Bidimensional Parameters and Local Treewidth

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Abstract

For several graph theoretic parameters such as vertex cover and dominating set, it is known that if their values are bounded by k then the treewidth of the graph is bounded by some function of k. This fact is used as the main tool for the design of several fixed-parameter algorithms on minor-closed graph classes such as planar graphs, single-crossing-minor-free graphs, and graphs of bounded genus. In this paper we examine the question whether similar bounds can be obtained for larger minor-closed graph classes, and for general families of parameters including all the parameters where such a behavior has been reported so far.

Given a graph parameter P, we say that a graph family $\mathcal{F}$ has the *parameter-treewidth property for* P if there is a function $f(p)$ such that every graph $G \in \mathcal{F}$ with parameter at most p has treewidth at most $f(p)$. We prove as our main result that, for a large family of parameters called *contraction-bidimensional parameters*, a minor-closed graph family $\mathcal{F}$ has the parameter-treewidth property if $\mathcal{F}$ has bounded local treewidth. We also show "if and only if" for some parameters, and thus this result is in some sense tight. In addition we show that, for a slightly smaller family of parameters called *minor-bidimensional parameters*, all minor-closed graph families $\mathcal{F}$ excluding some fixed graphs have the parameter-treewidth property. The bidimensional parameters include many domination and covering parameters such as vertex cover, feedback vertex set, dominating set, edge-dominating set, q-dominating set (for fixed q). We use these theorems to develop new fixed-parameter algorithms in these contexts.

Keywords: treewidth, local treewidth, graph minors, dominating set

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1 Introduction

The last ten years has witnessed the rapid development of a new branch of computational complexity, called parameterized complexity; see the book of Downey & Fellows [16]. Roughly speaking, a parameterized problem with parameter k is *fixed-parameter tractable (FPT)* if it admits an algorithm with running time $f(k)|I|^{O(1)}$. (Here f is a function depending *only* on k and $|I|$ is the size of the instance.)

A celebrated example of a fixed-parameter tractable problem is VERTEX COVER, asking whether an input graph has at most k vertices that are incident to all its edges. When parameterized by k, the k-VERTEX COVER problem admits a solution as fast as $O(kn + 1.285^k)$ [9]. Moreover, if we restrict k-VERTEX COVER to planar graphs then it is possible to design FPT-algorithms where the contribution of k in the non-polynomial part of their complexity is subexponential. The first algorithm of this type was given by Alber et al. (see [4]). Recently, Fomin and Thilikos reported a $O(k^4 + 2^{4.5\sqrt{k}} + kn)$ algorithm for planar k-VERTEX COVER [21].

However, not all parameterized problems are fixed-parameter tractable. A typical example of such a problem is DOMINATING SET, asking whether an input graph has at most k vertices that are adjacent to the rest of the vertices. When parameterized by k, the k-DOMINATING SET Problem is known to be $W[2]$-complete and thus it is not expected to be fixed-parameter tractable. Interestingly, the fixed-parameter complexity of the same problem can be distinct for special graph classes. During the last five years, there has been substantial work on fixed-parameter algorithms for solving the k-DOMINATING SET on planar graphs and different generalizations of planar graphs. For planar graphs Downey and Fellows [16], suggested an algorithm with running time $O(11^d n)$. Later the running time was reduced to $O(8^d n)$ [2]. An algorithm with a sublinear exponent for the problem with running time $O(4^{6\sqrt{34d}}n)$ was given by Alber et al. [1]. Recently, Kanj & Perkovuć [25] improved the running time to $O(2^{27\sqrt{d}}n)$ and Fomin & Thilikos to $O(2^{15.13\sqrt{d}}d + n^3 + d^4)$ [20]. The fixed-parameter algorithms for extensions of planar graphs like bounded-genus graphs and graphs excluding single-crossing graphs as minors are introduced in [11, 13, 17].

In the majority of these results, the design of FPT algorithms for solving problems such as k-VERTEX COVER or k-DOMINATING SET in a sparse graph class $\mathcal{F}$ is based on the following lemma: every graph G in $\mathcal{F}$ where the value of the parameter is at most p has treewidth bounded by $f(p)$, where f is a function depending only on $\mathcal{F}$. With some work (sometimes very technical), a tree decomposition of width $O(f(p))$ is constructed and standard dynamic-programming techniques on graphs of bounded treewidth are implemented. Of course this method can not be applied for any graph class $\mathcal{F}$. For instance, the n-vertex complete graph K_n has a dominating set of size one and treewidth equal to $n - 1$. So the emerging question is:

> For which (larger) graph classes and for which parameters can the "bounding treewidth method" be applied?

