Tight Bounds For Random MAX 2-SAT

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Abstract

For a conjunctive normal form formula F with n variables and $m = cn$ 2-variable clauses (c is called the *density*), denote by $\max F$ is the maximum number of clauses satisfiable by a single assignment of the variables. For the uniform random formula F with density $c = 1 + \varepsilon$, $\varepsilon \gg n^{-1/3}$, we prove that $\max F$ is in $(1 + \varepsilon - \Theta(\varepsilon^3))n$ with high probability. This improves the known upper bound $(1 + \varepsilon - \Omega(\varepsilon^3/\ln(1/\varepsilon)))$ due to [6]. The algorithm for the lower bound is also simpler. In addition, we present a simple unified algorithm which not only yields bounds mentioned above for $c = 1+\varepsilon$, but also provides a tight lower bound $(3/4c+\Theta(\sqrt{c}))n$ for large enough c's. To obtain the bounds for $c = 1 + \varepsilon$, we use the Poisson cloning model and analyze the pure literal algorithm, which is simpler than that of the unit clause algorithm used in [6] (Actually, in [6], it is conjectured that the "pure-literal" rule should give the same result using an alternative analysis.). The Poisson cloning model has been introduced in [13] to simplify analysis of certain algorithms with branching process natures. The model turns out to be almost the same as the uniform model with the same (mean) density.

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1 Introduction

There is so much interest at present to obtain phase transitions of probabilistic properties in physics, mathematics, and more recently in computer science. Coppersmith, Gamarnik, Hajiaghayi and Sorkin [6] considered the phase transition in random instances of optimization problems, in particular random MAX k-SAT and random MAX CUT. This work has been considered empirically [17] and extended to other optimizations problems such as approximating the chromatic number of random graphs [5]. The results for MAX k-SAT, $k \geqslant 3$, has been further sharpened by Achlioptas, Naor and Peres [1]. In the vein of the work due to Coppersmith, Gamarnik, Hajiaghayi and Sorkin [6], we mainly consider random MAX 2-SAT in this paper.

1.1 The model and related work

The random MAX k-SAT model can be considered as an extension of two well-known concepts of random k-SAT and MAX k-SAT. Below we consider each of these concepts separately and mention some of the related work briefly.

We denote by $\mathcal{F}_k(n, m)$ the set of all formulas with n variables and $m = cn$ (c is called the *density*) clauses, where each clause is proper, i.e., consisting k distinct variables, each of which may be complemented or not, and clauses may be repeated. By this definition, choosing a random formula $F_k(n, m) \in \mathcal{F}_k$ is equivalent to choosing m clauses uniformly at random, with replacement, from all $2^k \binom{n}{k}$ proper clauses. The random formula $F_k(n, p)$, in which each of $2^k \binom{n}{k}$ clauses independently appears with probability p, has been considered too and it behaves very similar to $F_k(n, m)$ with $m = p2^k \binom{n}{k}$.

It is conjectured that for any $k \geqslant 2$, there exists a constant c_k such that for all $\varepsilon > 0$, if $c \leqslant c_k - \varepsilon$ then with high probability the random formula with density c is satisfiable and if $c \geqslant c_k + \varepsilon$ then with high probability, the random formula is unsatisfiable. Calculating the value of c_k, besides being a combinatorial challenge, has appeared as an approach for better understanding of how the space of solutions of random formulas is structured. This has important practical applications in designing efficient algorithms for solving SAT formulas. Chvátal and Reed [4], Geordt [9] and Fernandez de la Vega [8] independently proved the above conjecture for $k = 2$ by showing that $c_2 = 1$ is the right constant for random 2-SAT formulas. The result was further sharpened by Bollobas, Borgs, Chayes, Kim and Wilson [3] who determined the "scaling window": For $c_2 = 1 + \lambda n^{-1/3}$, if $\lambda \to -\infty$, then the random formula is satisfiable with high probability (w.h.p.), and if $\lambda \to \infty$ it is not satisfiable with high probability. The exponent $1/3$ is shown to be the largest with the property. For $k > 2$, not only the value of c_k remains unknown, but even its existence has not been established yet. Recently, Achlioptas and Peres [2] proved the general lower bound $2^k \ln 2 - O(k)$ for any fixed value of $k \geqslant 2$. For the special case of $k = 3$, it is conjectured that there exists a critical threshold density c_3 around 4.2. The best current upper bound $c_3 \leqslant 4.506$ is due to Dubois, Boufkhad and Mandler [7] and the best lower bound 3.52 is due to Hajiaghayi and Sorkin [10] and Karporis, Kirousis and Lalas [12].

