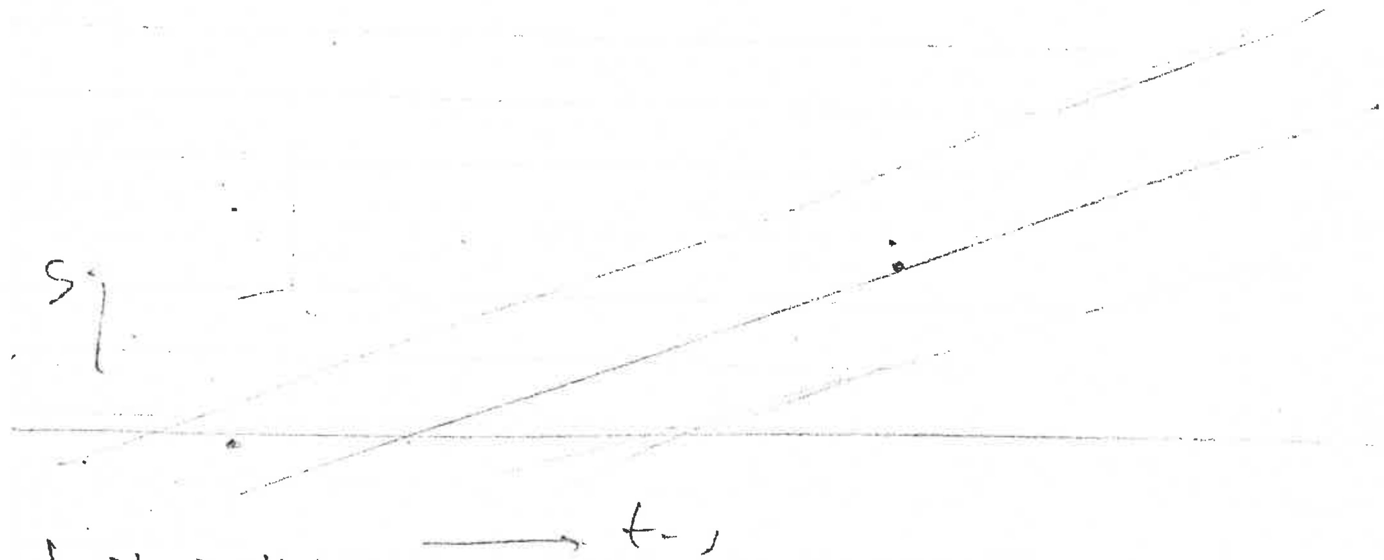
(iv) Prove that some power $x^{2^?}$ is a permanent cycle. This programme will the Adams spectral sequence

$$\mathrm{Ext}_A^{*,*}(H^*(h(i)_n Y \wedge DY); F_2) \Rightarrow h(i)_* (Y \wedge DY)$$

is zero outside a strip of the form

$$-c + \lambda s \leq t \leq c + \lambda s,$$

where $\lambda = 2^{i+1} - 1$ and $c = \dim Y$.



With $i = n$, suppose $[f] = t^{2^e} \eta + \nu$ with ν of one higher filtration; then ν is nilpotent.

Similarly if $i > n$.

Lemma 5. P^s_t is scalar on $\overline{H}^*(X;\mathbb{F}_2)$

if $\begin{cases} s < t & (\text{so that } (P^s_t)^2 = 0) \\ s+t \le n+1 & (\text{so that } P^s_t \in A_n) \\ t \le n & (\text{to rule out } P^0_{n+1} \text{ which acts as zero}) \end{cases}$.

P^s_t is also scalar for $s = t$, $2t = n+1$.

The point is that $(P^t_t)^2 \neq 0$ in A_n, but it is 0 in $A_n // \Lambda(P^0_{n+1})$ if $2t = n+1$;

each $t \le 1, n \le 1$,

eg $Sq^2 Sq^2 = Sq^1 Sq^{01}$.

complete the proof of Theorem 2.

So now we must begin by discussing the technology for proving vanishing lines. Basically this was built by Adams & Margolis, "Modules over the Steenrod Algebra", Topology 10 (1971) 271–282, with improvements due to later authors, especially Anderson & Davis, "A vanishing theorem in homological algebra", Comm. Math. Helv 48 (1973) 318–327 and Miller & Wilkerson, "Vanishing lines for modules over the Steenrod algebra", JPAA 22 (1981) 297–307.

The fundamental invariant is as follows. Let B be an algebra, which is free as a left module over an exterior subalgebra $\Lambda(x)$. Then any free B-module M will also be free over $\Lambda(x)$; in particular, $M \xrightarrow{x} M \xrightarrow{x}$ will be exact, alternatively, when we form $H(M;x) = \frac{\operatorname{Ker} x}{\operatorname{Im} x}$, we shall find $H(M;x) = 0$. Conversely, if x is exact on M, otherwise said if $H(M;x) = 0$, it gives an important restriction on the behaviour of x.

The P^s_t which do satisfy (…) ~~and so on again~~ are those with $s<t$. They lie in A_m if $s+t \leq m+1$.

preserves both filtrations. This is automatic because b commutes with C.

Secondly, I claim that in the associated graded W of $\bigotimes_i V$, the operation of P^s_t is given by

$$P^s_t(v_1 \otimes v_2 \cdots \otimes v_n)$$

$$= \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes v_{i-1} \otimes P^s_t v_i \otimes v_{i+1} \otimes \cdots \otimes v_n).$$

This is clear, because every other term in the Cartan expansion of P^s_t has the form

$$c_1 \otimes c_2 \cdots \otimes c_n$$

with $c_i \in C$ and $\sum \deg c_i = 2^s(2^t - 1)$, so it increases filtration by 1.