In this paper we give a *complete* characterization of minor-closed graph families for which the aforementioned "bounding treewidth method" can be applied for a wide family of graph parameters.

For a given parameter P, we say that a graph family $\mathcal{F}$ has the *parameter-treewidth property* for P if there is a function $f(p)$ such for every graph $G \in \mathcal{F}$ where $P(G) \le p$ implies that G has treewidth at most $f(p)$. Our main result is that for a large family of parameters called *contraction-bidimensional parameters*, a minor-closed graph family $\mathcal{F}$ has the parameter-treewidth property if $\mathcal{F}$ has bounded local treewidth. Moreover, we show that the inverse is also correct if some simple condition is satisfied by P. In addition we show that, for a slightly smaller family of parameters called *minor-bidimensional parameters*, every minor-closed graph family $\mathcal{F}$ excluding some fixed graph has the parameter-treewidth property. The bidimensional-parameter family includes many domination and covering parameters such as vertex cover, feedback vertex set, dominating set, edge-dominating set, and q-dominating set (for fixed q) (see also [11] for more examples).

The proof of the main result uses the characterization of Eppstein for minor-closed families of bounded local treewidth [18] and Diestel et al.'s modification of the Robertson & Seymour excluded-grid-minor theorem [15]. In addition, the proof is constructive and can be used for constructing fixed-parameter algorithms to decide bidimensional parameters on minor-closed families of bounded local treewidth. In this sense, we extend to fixed-parameter algorithms the result of Frick & Grohe [23] that, for each property ϕ definable in first-order logic, and for each class of minor-closed graphs of bounded local treewidth, there is a (non-fixed-parameter) $O(n^{1+\epsilon})$-time algorithm deciding whether a given graph has property ϕ.

A preliminary and special case of our result, concerning only the dominating set parameter, appeared in [22] with a different and more complicated proof. Also, another proof of the same result appeared in [10]. In this paper we present shorter and more elegant proofs of the combinatorial results of [22] and [10] while we extend their applicability to general families of parameters.

2 Definitions and preliminary results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We let n denote the number of vertices of a graph when it is clear from context. For every nonempty $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$. We define the *q-neighborhood* of a vertex $v \in V(G)$, denoted by $N_G^q[v]$, to be the set of vertices of G at distance at most q from v. Notice that $v \in N_G^q[v]$. We put $N_G[v] = N_G^1[v]$. We also often say that a vertex v *dominates* subset $S \subseteq V(G)$ if $N_G[v] \supseteq S$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge contractions is said to be a *contraction* of G. A graph H is a *minor* of a graph G if H is the subgraph of a contraction of G. We use the notation $H \preceq G$ [resp. $H \preceq_c G$] for H a minor [a contraction] of G. A family (or class) of graphs $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. A minor-closed graph family $\mathcal{F}$ is *H-minor-free* if $H \notin \mathcal{F}$.

The $m \times m$ grid is the graph on $\{1, 2, \ldots, m^2\}$ vertices $\{(i,j) \colon 1 \le i, j \le m\}$ with the edge set

$$\{(i,j)(i',j') \colon |i-i'| + |j-j'| = 1\}.$$

For $i \in \{1, 2, \ldots, m\}$ the vertex set (i, j), $j \in \{1, 2, \ldots, m\}$, is referred as the *ith row* and the vertex set (j, i), $j \in \{1, 2, \ldots, m\}$, is referred to as the *ith column* of the $m \times m$ grid. The vertices (i, j) of the $m \times m$ grid with $i \in \{1, m\}$ or $j \in \{1, m\}$ are called *boundary* vertices and the rest of the vertices are called *non-boundary* vertices.

The notion of treewidth was introduced by Robertson and Seymour [27]. A *tree decomposition* of a graph G is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, with $\{X_i \mid i \in I\}$ a family of subsets of $V(G)$ and T a tree, such that

1. $\bigcup_{i \in I} X_i = V(G)$;
2. for all $\{v, w\} \in E(G)$, there is an $i \in I$ with $v, w \in X_i$; and
3. for all $i_0, i_1, i_2 \in I$, if i_1 is on the path from i_0 to i_2 in T, then $X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$.