Having briefly surveyed random k-SAT, let us similarly consider MAX k-SAT. The MAX 2-SAT problem is NP-hard to solve exactly, it is hard even to approximate within a factor of $21/22$ [11]. A random assignment satisfies an expected $3/4$ths of the clauses. Using this fact and a simple derandomization technique, one can obtain a $3/4$ approximation for MAX 2-SAT. The best approximation ratio for MAX 2-SAT is 0.940 due to Lewin, Livnat and Zwick [15]. For MAX 3-SAT again a simple random assignment algorithm, which can be derandomized, gives a $7/8$ approximation, which is tight unless $P = NP$ [11].

Finally, having definitions of random k-SAT and MAX k-SAT, we consider random MAX k-SAT. For a given formula F, let $F(\vec{X})$ be the number of clauses satisfied by $\vec{X}$. The problem of MAX k-SAT asks for $\max F \doteq \max_{\vec{X}} F_k(\vec{X})$. We define $\operatorname{comax} F$ to be $m - \max F$ for a formula F with m clauses.

1.2 The Poisson cloning model

In the random 2-SAT model $F(n, p)$ with $p = \Theta(1/n)$, the degrees of literals are almost i.i.d Poisson random variables with mean $c = 2pn + O(1/n)$, where the degree of a literal is the number of clauses containing it. Though this fact is useful to understand the nature of the model, it has not been possible to fully utilize properties of i.i.d Poisson random variables. For example, the distribution of the number of variables that appear in the formula neither as itself nor as its complement is very close to the binomial distribution $B(n, e^{-2c})$. In a rigorous proof, however, one has to say how close the distribution is, and keep tracking the effect of the small difference to the next processes or computations, which are not needed if the degrees are exactly i.i.d Poisson. Since these kinds of small differences occur almost everywhere in the analysis of many algorithms, they make rigorous analysis significantly difficult, if not impossible. As an approach to minimize such non-essential processes and computations, the second author [13] introduced a model for the random 2-SAT problem (more generally for the k-SAT problem as well as for the random graphs), called the Poisson cloning model, in which all degrees are i.i.d Poisson random variables: First take i.i.d Poisson $p(2n-1)$ random variables d_y's indexed by all literals y. Take d_y copies, or clones, of each literal y and generate a uniform random perfect matching on the set of all clones. This is possible only if $\sum d_y$ is even. If the sum is odd, one may arbitrarily take one clone to make a 1-clause consisting of its underlying literal, and generate a uniform random perfect matching on the set of remaining clones. A clause $(y_1 \vee y_2)$ is in the formula if a clone of y_1 is matched to a clone of y_2.

A literal is called pure in a formula if its degree is not 0 while the degree of its negation is 0. The pure literal rule is an algorithm that keeps setting pure literals 'TRUE' and removing the clauses containing them. After showing that the Poisson cloning model is essentially the same as the classical model [13], it was possible to analyze the pure literal rule [14], which, in particular, yields the following theorem:

Theorem 1 ([14]) *Let $c = 1 + \varepsilon$ with $n^{-1/3} \ll \varepsilon < 0.01$. Then, with high probability, the pure literal rule applied to $F(n, cn)$ stops leaving $\Theta(\varepsilon^2 n)$ type $(1, 1)$ variables, $\Theta(\varepsilon^3 n)$ type $(2, 1)$ or $(1, 2)$ variables, and $O(\varepsilon^4 n)$ clones of other type variables, where a variable x is of type (i, j) if $d(x) = i$ and $d(\bar{x}) = j$. Moreover, the residual formula is the uniform random formula conditioned on the degree sequence.*