Now Σ_n acts on $\bigotimes^n V$ so as to preserve the filtration. In particular e and $(1-e)$ preserve it, ~~[illegible]~~ and

$e \bigotimes_i^n V$ is filtered by submodules over $A(P^s_t)$

so that the subquotients make up eW.

Now take the action of $A(P^s_t)$ on W, we can argue that

$$W = eW \oplus (1-e)W$$

$$H(W; P^s_t) = H(eW; P^s_t) \oplus H((1-e)W; P^s_t)$$

& each [illegible] a ( ) 0

This lemma follows take a bit of care, because the operation P^s_t aren't derivations for $s > 0$. We have to construct a quotient on which P^s_t acts as if it were primitive.

First on an operation P^s_t with $s < t$, $s + t \leq n+1$, $t \leq n$ are closed $s = t$, $2t = n+1$.

Let B be the subspace of A spanned by the $Sq^{0 \ldots 0 h 0 \ldots 0}$ (with h in position t) with $0 \leq h < 2^{s+1}$;

similarly for C, except we write $0 \leq h < 2^s$. Then B is a Hopf subalgebra of A; as an algebra it is an exterior algebra $\Lambda(P^0_t, P^1_t, \ldots P^s_t)$. Similarly for C, except we omit P^s_t.

Set $V = \tilde{H}^*(X; F_2) = \tilde{H}^*(RP^{\infty}/RP^{1}; F_2)$

Put on V a decreasing filtration $V(i)$ defined by

$$V(i)^d := \sum_{j \geq i} C^j V^{d-j}$$

Now filter $\bigotimes^m V$ with a decreasing filtration take

$$\left(\bigotimes^n V\right)(k) = \sum_{i_1 + i_2 + \cdots + i_n = 2^s(2^t - 1)k} V(i_1) \otimes V(i_2) \otimes \cdots \otimes V(i_n)$$

First, I claim that any operation be B

So now I must explain how you prove vanishing theorems. Let B be a subalgebra of A. The statement I want to prove is:

(V): there is a constant $c = c(B, \lambda)$ such that if M is a B-module, $M_t = 0$ for $t < m_0$ and P^s_t is exact on M for each P^s_t with $|P^s_t| < \lambda$, then

$$\mathrm{Ext}_B^{s,t}(M; F_2) = 0 \text{ for } t < m_0 + \lambda s + c.$$

One proves (V) for larger and larger algebras B by throwing in the generators one at a time. We must explain how to throw in one more generator.

Suppose then that B contains a subalgebra C, and we already know that C satisfies (V), and B is free as a left module over C on two generators $1, x$. Then we get an exact sequence of B-modules

$$0 \to F_2 \to \mathrm{Hom}_C(B, F_2) \to F_2 \to 0$$

(suitably graded). This exact sequence defines an element

$$\xi \in \mathrm{Ext}_B^{1,|x|}(F_2, F_2)$$

and also a long exact sequence

ie $H(eW; P^s_t) = e\, H(W; P^s_t)$

$= e \bigotimes^m H(V; P^s_t)$

$= 0$, since

$\dim H(V; P^s_t) < N$ (P^s_t acts non-trivially on $V = H^*(RP^{2^{n+1}}/RP^1; F_2)$).

Thus $e \bigotimes V$ is filtered by submodules over $A(P^s_t)$ so that the subquotients have $H(\text{---}, P^s_t) = 0$. Thus

$H(H^*(Y : F_2) ; P^s_t) = 0$. This proves the lemma 5.

Corollary 6. The spec operators P^s_t are exact in $H^*(Y, DY)$.

Proof. Filter according to degree in $H^*(DY)$.

Remark. In part (ii) you definitely must be prepared to ~~give up~~ lose something on c. For example, outside

A/ASq^1. This has $m_0 = 2$, & the action of Sq^1 on the cell is trivial, so the first non-trivial P^s_t is $Sq^{0,1}$, i.e. $\lambda = 3$; if Lemma 7 (ii) were true for $t < \lambda s + c$ you would infer

$$Ext_A^{s,t}(A/ASq^1, \mathbb{Z}_2) = 0 \text{ for } t < 2 + 3s;$$

in fact it is non-zero for $s = 2$, $t = 6$ so the best possible result is that it is 0 for $t < 3s$, so you have to be prepared to lose 2 units, and they get lost precisely by one application of 7 (ii) with $x = Sq^2$, $|x| = 2$.

$\to \mathrm{Ext}^{s+1,t}_B(M, F_2)$

$$\xi$$

$$\mathrm{Ext}^{s,t}_B(M, F_2) \to \mathrm{Ext}^s_B(M, \mathrm{Hom}_C(B, F_2)) \to \mathrm{Ext}^{s,t-|x|}_B(M, F_2)$$

$$\mathrm{Ext}^s_B(M, \mathrm{Hom}_C(B, F_2)) = \mathrm{Ext}^{s,t}_C(M, F_2)$$

Lemma 7. Assume $\mathrm{Ext}^{s,t}_C(M, F_2) = 0$ for $t < m_0 + \lambda s + c$, and

(i) If $|x| \geq \lambda$ then $\mathrm{Ext}^{s,t}_B(M, F_2) = 0$ for $t < m_0 + \lambda s + c$.

(ii) If $|x| < \lambda$ and ξ acts nilpotently on $\mathrm{Ext}^{*,*}_B(M, F_2)$ then $\mathrm{Ext}^{s,t}_B(M, F_2) = 0$ for $t < m_0 + \lambda s + c - |x|$.

Proof (i) Suppose $|x| \geq \lambda$, $t < m_0 + \lambda s + c$. Then $\mathrm{Ext}^{s,t}_C(M, F_2) = 0$ by hypothesis. Therefore

$$\mathrm{Ext}^{s-1,t-|x|}_B(M, F_2) \xrightarrow{\xi} \mathrm{Ext}^{s,t}_B(M, F_2)$$

is epi. Since $|x| \geq \lambda$, we have

$t - |x| < m_0 + \lambda(s-1) + c$, and so

a descending (?) recursion. So

$$0 = \mathrm{Ext}^{-1,?}_B(M, F_2) \xrightarrow{\xi^{s+1}} \mathrm{Ext}^{s,t}_B(M, F_2)$$

is epi.