The *width* of the tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The treewidth $\mathbf{tw}(G)$ of a graph G is the minimum width of a tree decomposition of G.

We need the following facts about treewidth. The first fact is trivial.

- For any complete graph K_n on n vertices, $\mathbf{tw}(K_n) = n - 1$.

The second fact is well known but its proof is not trivial. (See e.g., [14].)

- The treewidth of the $m \times m$ grid is m.

The next fact we need is the improved version of the Robertson & Seymour theorem on excluded grid minors [28] due to Diestel et al. [15]. (See also the textbook [14].)

Theorem 1 ([15]) *Let r, m be integers, and let G be a graph of treewidth at least $m^{4r^2(m+2)}$. Then G contains either K_r or the $m \times m$ grid as a minor.*

A *parameter* P is any function mapping graphs to nonnegative integers. The *parameterized problem* associated with P asks, for a fixed k, whether $P(G) \leq k$ for a given graph G.

A parameter P is *$g(r)$-minor-bidimensional* if (i) contracting an edge, deleting an edge, or deleting a vertex in a graph G cannot increase $P(G)$, and (ii) there exists a function g such that, for the $r \times r$ grid R, $P(R) \geq g(r)$. Similarly, a parameter P is *$g(r)$-contraction-bidimensional* if (i) contracting an edge in a graph G cannot increase $P(G)$, and (ii) there exists a function g such that, for any $r \times r$ augmented grid R of constant span, $P(R) \geq g(r)$[1]. Here an $r \times r$ *augmented grid of span s* is an $r \times r$ *grid* with some extra edges such that each vertex is attached to at most s non-boundary vertices of the grid (see an example in Figure 1.) We assume that $g(r)$ is monotone and invertible for $r \geq 0$. We note that a $g(r)$-minor-bidimensional parameter is also a $g(r)$-contraction-bidimensional parameter. One can easily observe that many parameters such as minimum sizes of dominating set, q-dominating set (distance q-dominating set for a fixed q), vertex cover, feedback

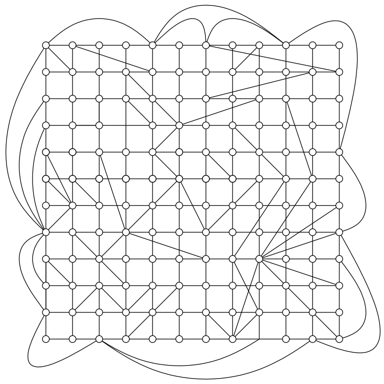

Figure 1: An augmented 12×12 grid with span 8.

vertex set, and edge-dominating set (see exact definitions of the corresponding parameters in [11]) are $\Theta(r^2)$-minor- or $\Theta(r^2)$-contraction-bidimensional parameters.

Here, we present a theorem for minor-bidimensional parameters on general minor-closed classes of graphs excluding some fixed graphs, whose intuition plays an important role in the main result of this paper.

Theorem 2 *If a $g(r)$-minor-bidimensional parameter P on an H-minor-free graph G has value at most p, then $\mathbf{tw}(G) \leq 2^{O(g^{-1}(p) \log g^{-1}(p))}$. (The constant in the O notation depends on H.)*

Proof: By Theorem 1, since G is H-minor-free (and thus $K_{|V(H)|}$-minor-free), we know if m is the largest integer such that $\mathbf{tw}(G) \geq m^{4|V(H)|^2(m+2)}$, then G has an $m \times m$ grid as a minor. Since P is $g(r)$-minor-bidimensional, $p \geq g(m)$ and thus we obtain the desired bound. $\square$

Theorem 2 can be applied for minor-bidimensional parameters such as vertex cover or feedback vertex set.