The theorem is actually proven for $F(n, p)$ with $2pn = cn$. It is well-known that the two models $F(n, p)$ and $F(n, cn)$ share most properties such as the one described in the theorem. The uniform random formula conditioned on the degree sequence $\{d(y)\}$ may be generated by a similar way the Poisson cloning model is generated. The probability space must be understood as the conditional space on the event that all clauses are proper. However, since the event occurs with probability uniformly bounded below from 0, provided c is bounded (see e.g. [18]), all events that occur asymptotically almost surely in the non-conditioned space occur asymptotically almost surely in the uniform random formula.

1.3 Our results

For $c = 1 + \varepsilon$ with $\varepsilon \gg n^{-1/3}$, we prove that

$$\max F(n, cm) = (1 + \varepsilon - \Theta(\varepsilon^3))n,$$

with high probability. First, this improves the known upper bound $(1 + \varepsilon - \Omega(\varepsilon^3/\ln(1/\varepsilon)))$ due to Coppersmith, Gamarnik, Hajiaghayi and Sorkin [6]. Here we use the Poisson cloning model and a more careful analysis. To obtain the lower bound $(1 + \varepsilon - O(\varepsilon^3))n$, we present a simple algorithm partially consisting of the pure literal rule and resolutions. Coppersmith, Gamarnik, Hajiaghayi and Sorkin obtain the same lower bound using a more complicated "unit-clause" rule and conjecture that the "pure-literal" rule should give the same result using an alternative analysis. Finally, we present a simple "unified approach" for random MAX 2-SAT for general random formulas with $m = cn$ clauses, which not only obtains the tight lower bound for small densities, i.e., $c = 1+\varepsilon$, but also provides the tight lower bound $(3/4c + \Theta(\sqrt{c}))n$ for large c's. In addition using the Poisson cloning model, we can analyze this algorithm for a particular value of c. Finally, we make some progress on a result of Scott and Sorkin on faster algorithms for MAX CSP.

2 Tight bounds

2.1 Upper Bound

In this subsection, we present an upper bound for the case that the density is close to 1. This upper bound matches the algorithmic lower bound that we will obtain in the next subsection.

Theorem 2 *There exists absolute constants α_0 and ε_0, such that for any ε with $n^{-1/3} \ll \varepsilon < \varepsilon_0$, $f(n, (1+\varepsilon)n) \leqslant (1 + \varepsilon - \alpha_0\varepsilon^3)n$.*

Proof: We will use a similar argument used in [14]. By Theorem 1 and the statement mentioned just after the theorem, it is enough to consider the probability space generated by the uniform random perfect matching on all clones for $\Theta(\varepsilon^2 n)$ type $(1,1)$ variables, $\Theta(\varepsilon^3 n)$ type $(1,2)$ or $(2,1)$ variables and $O(\varepsilon^4 n)$ clones of other type variables. We first take resolutions of all variables of type $(1,1)$, that is, keep replacing the two clauses $(x \vee y), (\bar{x} \vee z)$ by $(y \vee z)$ for type $(1,1)$ variables x. The clauses like $(x \vee \bar{x})$ for type $(1,1)$ variables x just disappear. Thus, there remains no variable of type $(1,1)$ after all, and the types of other variables remain the same. It is easy to check that such resolutions do not change the maximum number of satisfiable clauses of the formula.

To make our estimation even simpler, the following procedure is taken. Remove all clauses containing variables of types other than $(1,2)$ and $(2,1)$. Clauses containing negations of such variables are removed too. This produces at most $O(\varepsilon^4 n)$ pure literals since, by removing one clause, at most one variable of type $(1,2)$ or $(2,1)$ changes its type. Considering the random perfect matching on the remaining clones, a pure clone can be matched to either another pure clone or a clone of a literal of a type $(1,2)$ or $(2,1)$ variable. Thus, after removing the clause containing the pure clone, no new pure clone is produced with probability at least $2/3$ and 2 new pure clones are produced with the other probability. In other words, at each step, the number of pure clones decreases at least by $1/3$, in expectation, provided that a resolution is taken as soon as a variable becomes type $(1,1)$. Thus, with high probability, all pure clones disappear within $O(\varepsilon^4 n)$ steps and there are $\Theta(\varepsilon^3) - O(\varepsilon^4 n)$ variables left, all of which are of type $(1,2)$ or $(2,1)$.