This is an example of the second case: ξ is Gelfand–not nilpotent.

Notice that a freeness result is a result of the form $Ext^{s,t}_B(\Gamma, F_2) = 0$ for $s \geq 1$, any t; i.e. it is essentially a vanishing line with $\lambda \Rightarrow \infty$, any c.

Proof. By induction on n. It's clear that for $n = 1$, so let x_1 be the generator of least degree, & assume the result true for $C = \Lambda(x_2 \ldots x_n)$.

If x_1 is central in Γ, then Γ is free over $\Lambda(x_1)$. Now B is free over $\Lambda(x_1)$ on 1 & basis elements of degree $\geq d > |x_1|$, so by a result we have already done

$Ext^{s,t}_B(\Gamma, F_2) = 0$

for $t < sd + m_0$.

(iii) Suppose $|x| < \lambda$. Then we can argue thus: if $t + |x| < \min \lambda s, c$ then $\mathrm{Ext}_B^{s,t}(M, F_2) \xrightarrow{\xi} \mathrm{Ext}_B^{s+1, t+|x|}(M; F_2)$ is iso, and we can repeat the process, so

$$\mathrm{Ext}_B^{s,t}(M, F_2) \xrightarrow{\xi^r} \mathrm{Ext}_B^{s+r, t+r|x|}(M, F_2)$$

is iso; if ξ acts nilpotently, we must have

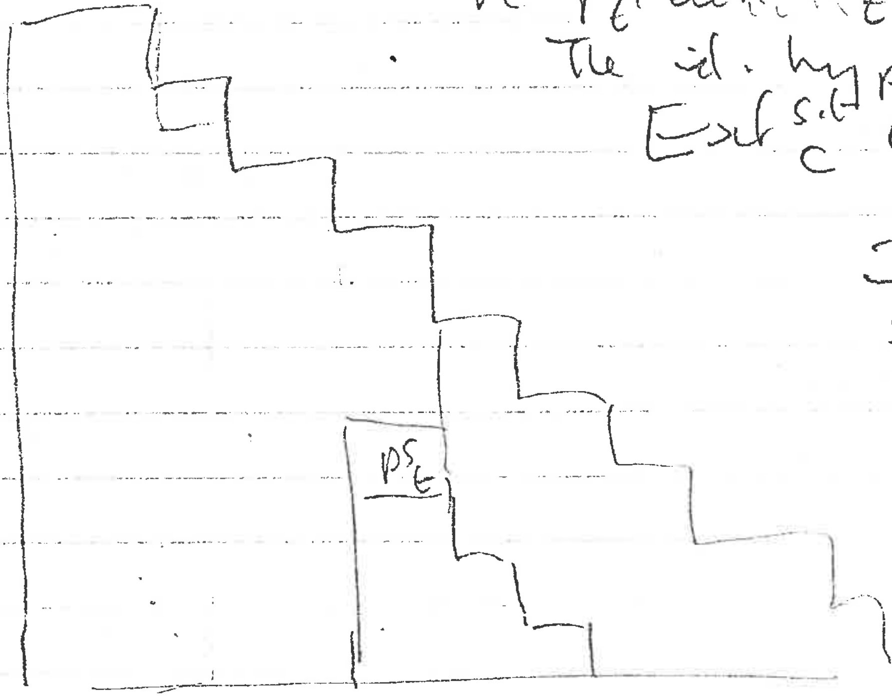
$$\mathrm{Ext}_B^{s,t}(M, F_2) = 0$$

Basically there are 2 cases in which we can use 7(ii). ~~Either~~ In one case we can prove that ξ is nilpotent in $\mathrm{Ext}_B^{**}(F_2; F_2)$, and then it has to act nilpotently. In the other we can't, so we have to know something about the way ξ acts on M.

<u>Corollary 8</u> (Adams - Margolis, 2.1)

Let B be an exterior algebra $\Lambda(x_1, x_2 \ldots x_n)$ on generators of distinct degrees. If each x_i is scalar $M\!\!/\!\!\!\!\!\!\;\;\!$ (hold below) then M is B-free.

in increasing order of s. So the intermediate steps [illegible] $\alpha \cup \lambda = \{p^s_t$ [illegible] with $s+t \le n+1$, and C_i, the subextraction generated by the $p^{s'}_t$ with $s' < s$, plus the $p^{s'}_{t'}$ with [illegible] $t < t'$, $s'+t' \le n+1$.



The ind. hyp. is that with M [illegible]

$\mathrm{Ext}^{s,t}_C(M, F_2) = 0$ for [illegible] all $t <$ [illegible]

If $|x| \geq \lambda$, we simply apply (ii), so we may assume $|x| < \lambda$. [illegible] After that there are 2 cases [illegible] [illegible], & the first is $s <$ [illegible]

~~If $|x| > \lambda$, we simply apply (ii), so we may assume $|x| \le \lambda$.~~ In this case we copy the argument of Coroll. 8.

Let D be the subextraction gen. by the p^s_t and the $p^{s'}_{t'}$ in C of less degree. Then D is exterior, on generators of distinct degree $\leq \lambda$. We are told to assume that each such generator is [illegible] in D, so by Coroll. 8, M is free over D. By the same result we concluded in proving Coroll. 8,

$\mathrm{Ext}^{s,t}_{C}(M, F_2) = 0$ [illegible]

where d is the least degree of a generator in B not in D. But here $d > |x|$, so ξ [illegible] nil [illegible] on $\mathrm{Ext}^{s,t}_B(M$ [illegible]

But $\xi_t \in Ext_B^{1,|x_1|}(F_2, F_2)$

with $|x_1| < d$,

---

so ξ_t certainly acts nil potently.