The notion of local treewidth was introduced by Eppstein [18] (see also [24]). The *local treewidth* of a graph G is

$$\mathbf{ltw}(G, r) = \max\{\mathbf{tw}(G[N_G^r[v]]) : v \in V(G)\}.$$

For a function $f : N \to N$ we define the minor-closed class of graphs of bounded local treewidth

$$\mathcal{L}(f) = \{G : \forall H \preceq G \ \forall r \geq 0,\ \mathbf{ltw}(H, r) \leq f(r)\}.$$

Also we say that a minor-closed class of graphs $\mathcal{C}$ has bounded local treewidth if $\mathcal{C} \subseteq \mathcal{L}(f)$ for a function f.

Well-known examples of minor-closed classes of graphs of bounded local treewidth are graphs of bounded treewidth, planar graphs, graphs of bounded genus, and single-crossing-minor-free graphs.

Many difficult graph problems can be solved efficiently when the input is restricted to graphs of bounded treewidth (see e.g., Bodlaender's survey [7]). Eppstein [18] made a step forward by proving that some problems like subgraph isomorphism and induced subgraph isomorphism can be

[1]Closely related notions of bidimensional parameters are introduced by the authors in [13].

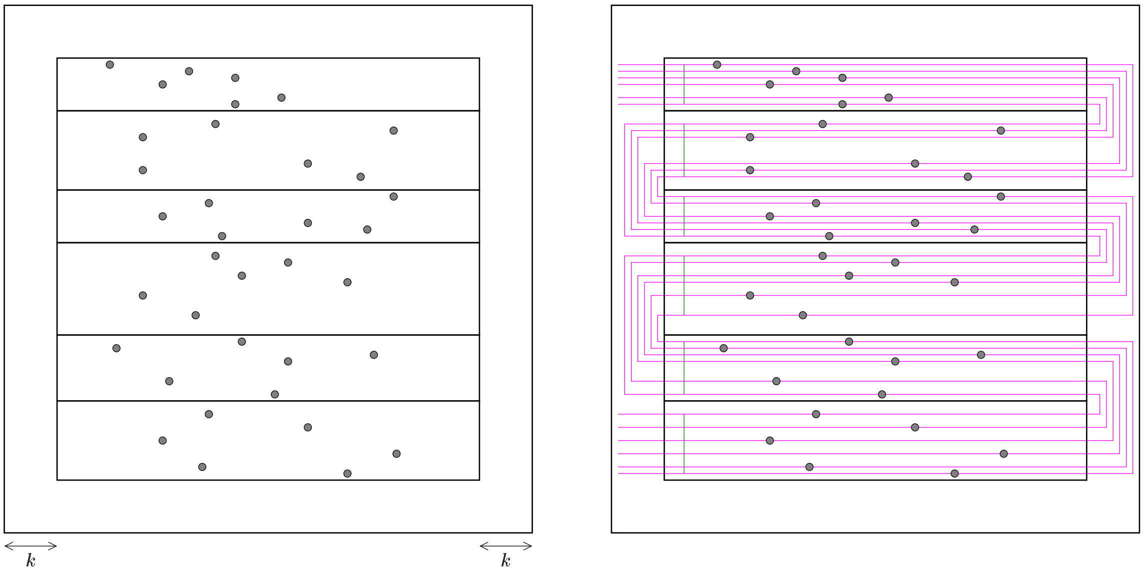

Figure 2: Left: The grid H, the points in S'', and their grouping. Here $k = 6$. Right: Construction of the minor $k \times k$ grid R passing through the points in S''.

solved in linear time on minor-closed graphs of bounded local treewidth. Also the classic Baker's technique [6] for obtaining approximation schemes on planar graphs for different NP-hard problems can be generalized to minor-closed families of bounded local treewidth. (See [18] for a generalization of these techniques.)

An *apex graph* is a graph G such that, for some vertex v (the *apex*), $G - v$ is planar. The following result is due to Eppstein [18].

Theorem 3 ([18]) *Let $\mathcal{F}$ be a minor-closed family of graphs. Then $\mathcal{F}$ is of bounded local treewidth if and only if $\mathcal{F}$ does not contain all apex graphs.*

3 Combinatorial lemmas

In this section we prove two combinatorial lemmas regarding grids and graphs of bounded local treewidth.