Suppose there are b such variables. Then $b = \Theta(\varepsilon^3) - O(\varepsilon^4 n) = \Theta(\varepsilon^3)$ assuming ε is small enough. Changing the roles of x_i and $\bar{x}_i$ if necessary, we may assume that all b variables are of type $(2,1)$. It is now enough to show that the formula F induced by the uniform random perfect matching on $3b$ clones has the property comax$F > \delta b$ with high probability, for a universal constant δ in the range $0 < \delta < 0.1$. For the proof, assignments will be regarded as $0,1$ vectors of length b so that the i^{th} coordinates of them tell the truth value of the i^{th} variable x_i. An assignment with αb 0's yields $2\alpha b + (1-\alpha)b$ clones that are set to be 0. These clones are to be called negative. The other clones are set to be 1 and will be called positive. An assignment is a satisfying assignment if and only if there is no edge connecting two negative clones in the uniform random perfect matching. We call such an edge *bad*. A clause corresponding to a bad edge is also called *bad*.

If $\mathrm{comax}F \leqslant \delta b$, then there is an assignment that yields δb or less bad clauses. Among all such assignments, we may take one with maximum number of 1's. Those assignments are called maximal. Suppose an assignment $s = (s_i)$ is maximal with $\mathrm{comax}F(s) \leqslant \delta b$. Then, for i with $s_i = 0$, the clone of $\bar{x}_i$ must be matched to a negative clone. Otherwise, one may set $s_i = 1$ without increasing $\mathrm{comax}F$ since $\bar{x}$ is contained in only one clause. Provided s has αb 0's, the number S of negative clones is $2\alpha b + (1-\alpha)b = (1+\alpha)b$ and the number T of positive clones is

$2(1-\alpha)b+\alpha b=(2-\alpha)b$. If s is maximal with $\mathrm{comax}F(s)\leqslant\delta b$, then there are at most δb bad clauses and $\bar{x}_i$-clones with $s_i=0$ must be matched to negative clones. Clearly, the number T^* of such $\bar{x}_i$-clones is αb. (Notice that T^* is determined if s is given.) If $\alpha\geqslant 0.8$, then $S\geqslant 1.8b$ and $T\leqslant 1.2b$ imply that there are at least $0.3b$ bad clauses, which is not possible as $\delta<0.1$.

Let $0\leqslant\alpha\leqslant 0.8$ and R be the number of bad clauses with respect to s, $R=0,1,...,\delta b$. Given $R=r$, there are $\binom{S}{2r}$ different collections of clones consisting of the bad edges. The remaining $S-2r$ negative clones must be matched to positive clones, and the T^* positive clones mentioned above must be matched to negative clones. Since the number of perfect matchings on m vertices for even m is

$$(m-1)!!=\frac{m!}{2^{m/2}(m/2)!},$$

we have that

$$\begin{aligned} P(s) &:= \Pr[s \text{ is maximal with } \mathrm{comax}F(s)\leqslant\delta b] \\ &\leqslant \sum_{r=0}^{\delta b}\frac{\binom{S}{2r}(2r-1)!!\binom{T-T^*}{S-2r-T^*}(S-2r)!(T-S+2r-1)!!}{(S+T-1)!!} \end{aligned}$$

where

$$S=(1+\alpha)b,\ \ T=(2-\alpha)b,\ \ T^*=\alpha b,$$

provided s has αb 0's. Using Stirling formula and $\delta<0.1$,

$$\begin{aligned} P(s) &\leqslant b\exp\Big((1+\alpha)bH(2\delta)+2(1-\alpha)bH(\tfrac{1}{2(1-\alpha)}+O(\delta)) \\ &+ S\ln\frac{S}{S+T}+\frac{T-S}{2}\ln\frac{T-S}{S+T}+O(\delta b)\Big) \\ &= b\exp\Big(2(1-\alpha)bH(\tfrac{1}{2(1-\alpha)})+(1+\alpha)b\ln\frac{1+\alpha}{3} \\ &+ \frac{(1-2\alpha)b}{2}\ln\frac{1-2\alpha}{3}+O(\delta b\ln(1/\delta))\Big), \end{aligned}$$