Then given $Ext_C^{s,t}(M, F_2) = 0$ for $s \geq 1$,

the analogue of Lemma 7 (ii) shows

$Ext_B^{s,t}(M, F_2) = 0$ for $s \geq 1$.

Thus M is B-free.

Proposition 8. The subalgebra A_n of A has property (V).

Proof. We know in the accordance P^s_t in lexicographic order; here we regard t as the most significant digit & then the ones with larger t first, then again the the rule the P^s_t in

Apply Sq^2, we get

$$h_s^{4t} h_{s-t+2} = 0$$

Continue, after t steps we get

$$h_s^{2t} h_s = 0.$$

So $g = h_s$ is nilpotent.

We can now apply 7(ii) again and get

$$Ext_B^{s,t}(M, F_2) = 0$$

for $t < m_0 + \lambda s + c'$.

This completes the induction, and proves Prop. 9.

Corollary 10. For each subalgebra A_m we have

$$Ext_{A_m}^{s,t}(H^*(Y_n D Y); F_2) = 0$$

for $t < s(2^{n+1} - 1) + c(m)$.

In fact we can get $c(m)$ to be minus the bottom dimension of $H^*(Y_n D Y)$, but in … cases.

But now Lemma 7.(ii) says that

$\mathrm{Ext}^{s,t}_B(\mathbb{Z}, F_2) = 0$ for $t < m_0 + \lambda s \epsilon'$.

The second case is the case $s \geq t$.

<u>Lemma 10</u>. In this case ξ is nilpotent in $\mathrm{Ext}^{**}_B(F_2, F_2)$.

<u>Proof</u>. Set $h_r = [\zeta_t^{2^r}]$ so that we have $h_0, h_1, \ldots h_s = \xi$, $h_{s+1} = 0$ in $\mathrm{Ext}^{**}_B(F_2, F_2)$. In the standard complex we have

$$\delta[\zeta_{2t}^{2^{s-t}}] = [\zeta_t^{2^s} \mid \zeta_t^{2^{s-t}}]$$

ie $h_s . h_{s-t} = 0$.

Now, the cohomology of a Hopf algebra like B has Steenrod squares acting on it. The only property, derivation from standard, is that we don't have $Sq^0 = $ identity, in fact we have

$$Sq^0 h_i = h_{i+1}$$

including

$$Sq^0 h_s = h_{s+1} = 0$$

Take our rel. & apply Sq^1, we get

$$h_s^2 h_{s-t+1} = 0.$$

So the inductive hypothesis applies to L with ~~m_0~~ $+c$ replaced by ~~m_0~~ $+ \lambda + c$. We have

$$\mathrm{Ext}^{s,t}_A (L, F_2) \xrightarrow{\text{epi}} \mathrm{Ext}^{s+1,t}_A (M, F_2),$$

and the inductive hypothesis says

$$\mathrm{Ext}^{s,t}_A (L, F_2) = 0 \text{ for } t < \lambda s + \lambda + c,$$

so we get

$$\mathrm{Ext}^{s+1,t}_A (M, F_2) = 0 \text{ for } t < \lambda(s+1) + c.$$

This completes the proof.

<u>Corollary 13</u> So $\mathrm{Ext}^{s,t}_A (H^*(Y \wedge DY), F_2) = 0$ for $t < s(2^{n+1} - 1) + c$.

<u>Proof.</u> As an A_{n}-module we have

$$M \cong A_n \otimes_E V$$

where V is an E-module with P^0_{n+1} acting ...

Then

$$\mathrm{Ext}^{s,t}_{A_n} (M, F_2) = \mathrm{Ext}^{s,t}_E (V, F_2)$$

$$= 0 \text{ for } t < m_0 + s(2^{n+1} - 1)$$

Now we use the lemma.

<u>Lemma 14</u> Suppose M is an A-module bdd below, and $\mathrm{Ext}^{s,t}_A (M, F_2) = 0$ for $t <$ ~~m_0~~ $+ \lambda s + c$ ~~Suppose let N ... A ... bdd above~~ and k be a given constant. Then there exists m_0 s.t. the map

$$\mathrm{Ext}^{s,t}_A (M, F_2) \longrightarrow \mathrm{Ext}^{s,t}_{A_m} (M, F_2)$$

is iso for $m \geq m_0$, $t < \ldots + \lambda s + k$.

Now we want to get theorems about the behaviour of $\tilde{H}^*(Y; F_2)$ as a module over A rather than A_n; but we can begin with some crude results.

<u>Lemma 12</u> Suppose M is an A-module bdd below and $\mathrm{Ext}^{s,t}_{A_n}(M, F_2) = 0$ for $t <$ ~~m_0~~ $+ \lambda s + c$ where $\lambda \leq 2^{n+1}$. Then $\mathrm{Ext}^{s,t}_{A}(M, F_2) = 0$ for $t <$ ~~m_0~~ $\lambda s + c$

<u>Proof</u>. It is true for $s = 0$; the statement about $\mathrm{Ext}^0_{A_n}$ implies that $M_t = 0$ for $t <$ ~~m_0~~ $+ c$, which implies the statement about Ext^0_A. Proceed by induction over s and assume the result true for s, $s \geq 0$.

We have

$\mathrm{Ext}^{0t}_{A_n}(M, F_2) = 0$ for $t <$ ~~m_0~~ $+ c$

$\mathrm{Ext}^{1t}_{A_n}(M, F_2) = 0$ for $t <$ ~~m_0~~ $+ \lambda + c$

This shows that we can construct a free resolution of M over A_n starting

$$0 \to K \to C_0 \to M \to 0$$

where C_0 is zero in dims $<$ ~~m_0~~ $+ c$

K is zero in dims $<$ ~~m_0~~ $+ \lambda + c$.