Lemma 4 *Suppose we have a $m \times m$ grid H and a subset S of vertices in the central $(m - 2k) \times (m - 2k)$ subgrid H', where $s = |S|$ and $k = \lfloor \sqrt[4]{s} \rfloor$. Then H has as a minor the $k \times k$ grid R such that each vertex in R is a contraction of at least one vertex in S and other vertices in H.*

Proof: Let s_x denote the number of distinct x coordinates of the vertices in S, and let s_y denote the number of distinct y coordinates of the vertices in S. Thus, $s \le s_x \cdot s_y$. Assume by symmetry that $s_y \ge s_x$, and therefore $s_y \ge \sqrt{s}$.

We define the subset S' of S by removing all but one point that share a common y coordinate, for each y coordinate. Thus, all y coordinates of the vertices in S' are distinct, and $|S'| = s_y$. We discard all but k^2 vertices in S' to form a slightly smaller set S'', because $|S'| = s_y \geq \sqrt{s} \geq (\lfloor \sqrt[4]{s} \rfloor)^2 = k^2$. We divide these k^2 vertices into k groups each of exactly k consecutive vertices according to the order of their y coordinates. Now we have the situation shown on the left of Figure 2.

We construct the minor grid R as shown on the right of Figure 2. Because each y coordinate is unique, we can draw long horizontal segments through every point. The k columns on the left-hand and right-hand sides of H allow us to connect these horizontal segments together into k vertex-disjoint paths, each passing through exactly k vertices of S''. These paths can be connected by vertical segments within each group. By contracting each horizontal segment into a single vertex, and some further contraction, we can obtain the desired $k \times k$ grid R as a minor. Each vertex of this grid R is a contraction of at least one vertex in S'' (and hence in S) and other vertices in H.

□

Lemma 5 *Let $G \in \mathcal{L}(f)$ be a graph containing the $m \times m$ grid H as a subgraph, $m > 2k$, where $k = f(2) + 1$. Then the central $(m - 2k) \times (m - 2k)$ subgrid H' has the property that every vertex $v \in V(G)$ is adjacent to less than k^4 vertices in H'.*

Proof: Suppose for contradiction that there is a vertex $v \in V(G)$ such that $S = N_G[v] \cap V(H)$ has size $s = |S| \geq k^4$. By Lemma 4, H' has as a minor a $k \times k$ grid R such that, if we perform these contractions and deletions in G, then v is adjacent to all vertices in R. Define $R + v$ to be the grid R plus the vertex v (if v is not already in R) and the star of connections between v and all vertices in R. Then $R + v$ is a minor of G, but has diameter 2 and treewidth $k \geq f(2) + 1$, a contradiction.

□

4 Main theorem

Now we are ready to present the main result of this paper.

Theorem 6 *Let P be a $g(r)$-contraction-bidimensional parameter. Then for any function $f \colon \mathbb{N} \to \mathbb{N}$ and any graph $G \in \mathcal{L}(f)$ on which parameter P has value at most p, we have $\mathbf{tw}(G) \leq 2^{O(g^{-1}(p) \log g^{-1}(p))}$. (The constant in the O notation depends on $f(1)$ and $f(2)$.)*

Proof: Let $r = f(1) + 1$ and $k = f(2) + 1$. Let $G \in \mathcal{L}(f)$ be a graph on which the parameter P has value p. Let m be the largest integer such that $\mathbf{tw}(G) \geq m^{4r^2(m+2)}$. Without loss of generality, we assume G is connected, and $m > 2k$ (otherwise, $\mathbf{tw}(G)$ is a constant since both r and k are constants.) Then G has no complete graph K_r as a minor. By Theorem 1, G contains an $m \times m$ grid H as a minor. Thus there exists a sequence of edge contractions and edge/vertex deletions reducing G to H. We apply to G the edge contractions from this sequence, we ignore the edge deletions, and instead of deletion of a vertex v, we only contract v into one of its neighbors. Call the new graph G', which has the $m \times m$ grid H as a subgraph and in addition $V(G') = V(H)$. Since parameter P is contraction-bidimensional, its value on G' will not increase. By Lemma 5, we know that the central $(m-2k) \times (m-2k)$ subgrid H' of H has the property that every vertex $v \in V(G')$ is adjacent to less than k^4 vertices in H'.