where the entropy function $H(a)=-a\ln a-(1-a)\ln(1-a)$. Counting the number of s with αb 0's, i.e., $\binom{b}{\alpha b}=O(\exp(bH(\alpha)))$, it is not difficult to see that

$$H(\alpha)+2(1-\alpha)H(\tfrac{1}{2(1-\alpha)})+(1+\alpha)\ln\frac{1+\alpha}{3}+\frac{1-2\alpha}{2}\ln\frac{1-2\alpha}{3}<-0.02.$$

Thus

$$\Pr[\mathrm{comax}F\leqslant\delta b]\leqslant be^{-0.02b}=e^{-(0.02+o(1))b}$$

for large enough b and small enough δ. □

2.2 Lower Bounds

In this subsection, we present algorithms which obtain tight lower bounds for random formulas with bounded density c. However, we mainly focus on the case in which c is small or large, since in this case the problem is much more interesting.

Theorem 3 *There is an algorithm that with high probability satisfies at least $(1+\varepsilon-\Theta(\varepsilon^3))n$ clauses for the random formula $F(n,(1+\varepsilon)n)$, where $\varepsilon \gg n^{-1/3}$. (Note that this is a tight bound according to Theorem 2.)*

Proof: Our algorithm is very simple and has two steps. First, run the pure-literal rule. Next, take resolutions of type $(1,1)$ variables. By theorem 1, after running the first step, i.e., applying pure literal rule, we are left with $\Theta(\varepsilon^2 n)$ type $(1,1)$ variables, $\Theta(\varepsilon^3 n)$ type $(2,1)$ or $(1,2)$ variables, and $O(\varepsilon^4 n)$ clones of other type variables. It is easy to see that after the second step, i.e., the resolutions of $(1,1)$-degree variables, the minimum number of unsatisfied clauses will remain the same. Suppose in the worst case, all clauses after the resolutions become unsatisfied. Since the number of clauses in this step is at most $3\Theta(\varepsilon^3 n)+\Theta(\varepsilon^4 n)=\Theta(\varepsilon^3 n)$, w.h.p. in total we do not satisfy at most $\Theta(\varepsilon^3 n)$ clauses, as desired. $\square$

Theorem 4 *There is an algorithm that with high probability satisfies at least $(\frac{3}{4}c+\Theta(\sqrt{c}))n$ clauses for $F(n,cn)$, where c is a large enough constant.*

Proof: Again the algorithm is very simple. It sets a variable True if the number of its positive appearances is greater than or equal to the number of its negative appearances; it sets False otherwise.

With high probability, the number of variables of type (i,j) of a random 2-SAT formula in $F(n,cn)$ is $n_{i,j}=e^{-2c}\frac{c^i}{i!}\frac{c^j}{j!}n+o(n)$. Now if we consider the Poisson cloning model corresponding to $F(n,cn)$, the above algorithm sets $2\alpha m$ clones positive and $2\beta m$ clones negative ($m=cn$), where $\alpha=\frac{\sum \max(i,j)n_{ij}}{2cn}$ and $\beta=1-\alpha=\frac{\sum \min(i,j)n_{ij}}{2cn}$. Since a random formula in $F(n,cn)$ corresponds to a perfect matching on the $2m$ clones with the aforementioned property, we may expect that the number of edges connecting two negative clones, which are corresponding to unsatisfied clauses, is at most $\beta^2 m+m^{2/3}$ with high probability. This can be actually proven using the second moment method: For each u of the $2\beta m$ negative clones, let X_u be the indicator random variable for the event that u is matched to a negative clone. Then, $E[X_u]=\frac{2\beta-1}{2m-1}=\beta+O(1/m)$, and, for distinct negative clones u,w, it is easy to see that $E[X_uX_w]=\frac{1}{2m-1}+\beta^2+O(1/m)=\beta^2+O(1/m)$, and $\sum_u E[X_uX_w]-\beta^2=O(m)$. The second moment method yields

$$\Pr\Big[\sum_u (X_u-\beta)\geqslant m^{2/3}\Big]\leqslant\frac{E[(\sum_u(X_u-\beta))^2]}{m^{4/3}}=O(\frac{m}{m^{4/3}})=O(m^{-1/3}).$$