Now form

$$0 \to L \to A \otimes_{A_n} C_0 \to M \to 0.$$

(the middle term is labelled F_{11})

Then $A \otimes_{A_n} C_0$ coincides with C_0 in dims $<$ ~~m_0~~ 2^{n+1} ... so theream kernel L coincides with K in those dimensions and is 0 in dims $<$ ~~m_0~~ $+ \lambda + c$.

For $s \geq 1$, we have

$$\mathrm{Ext}^s_{A_n}(L, F_2) \cong \mathrm{Ext}^{s+1}_{A_n}(M, F_2)$$

(F is A_n-free) so

$$\mathrm{Ext}^s_{A_n}(L, F_2) = 0 \text{ for } t < \text{~~}m_0\text{~~} + \lambda(s+1) + c$$

We have to prove everything else A(?) ... & it will be done by induction, using some quotient algebras of A_m.

~~Comment~~ $F_2[\xi_1, \xi_2 \dots \xi_n]$ is a sub-Hopf-algebra of $F_2[\xi_1, \xi_2 \dots \xi_n, \xi_{n+1} \dots]$, the dual of A. Therefore $F_2[\xi_1 \dots \xi_n]^*$ is a quotient Hopf algebra of A. Similarly, we call a quotient algebra $A_m(n)$ of A_m; the elements $Sq^{h_1 \dots h_s \dots}$ with $h_s > 0$ for some $s > n$ map to zero in it, and the ones with $h_s = 0$ for all $s > n$, $h_s < 2^{m+2-s}$ form a base.

215

Prop 17 (after Wilkerson) $Ext^{**}_{A_m(n)}(F_2, F_2)$ is a finitely-generated algebra.

Proof Let us write $H^*(B) = Ext^*_B(F_2, F_2)$ suppressing the notation for the second grading t.

We prove the result by induction over n. For n = 1 we have

$$A_m(1) = \Lambda(Sq^1, Sq^2, \dots Sq^{2^m});$$

so $H^*(A_m(1)) = F_2[h_0, h_1, \dots h_m]$ & the result is true.

Assume the result for n. Then we have an extension of Hopf algebras

$$E = \Lambda(P^0_{n+1}, P^1_{n+1} \dots P^{m-n}_{n+1}) \to A_m(n+1) \to A_m(n).$$

This gives a spectral sequence

$$H^p(A_m(n)) \otimes H^q(E) \Rightarrow H^{p+q}(A_m(n+1))$$

By induction hypothesis, $E_2^{*0} = H^*(A_m(n))$ is a fin. gen. algebra, and hence Noetherian. It contains an invariant sequence

~~Suppose $N_t = 0$ for $t > n_0$.~~

Proof. Under the hypotheses on M we can construct a free resolution

$$\cdots \to C_s \to C_{s-1} \longrightarrow \cdots \to C_0 \to M \to 0$$

with $C_s = 0$ in dim $< \lambda s + c$. ~~Suppose $N_t = 0$~~ Then

$$Hom^t_A(C_s, F_2) \longrightarrow Hom^t_{A_m}(C_s, F_2)$$

is iso if we arrange

$$\lambda s + c + 2^{m+1} > t + n_0,$$

so we just want to have $2^{m+1} > k - n_0^c$

Proposition 16. For any m the image of the restriction

$$Ext^{s,t}_{A_m}(F_2, F_2) \longrightarrow Ext^{s,t}_{A(P^0_{n+1})}(F_2, F_2)$$

contains some power $[\xi_{n+1}]^{2^e}$ of the generator.

Let's ... imp. Pb

~~Note~~ Corollary 15. $\exists$ m s.t. the map

$$Ext^{s,t}_A(H^*(Y \wedge DY), F_2)) \to Ext^{s,t}_{A_m}(H^*(Y \wedge DY), F_2))$$

is iso for $t \leq s(2^{n+1} - 1)$.

cycles. It follows that the submodule of cycles Z_∞ is a fin. gen. module over this ring; hence $E_\infty = Z_\infty / B_\infty$ is a fin. gen. module over this ring. From this it follows that ~~[illegible]~~ $H^*(A_m(n+1))$ is fin. gen. as an algebra. This completes the induction.

Corollary 18 (of the proof). The image of the restriction

$$H^*(A_m(n+1)) \longrightarrow H^*(\Lambda(P^0_{n+1}))$$

contains some power $h_0^{2^e}$ of the [illegible]

Proof of ~~main proposition~~ (16).

$$\begin{array}{ccc} & A_m & \\ \nearrow & & \searrow \\ \Lambda(P^0_{n+1}) & \longrightarrow & A_m(n+1) \end{array}$$

induces

$$\begin{array}{ccc} & H^*(A_m) & \\ \swarrow & & \nwarrow \\ H^*(\Lambda P^0_{n+1}) & \longleftarrow & H^*(A_m(n+1)) \end{array}$$

of ideals

$$B_2 \subset B_3 \subset B_4 \quad \cdots \quad \subset B_\infty = \bigcup_r B_r .$$

$B_2 = \mathrm{Ind}_2$

By the Noetherian property B_∞ must be a fin. gen. ideal, and the sequence converges after a finite no. of terms, say $B_r = B_\infty$.