Now, suppose in graph G', we further contract all $2k$ boundary rows and $2k$ boundary columns into two boundary rows and two boundary columns (one on each side) and call the new graph G''. Note that here G'' and H' have the same set of vertices. The degree of each vertex of G'' to the vertices that are not on the boundary is at most $(k+1)^2 k^4$, which is a constant since k is a constant. Here the factor $(k+1)^2$ is for the boundary vertices each of which is obtained by contraction of at most $(k+1)^2$ vertices. Again because parameter P is contraction-bidimensional, its value on G'' does not increase and thus it is at most p. On the other hand, since the parameter is $g(r)$-contraction-bidimensional, its value on graph G'' is at least $g(m-2k)$. Thus $g^{-1}(p) \geq m-2k$, so $m = O(g^{-1}(p))$. Therefore, the treewidth of the original graph G is at most $2^{O(g^{-1}(p) \log g^{-1}(p))}$ as desired. $\square$

A direct corollary of Theorem 6 is the following.

Lemma 7 *Let P be a contraction-bidimensional parameter. A minor-closed graph class $\mathcal{F}$ has the parameter-treewidth property for P if $\mathcal{F}$ is of bounded local treewidth.*

The *apex graphs* A_i, $i = 1, 2, 3, \ldots$, are obtained from the $i \times i$ grid by adding a vertex v adjacent to all vertices of the grid. It is interesting to see that, for a wide range of parameters, the inverse of Lemma 7 also holds.

Lemma 8 *Let P be any contraction-bidimensional parameter where $P(A_i) = O(1)$ for any $i \geq 1$. A minor-closed graph class $\mathcal{F}$ has the parameter-treewidth property for P only if $\mathcal{F}$ is of bounded local treewidth.*

Proof: The proof follows from Theorem 3. The apex graph A_i, has diameter ≤ 2 and treewidth $\geq i$. So a minor-closed family of graphs with the parameter-treewidth property for P cannot contain all apex graphs and hence it is of bounded local treewidth. $\square$

Typical examples of parameters satisfying Lemmas 7 and 8 are dominating set and its generalization q-dominating set, for a fixed constant q (in which each vertex can dominate its q-neighborhood). These parameters are $\Theta(r^2)$-contraction-bidimensional and their value is 1 for any apex graph $A_i, i \geq 1$.

We can strengthen the "if and only if" result provided by Lemmas 7 and 8 with the following lemma. We just need to use the fact that if the value of P is less than the value of P' then the parameter-treewidth property for P implies the parameter-treewidth property for P' as well.

Lemma 9 *Let P be a parameter whose value is lower bounded by some contraction-bidimensional parameter and let $P(A_i) = O(1)$ for any $i \geq 1$. Then a minor-closed graph class $\mathcal{F}$ has the parameter-treewidth property for P if and only if $\mathcal{F}$ is of bounded local treewidth.*

Lemma 9 can apply for parameters that are not necessarily contraction-bidimensional. As an example we mention the *clique-transversal number* of a graph, i.e., the minimum number of vertices meeting all the maximal cliques of a graph.[2] It is easy to see that this parameter always exceeds the domination number (the size of a minimum dominating set) and that any graph in A_i has a clique-transversal set of size 1.

Another application is the Π-*domination number*, i.e., the minimum cardinality of a vertex set that is a dominating set of G and satisfies some property Π in G. If this property is satisfied for any one-element subset of $V(G)$ then we call it *regular*. Examples of known variants of the parameterized dominating set problem corresponding to the Π-domination number for some regular property Π are the following parameterized problems: the independent dominating set problem, the total dominating set problem, the perfect dominating set problem, and the perfect independent dominating set problem (see the exact definitions in [1]).

We summarize the previous observations with the following:

Corollary 10 *Let P be any of the following parameters: the minimum cardinality of a dominating set, the minimum cardinality of a q-dominating set (for any fixed q), the minimum cardinality of a clique-transversal set, or the minimum cardinality of a dominating set with some regular property Π. A minor-closed family of graphs $\mathcal{F}$ has the parameter-treewidth property for P if and only if $\mathcal{F}$ is of bounded local treewidth. The function $f(p)$ in the parameter-treewidth property is $2^{O(\sqrt{p}\log p)}$.*

5 Algorithmic consequences and concluding remarks

Courcelle [8] proved a meta-theorem on graphs of bounded treewidth; he showed that, if ϕ is a property of graphs that is definable in monadic second-order logic, then ϕ can be decided in linear time on graphs of bounded treewidth. Frick and Grohe [23] extended this result to graphs of bounded local treewidth; they showed that, for each property ϕ that is definable in first-order logic and for each minor-closed class of graphs of bounded local treewidth, there is an $O(n^{1+\epsilon})$-time algorithm deciding whether a given graph has property ϕ. However Frick & Grohe's proof is not constructive. It uses a transformation of a first-order logic formula into a "local formula" according to Gaifman's theorem and even the complexity of this transformation is unknown.