As the number of unsatisfied clauses is $\frac{1}{2}\sum_u X_u$, the desired result follows.

For large enough c, with high probability, all but $o_c(n)$ variables are of types $(c+\Theta(\sqrt{c}), c-\Theta(\sqrt{c}))$ or $(c-\Theta(\sqrt{c}), c+\Theta(\sqrt{c}))$ and thus $\beta \leqslant \frac{1-\Theta(c^{-1/2})}{2}$, with high probability. It means the number of unsatisfied clauses is $cn - (\frac{1-\Theta(c^{-1/2})}{2})^2 cn = (\frac{3}{4}c + \Theta(\sqrt{c}))n$ as desired. $\square$

Finally, we are ready to demonstrate our unified approach for random 2-SAT formulas with bounded density c.

Algorithm A

Input: A random 2-CNF formula with n variables and $m = cn$ clauses.

// step 1 **while** there exists an unset literal y whose negation $\bar{y}$ has no appearance
    set literal y True
// step 2 **while** there exists a $(1,1)$-degree variable x
    replace two clauses (y, x) and $(\bar{x}, z)$, in which x and $\bar{x}$ appear, respectively, with a clause (y, z)
// step 3 **for** each remaining (i, j)-degree variable x
    set x True if $i \geqslant j$ and False otherwise
set the values of $(1,1)$-degree variables resolved in Step 2 appropriately
to minimize the number of unsatisfied clauses.

In fact, it is not hard to observe that Algorithm A above stochastically dominates the algorithm mentioned in the proof of Theorem 3, since the latter only runs steps 1 and 2 of the former. In addition, Algorithm A is essentially the same as (actually slightly better than) the algorithm in the proof of Theorem 4, as the latter only runs Step 3 of the former and for large c the pure literal rule stops almost immediately. Thus Algorithm A at least satisfies the lower bounds mentioned in Theorems 3 and 4. In general, for any random formula with bounded density c, if we run the algorithm A, after Steps 1 and 2, we can compute α (and thus β), n^*(the number of remaining variables), m^*(the number of remaining clauses) with high probability using the Poisson cloning model. It means we can compute the number of unsatisfied clauses reported from algorithm A for any bounded density c almost exactly with high probability. The details of this computation rely on the analysis of the pure literal rule via the Poisson cloning model. In addition for any random formula with bounded density c, we can compute the number of unsatisfied clauses reported from algorithm A almost exactly with high probability. The proof is omitted as it is straightforward (modulo [14]) and somewhat tedious.

3 Discussion and Further Results

Scott and Sorkin [16] show that for any $c \leqslant 1/2$, MAX 2-SAT for a random formula in $\mathcal{F}(n, cn)$ can be solved in polynomial expected time. They conjecture that we should be able to extend this result through the scaling window, i.e., $c = 1+\lambda n^{-1/3}$. Interestingly steps 1 and 2 of their algorithm is exactly the same as those of Algorithm A (i.e., using pure literal rule and then removing $(1,1)$-degree variables). Then they use backtracking for the rest of the formula. However, they are only able to show the expected polynomial time when $c \leqslant 1/2$. As we mentioned for $c = 1 + \lambda n^{-1/3}$, with high probability, the number of remaining variables after steps 1 and 2 is in $\Theta(\lambda^3)$. Thus their algorithm, which is in fact our algorithm by replacing the last step by a naive backtracking algorithm, has polynomial running time when $c \leqslant 1 + O(n^{-1/3} \ln^{1/3} n)$, with high probability. We did not attempt to estimate the expectation in this paper though.