On the other edge

$$E_2^{0,*} = H^{*}(E) = F_2[h_0, h_1, \ldots h_{m-n}]$$

with each h_i of degree 1. Now the usual reasoning shows that successive powers $h_i^{2^e}$ are transgressive:

$$d_3 h_i^2 = d_3 Sq^1 h_i = Sq^1 d_2 h_i \in E_3^{0,3}$$

$$d_5 h_i^4 = d_5 Sq^2 h_i^2 = Sq^2 d_3 h_i^2 \in E_5^{0,5} \ \text{etc.}$$

But then because $B_r = B_\infty$, the transgression of $h_i^{2^e}$ must be zero as soon as $2^e+1 > r$. It follows that for such an e the powers

$$h_0^{2^e}, h_1^{2^e}, \ldots h_{m-n}^{2^e}$$

are permanent cycles.

Now the whole E_2-term is a fin. gen. module over

$$H^*(A_m(n)) \otimes F_2[h_0^{2^e}, h_1^{2^e}, \ldots h_{m-n}^{2^e}]$$

which is Noetherian and consists of permanent

$$d_2(X^2) = (d_2 X)X + X(d_2 X)$$

$$= 0 \quad \text{because } X \text{ is central.}$$

We can then [consider] $d_3(X^2)$. It may not be zero, but

$$d_3(X^4) = (d_3 X^2)X^2 + X^2(d_3 X^2)$$

$$= 0 \quad \text{because } X^2 \text{ is central.}$$

Continuing, we find $d_r(X^{2^{r-1}}) = 0$.

After a finite number of steps we reach past our vanishing line, and find that X^{2^e} is a permanent cycle for some suitable power. This permanent cycle lies in

the central image in $\mathrm{Ext}^{*,*}_{\Lambda(P^0_{n+1})}(H^*(Y_n DY), \mathbb{F}_2)$

In view of Lemma 4, this completes the proof of Theorem 2.

Proof of thm, completed. By Coroll. 15 we can

~~tct~~ choose m so that the restriction 218

$$\mathrm{Ext}^{s,t}_{A}(\tilde{H}^*(Y_n DY); \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{A_m}(\tilde{H}^*(Y_n DY); \mathbb{F}_2)$$

is iso for $t \leq s(2^{n+1} - 1)$. Then using Prop. 18 ~~tct, choose~~ we conclude

$$x \in \mathrm{Ext}^{*,*}_{A_m}(\mathbb{F}_2, \mathbb{F}_2)$$

~~[illegible]~~ which restricts to $[\xi^{a}_{n+1}]^{2c}$ is

$$\mathrm{Ext}^{*,*}_{A(P^0_{n+1})}(\mathbb{F}_2, \mathbb{F}_2).$$ The ~~[illegible]~~ unit map

$S^0 \xrightarrow{?} \tilde{H}^*(Y_n DY)$ gives us an element

$$\eta^* \in \mathrm{Ext}^{0,0}_{A_m}(\tilde{H}^*(Y_n DY), \mathbb{F}_2),$$ so

we can form $x\eta^* \in \mathrm{Ext}^{s,t}_{A_m}(\tilde{H}^*(Y_n DY); \mathbb{F}_2)$

This element is central, i.e. it commutes with the other elements of $\mathrm{Ext}^{*,*}_{A_m}(\tilde{H}^*(Y_n DY); \mathbb{F}_2)$.

Let $X \in \mathrm{Ext}^{s,t}_{A}(\tilde{H}^*(Y_n DY); \mathbb{F}_2)$ be the elem. which corresponds to it. Then this element must be central, in the sense that it commutes with other elements of $\mathrm{Ext}^{s,t}_{A}(\tilde{H}^*(Y_n DY); \mathbb{F}_2)$ provided they live $t \leq s(2^{n+1} - 1)$

We can form $d_2 X \in \mathrm{Ext}^{s+2,t+1}_{A}(\tilde{H}^*(Y_n DY); \mathbb{F}_2)$

This may not be zero, but we have

and such an element is invertible because the series

$$1 - x + x^2 - x^3 \dots$$

converges to an inverse.

<u>Lemma 1.</u> By replacing f with a suitable power f^e we may arrange that

(i) $H(n)_* f = t^? \cdot ?$

(ii) $H(i)_* f = 0$ for $i \neq n$.

<u>Proof</u> (a) First consider the case $i \gg \dim K$, $r = |f|$.

Then the AHSS, $H(i)_*(\mathbb{R})$ is zero except for a small number of residue classes mod $2(p^i - 1)$

~~[illegible]~~

Then a suitable power f^e will act into the range where $H(i)_*(\mathbb{R}) = 0$. So one choice of (e) will secure (ii) for all suff. large i.

(b) Since $H(i)_* f$ is nilpotent for $i > n$, we can secure (ii) for the remaining finite set of i by passing from f to f^e for e suff. large.

These 2 arguments secure (ii)

For (i) the action of $\pi_* H(n) = F_p[t, t^{-1}]$

§10. Study of periodicity maps.

The conclusion of §9 Theorem 2 delivers a map with much less than the full range of beautiful properties that might be asked for; and the same is true of some of the constructions which follow. So it will be convenient if our definition of "periodicity map" asks for the minimum, but we have devices, usually, to clean up our periodicity maps and improve their properties.

It doesn't much matter whether we talk about maps $f : S^r X \to X$, where X is a finite spectrum, or elements $f \in \pi_r(R)$ where R is a finite ring-spectrum; the latter includes the former, by taking $R = X \wedge DX$, so let's adopt the latter point of view. We may assume $H(i)_*(R) = 0$ for $i < n$ & $H(n)_*(R) \neq 0$; we may safely assume $n > 0$, and R is a p-torsion spectrum; we may safely assume that $r = |f| > 0$ and is a multiple of $2(p^n - 1)$.