Using Theorems 2 and 6, we can extend the result of Frick & Grohe for fixed-parameter algorithms and show that any minor-bidimensional property that is solvable in polynomial time on graphs of bounded treewidth is also fixed-parameter tractable on general minor-closed graph families excluding some fixed graphs, and similarly for any contraction-bidimensional property on minor-closed graph families of bounded local treewidth. In contrast to the work of Frick & Grohe, the running time of our algorithm is explicit.

[2] The clique-transversal number is not contraction-bidimensional because an edge contraction may create a new maximal clique and the value of the clique-transversal number may increase.

Theorem 11 *Let P be a parameter such that, given a tree decomposition of width at most w for a graph G, the parameter can be decided in $h(w)n^{O(1)}$ time. Now, if P is a $g(r)$-minor-bidimensional parameter and G belongs to a minor-closed graph family excluding some fixed graphs, or P is a $g(r)$-contraction-bidimensional parameter and G belongs to a minor-closed family of graphs of bounded local treewidth, then we can decide P on G in $h(2^{O(g^{-1}(k)\log g^{-1}(k))})n^{O(1)} + 2^{2^{O(g^{-1}(k)\log g^{-1}(k))}}n^{3+\epsilon}$ time.*

Proof: The algorithm is as follows. First we check whether $\mathbf{tw}(G)$ is in $2^{O(g^{-1}(k)\log g^{-1}(k))}$. By Theorems 2 and 6, if it is not, parameter P has value more than k on graph G. This step can be performed by Amir's algorithm [5], which for a given graph G and integer ω, either reports that the treewidth of G is at least ω, or produces a tree decomposition of width at most $(3+\frac{2}{3})\omega$ in time $O(2^{3.698\omega}n^3\omega^3\log^4 n)$. Thus by using Amir's algorithm we can either compute a tree decomposition of G of size $2^{O(g^{-1}(k)\log g^{-1}(k))}$ in time $2^{2^{O(g^{-1}(k)\log g^{-1}(k))}}n^{3+\epsilon}$, or conclude that the treewidth of G is not in $2^{O(g^{-1}(k)\log g^{-1}(k))}$.

Now if we find a tree decomposition of the aforementioned width, we can decide P on G in time $h(2^{O(g^{-1}(k)\log g^{-1}(k))})n^{O(1)}$ time. The running time of this algorithm is the one mentioned in the statement of the theorem. $\square$

For example, let G be a graph from a minor-closed family $\mathcal{F}$ of bounded local treewidth. Since the dominating set of a graph with a given tree decomposition of width at most ω can be computed in time $O(2^{2\omega}n)$ [1], Theorem 11 gives an algorithm which either computes a dominating set of size at most p, or concludes that there is no such a dominating set in $2^{2^{O(\sqrt{p}\log p)}}n^{O(1)}$ time. The same result holds also for computing the minimum size of a q-dominating set. Indeed, Theorem 11 can be applied because the q-dominating set of a graph with a given tree decomposition of width at most ω can be computed in time $O(q^{O(\omega)}n)$ [12]. Also, algorithms on graphs of bounded treewidth for clique-transversal set, and Π-domination set appeared in [26] and [3] respectively. Using these algorithms, and the fact that all these parameters are lower bounded by the domination number, the methodology of the proof of Theorem 11 can give algorithmic results for clique-transversal set and Π-domination set with the same running times as in the case of dominating set (i.e., $2^{2^{O(\sqrt{p}\log p)}}n^{O(1)}$).

Finally, we mention some open problems. For planar graphs and for some of their extensions, it is known that for any graph G from these classes with dominating set of size at most p, we have $\mathbf{tw}(G) = O(\sqrt{p})$. It is tempting to ask if such a much smaller bound holds for all minor-closed families of bounded local treewidth. This will provide subexponential fixed-parameter algorithms on graphs of bounded local treewidth for the dominating set problem.

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