References

[1] D. ACHLIOPTAS, A. NAOR, AND Y. PERES, *On the maximum satisfiability of random formulas*, in 44th Annual Symposium on Foundations of Computer Science (Cambridge, MA, 2003), IEEE Comput. Soc. Press, Los Alamitos, CA, 2003, pp. 362–371.

[2] D. ACHLIOPTAS AND Y. PERES, *The threshold for random k-sat is 2k (ln 2 - o(k))*, in Proceedings of the thirty-fifth ACM symposium on Theory of computing, ACM Press, 2003, pp. 223–231.

[3] B. BOLLOBAS, C. BORGS, J. T. CHAYES, J. H. KIM, AND D. B. WILSON, *The scaling window of the 2-SAT transition*, Random Struct. Algorithms, 18 (2001), pp. 201–256.

[4] V. CHVÁTAL AND B. REED, *Mick gets some (the odds are on his side)*, in 33th Annual Symposium on Foundations of Computer Science (Pittsburgh, PA, 1992), IEEE Comput. Soc. Press, Los Alamitos, CA, 1992, pp. 620–627.

[5] A. COJA-OGHLAN, C. MOORE, AND V. SANWALANI, *MAX k-CUT and approximating the chromatic number of random graphs*, in 30th International Colloquium Automata, Languages and Programming, Springer, Berlin, 2003, pp. 200–211.

[6] D. COPPERSMITH, D. GAMARNIK, M. T. HAJIAGHAYI, AND G. B. SORKIN, *Random MAX SAT, random MAX CUT, and their phase transitions*, Random Structures Algorithms, 24 (2004), pp. 502–545.

[7] O. DUBOIS, Y. BOUFKHAD, AND J. MANDLER, *Typical random 3-SAT formulae and the satisfiability threshold*, in 11th Annual ACM-SIAM Symp. on Disc. Alg. (San Francisco, CA, 2000), ACM, New York, 2000, pp. 126–127.

[8] W. FERNANDEZ DE LA VEGA, *On random 2-SAT.* Manuscript, 1992.

[9] A. GOERDT, *A threshold for unsatisfiability*, J. Comput. System Sci., 53 (1996), pp. 469–486.

[10] M. HAJIAGHAYI AND G. B. SORKIN, *The satisfiability threshold of random 3-SAT is at least 3.52*, Research Report RC22942, IBM, October 2003.

[11] J. HASTAD, *Some optimal inapproximability results*, J. ACM, 48 (2001), pp. 798–859.

[12] A. C. KAPORIS, L. M. KIROUSIS, AND E. LALAS, *Selecting complementary pairs of literals*, in Electronic Notes in Discrete Mathematics, E. Kranakis and L. Kirousis, eds., vol. 16, Elsevier, 2003.

[13] J. H. KIM, *Poisson cloning model for random graph*. manuscript, http://research.microsoft.com/theory/jehkim/papers/PCMRG.pdf, 2005.

[14] ——, *Finding cores of random 2-sat formulae via poisson cloning*. manuscript, http://research.microsoft.com/theory/jehkim/papers/PCSat.pdf, 2006.

[15] M. LEWIN, D. LIVNAT, AND U. ZWICK, *Improved rounding techniques for the MAX 2-SAT and MAX DI-CUT problems*, in Proc. of the 9th International Integer Programming and Combinatorial Optimizaiton Conference, 2002, pp. 67–82.

[16] A. D. SCOTT AND G. B. SORKIN, *Faster algorithms for MAX CUT and MAX CSP, with polynomial expected time for sparse instances*, in Proc. 7th International Workshop on Randomization and Approximation Techniques in Computer Science, 2003, pp. 382–395.

[17] H. SHEN AND H. ZHAN, *An empirical study of MAX-2-SAT phase transitions*, in Electronic Notes in Discrete Mathematics, E. Kranakis and L. Kirousis, eds., vol. 16, Elsevier, 2003.

[18] N. C. WORMALD, *Models of random regular graphs*, in Surveys in combinatorics, 1999 (Canterbury), vol. 267 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1999, pp. 239–298.