Defn (Hopkins) $f \in \pi_r(R)$ is a v_n-elt if
(i) $H(n)_* f$ is invertible
(ii) $H(i)_* f$ is nilpotent (i ≠ n)

The output of Th 2 surely has this property, because $H(n)_* f \doteq 1 \otimes v$ with v nilpotent,

223

composition. Also, of course, we have

$$(\text{ad} f)_* x = fx - xf$$
$$= (\text{ad} f)x \quad \text{for } x \in \Pi, R.$$

~~Sublemma 3~~ ~~$f^n x = \sum_{i+j=n} \frac{n!}{i!j!} (\text{ad} f)^i$~~

<u>Sublemma 3</u>. If $\alpha x = fx - xf$, then

$$f^n x = \sum_{i+j=n} \frac{n!}{i!j!} (\alpha^i x) f^j.$$

Proof by induction over n: ~~assume~~ for $n=1$ we get

$$fx = (fx - xf) + xf.$$

Assume the induction given. Then

$$f^{n+1} x = \sum_{i+j=n} \frac{n!}{i!j!} f(\alpha^i x) f^j$$

$$= \sum_{i+j=n} \frac{n!}{i!j!} (\alpha^{i+1} x) f^j + \frac{n!}{i!j!} (\alpha^i x) f^{j+1}$$

$$= \sum_{r+s=n+1} \left(\frac{n!}{(r-1)!s!} + \frac{n!}{r!(s-1)!} \right) (\alpha^r x) f^s$$

$$\left(= \frac{n!}{r!s!} (r+s) \right)$$

$$= \sum_{r+s=n+1} \frac{(n+1)!}{r!s!} (\alpha^r x) f^s.$$

suffices to identify all the groups $K(n)_a(R)$ with $a \equiv 0 \mod 2(p^n - 1)$. So it is sufficient to talk about $K(n)_0(R)$. This is a fin-dim'l algebra over F_p, & the idempotent elements for a finite group, so there is an e s.t. f^e acts as 1 in it. This proves L.1.

L.2 (Centrality) If $f \in \pi_* R$ is a v_n-element, then some power f^e of f is central in $\pi_* R$.

Proof We can assume that f is as in Lemma 1. Let

$$Ad\, f : S^r \wedge R \longrightarrow R$$

be the map (left mult by f) − (right mult by f)

ie
$$S^r \wedge R \xrightarrow[-T(1 \wedge f)]{f \wedge 1} R \wedge R \xrightarrow{1-T} R \wedge R \xrightarrow{\mu} R.$$

Then $K(i)_*(Ad\, f) = 0 \quad \forall i$.

Now by a version of the nilpotence theorem which unfortunately I omitted from §7, it follows that $Ad\, f$ is nilpotent under

225

<u>Proof</u> By Lemma 1.2 we may suppose, after passing to suitable powers, that $|f| = |q|$, $H(i)_*(f - q) = 0$ for all i and $fq = q.f$. From the fact about $H(i)$, we infer that $f - q$ is nilpotent, say $(f-q)^{p^a} = 0$. Since all the binomial coefficients are divisible by p, we get

$$f^{p^a} = q^{p^a} + ph.$$

Thus $f^{p^{a+1}} = q^{p^{a+1}} + p^2 h$ & so on by induction,

$$f^{p^{a+b}} = q^{p^{a+b}} + p^b k$$

Since R is a p-torsion spectrum, we can choose p^b so that $p^b.1 = 0$, and then

$$f^{p^{a+b}} = q^{p^{a+b}}.$$

$\triangleleft$

At this point we can switch to self-maps. ~~$f : S^r X \to X$~~ Let X be a finite spectrum and $f : S^r X \to X$ a [illegible] cell-map. We may [illegible] $= 0$ for $i < n$ & $H(X) \neq 0$; we may safely assume $n > 0$, and X is a p-torsion spectrum; we may safely assume that $r = |f| > 0$ and is

Subtracting off the term $i=0,\ j=h$ to the right, we get:

$$f^h x - x f^h = \sum_{\substack{i+j=h \\ i>0,\ j<h}} \frac{h!}{i!\,j!}\left(\left((\mathrm{Ad} f)_*\right)^i x\right) f^j$$

Now, R is a p-torsion spectrum; in particular $1 \in \pi_*(R)$ is annihilated by some power p^a of p, & then p^a annihilates $\pi_*(R)$.

Since $\mathrm{Ad} f$ is nilpotent, we have $(\mathrm{Ad} f)_*^i = 0$ for $i > i_0$, and we need only worry about a finite number of values $i = 1, 2, \ldots i_0$. For these the binomial coeff't

$$\frac{h(h-1)\cdots(h-i+1)}{1\cdot 2 \cdots i}$$

~~is this the p-adic~~
~~is a cty fn of this to p~~
will be divisible by p^a if we ensure that h is divisible by p^a & the p-primary factor of $i_0!$, so the outcome will be 0, & then f^h will be central.

<u>L4 (Uniqueness)</u> Let $f, g \in \pi_* R$ be non-nilpotent. Then $\exists$ j, k with $f^j = g^k$.

In other words, we may assume they are equal by passing to suitable powers.

In other words, you have to define the mult. in $\pi(n)_*(X \wedge DX)$ using the opposite of the mult. in $\pi(n)$ you use for the usual manipulations on left module-spectra, & for $p = 2$ this makes a difference.

However, it doesn't matter. Our previous theory doesn't care which product you use on $\pi(n)$, so they are all just as valid for the opposite of the product you first thought of.

With this interpretation, the equivalence of conds (i) is formal, & the equiv. of conds (iii) needs the fact that $\pi_* \pi(i)$ is a graded field to show that the map of left module-spectra is null-htpic iff the induced map of htpy groups is zero.

Alternatively, of course, you can define the product in $\pi(n)_*(X \wedge DX)$ by using the product you first thought of in $\pi(n)$ & the opposite of the product you first thought of in $X \wedge DX$. Our results apply to any ring-spectrum R, so there can't be any objection to that, except that it increases confusion.

276

a multiple of $2(p^n - 1)$.

<u>Defn</u> (Hopkins) $f : S^r X \to X$ is a v_n-map if
(i) $K(n)_* f$ is iso
(ii) $K(i)_* f$ is nilpotent for $i > n$.

<u>L+S</u>. This is equivalent to asking that the corr. elt $\bar{f} \in \pi_r(X \wedge DX)$ is a v_n-elt.

At this point I should explain a slight snag. You want to say that elts
$$\alpha^* \in \pi_r(K(n) \wedge Y \wedge DX)$$
are in 1-1 corr. with maps of left $K(n)$-module-spectra ~~[illegible]~~
$$K(n) \wedge S^r \wedge Y \longrightarrow K(n) \wedge X.$$
also they are. And now you want to say that composition of maps of left $K(n)$-module-spectra corresponds to the obvious product operation; and when I checked the comm. diagrams, I found that the wrong product operations,

$$S^a \wedge S^r \xrightarrow{\ e \wedge f\ } K(n) \wedge Z \wedge DY \wedge K(n) \wedge Y \wedge DX$$

$K(n) \wedge Z \wedge DX : \leftarrow K(n) \wedge K(n) \wedge Z \wedge DY \wedge Y \wedge DX$ ...

<u>Proof</u> Consider $Y \wedge DX$. This satisfies our standing assumptions, viz. $H(i)_*(Y \wedge DX) = 0$ for $i < n$ & $H(n)_*(Y \wedge DX) \neq 0$.

Consider the maps $S^n Y \wedge DX \xrightarrow{g \wedge 1} Y \wedge DX$

$S^{a} Y \wedge DX \xrightarrow{1 \wedge Df} Y \wedge DX.$

Both are v_n-maps; in detail, if f & g are as provided by Coroll. 6, so are these.

By the uniqueness result (Coroll 7), ∃ powers s.t.

$$g^{ae} \wedge 1 = 1 \wedge (Df)^{ve} : S^{ave} Y \wedge DX \to Y \wedge DX$$

Consider the effect of both on

$$S^s \xrightarrow{h} Y \wedge DX.$$

We get an equation in $\pi_{ave+s}(Y \wedge DX)$, and translating it back into $[S^{ave+s} X, Y]$, we get the required result.

Coroll 6. By replacing f with a suitable power f^e, we may arrange that $H(n)_*f$ is mult. by some t? and $H(i)_*f = 0$ for $i \neq n$.

Proof Translate L1, or else can be proved...

Coroll 7 (Uniqueness). Let $f : S^a X \to X$, $g : S^b X \to X$ be v_n-maps. Then $\exists\, i, j$ s.t. $f^i = g^j$.

Proof Translate L4.

Prop 8 (Strong centrality). Suppose X & Y have v_n-maps

$S^a X \xrightarrow{f} X$, $S^b Y \xrightarrow{g} Y$.

Then $\exists$ powers f^{re}, g^{ae} s.t. $\forall h : S^i X \to Y$ we have

$$\begin{array}{ccc} S^{are+i} X & \xrightarrow{h} & S^{are} Y \\ \downarrow f^{re} & & \downarrow g^{ae} \\ S^i X & \xrightarrow{h} & Y \end{array}$$

com.

Here $H(n)_* C(n)$ is iso by the 5 lemma. Coroll 6 allows us to

~~For~~ $i > n$ ~~we~~ we can assume $H(i)_* f = 0$,

$H(i)_* g = 0$ and then diagram-chasing

shows $H(i)_* h^2 = 0$. (This proves (iii))

(vii) Suppose $X \vee Y \xrightarrow{f} X \vee Y$

is a v.n-map. By Coroll. 6 we can

assume $H(n)_* f$ is well (by some power of ?) of the arrow and $H(i)_* f = 0$ for $i \neq n$

then $X \xrightarrow{i} X \vee Y \xrightarrow{f} X \vee Y \xrightarrow{j} X$

has the same properties.

<u>Theorem 10.</u> Every finite p-local spectrum X with $H(i)_* X = 0$ for $i < n$,

$H(n)_* X \neq 0$ admits a v_n-map.

<u>Proof.</u> The class $\mathcal{C}$ considered in Theorem 9

containing every X with $H(n)_*(X) \neq 0$ by §9 Thm 2. Thus by §7 Thm 9

$\mathcal{C}$ is the class C_n of all finite p-local spectra st. $H(i)_*(X) = 0$ for $i < n$.

Prop 9. The class $\mathscr{C}$ of finite p[illegible].

Spaces X with $\pi(i)X = 0$ for $i < n$ which admit a map $f : S^{?}X \longrightarrow X$ s.t. $H(n)_* f$ is iso & $\pi(m)_* f$ is nilpotent for $i > n$ is closed under the following operations

(i) If $X \in \mathscr{C}$ & $X \simeq X'$ then $X' \in \mathscr{C}$.

(ii) If $X \in \mathscr{C}$ then $S^n X \in \mathscr{C}$ for $n \in \mathbb{Z}$.

(iii) If $X \xrightarrow{i} Y \longrightarrow Z$ is a cofibration & two of X, Y, Z lie in $\mathscr{C}$ so does the third.

(iv) If $X \vee Y \in \mathscr{C}$ then $X \in \mathscr{C}$.

Proof. — (i) clear (ii) clear.

(iii). Suppose $X, Y \in \mathscr{C}$. By Prop 8 we can assume wlog that the diagram

$$\begin{array}{ccc} S^{?}X & \xrightarrow{i} & S^{?}Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & Y \end{array}$$

comm.

Then we get a comm. diagram

$$\begin{array}{ccccccc} S^{?}X & \xrightarrow{i} & S^{?}Y & \xrightarrow{j} & S^{?}Z & \longrightarrow & S^{?+1}X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle ?f} \\ X & \xrightarrow{i} & Y & \xrightarrow{j} & Z & \longrightarrow & SX \end{array}$$