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A note on the bounded fragmentation property and its applications in network reliability

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Abstract

In this paper, we introduce a new property for graphs called *bounded fragmentation*, by which we mean after removing any set of at most k vertices the number of connected components is bounded only by a function of k. We demonstrate how bounded fragmentation can be used to measure the reliability of a network and introduce several classes of bounded fragmentation graphs. Finally, we pose some open problems related to this concept.

Keywords: Bounded fragmentation; Connectivity; Network reliability

1. Introduction

This paper is devoted to a new concept called bounded fragmentation. In fact, this property can be considered as a generalization of connectivity and can be applied to measure the reliability and robustness of a network. In addition, this concept has been used implicitly in other areas such as solving the subgraph isomorphism problem for special kinds of graphs [2, 4, 5].

This paper is organized as follows. We start with the terminology and the formal definition of bounded fragmentation in Section 2. In Section 3, we explain how this property can be applied in network reliability. We present some classes and properties which guarantee a graph G to be a bounded fragmentation graph in Section 4. In Section 5, we consider the number of edges of a bounded fragmentation graph. Finally in Section 6, we conclude with a list of open problems and potential extensions for future work.

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2. Basic definitions

We assume the reader is familiar with general concepts of graph theory such as trees and planar graphs. The reader is referred to standard references for an appropriate background [1].

Our graph terminology is as follows. All graphs are finite, simple and undirected, unless indicated otherwise. A graph G is represented by $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We denote an edge e in a graph G between u and v by $\{u, v\}$. The maximum degree of G is denoted by $\Delta(G)$ and the minimum degree of G is denoted by $\delta(G)$. An *n-clique* (K_n) is a graph G with n vertices in which every pair of vertices is connected by an edge. A graph G is represented by $K_{n,m}$ if its vertices can be partitioned into sets V_1 and V_2 such that $|V_1| = n$, $|V_2| = m$ and edge $\{u, v\} \in E(G)$ if and only if $u \in V_1$ and $v \in V_2$ or vice versa.

A graph $G' = (V', E')$ is a *subgraph of* G if $V' \subseteq V$ and $E' \subseteq E$. A graph $G' = (V', E')$ is an *induced subgraph of* G, denoted by $G[V']$, if $V' \subseteq V$ and E' contains all edges of E which have both end vertices in V'.

The set of components of a graph G is represented by $\mathcal{C}(G)$, where each element of $\mathcal{C}(G)$ is a connected graph. The graph resulting from removal of a set S of vertices and all adjacent edges from G is denoted by $G[V - S]$. A set S is called a *separator* if $|\mathcal{C}(G[V - S])| > 1$. For $k > 0$, graph G is called *k-connected* if every separator has size at least k.

Definition 1. A graph G is a $(k, g(k))$-*bounded fragmentation* graph if $|\mathcal{C}(G[V - S])| \leq |g(k)|$ for every $S \subseteq V(G)$ of size at most k, where g is a function of k. A graph G is a *totally* $g(k)$-*bounded fragmentation* graph if it is a $(k, g(k))$-bounded fragmentation graph for all $0 \leq k \leq n$.

Here, we note that by our definition the number of components of $G[V - S]$ is constant when S has at most k vertices for some constant k. We mainly focus on this property in the rest of this paper.

3. Some applications of bounded fragmentation graphs

Connectivity can be considered as a measure of the reliability of a network. We suppose a network N is represented by an undirected graph G, in which two computers, namely nodes of the network, can communicate if and only if there is a path in G from one to the other. If G is k-connected, after removing at most $k-1$ vertices of G, the rest of G (which has $n - k + 1$ vertices) is still connected. This means that if at most $k - 1$ nodes of the network N fail, the rest of the nodes of the network can communicate with each other.

Bounded fragmentation can play a similar role in the reliability of a network. If G is a $(k, g(k))$-bounded fragmentation graph, after removing at most k vertices we have at least one component which has $\Omega(n)$ vertices. The reason is that after removing at most k vertices the rest of the nodes fall into at most a constant number of connected components ($g(k)$) and thus one component has at least $\Omega(n)$ vertices. Thus, after the failure of at most $k - 1$ nodes of N, $\Omega(n)$ nodes in the rest of N (and not necessarily $n - k$) still can communicate with each other. Using these facts, bounded fragmentation can be considered as a generalization of connectivity.

Bounded fragmentation can also have another application in the reliability of a network. Suppose that we need to repair the network N temporarily by adding several links between the current nodes of the network (not by adding any new node because of its high cost) when the number of failing nodes in the network is at most constant k. If G is a $(k, g(k))$-bounded fragmentation graph, then we can simply repair the network by adding at most $g(k) - 1$ numbers of links, which is constant. Here, after removing the failing nodes, we find the connected components of G in $O(|V(G)|)$ time. Then we can connect these at most $g(k)$ connected components in the form of a tree, by adding at most $g(k) - 1$ edges among them. These two simultaneous properties of bounded fragmentation graphs cause their corresponding networks to be more reliable and robust.

4. Bounded fragmentation graphs

In this section, we focus on classes of bounded fragmentation graphs.

Lemma 2. *Connected graphs with constant maximum degree c are totally ck-bounded fragmentation graphs.*

Proof. The proof follows from the fact that if $\Delta(G) = c$, after removing any k vertices, $0 \leq k \leq n$, the number of connected components is at most $g(k) = ck$. $\square$

Theorem 3. *If graph G has a maximum independent set of constant size c, then it is a totally c-bounded fragmentation graph.*

Proof. For any set $S \subseteq V(G)$ of size k, $0 \leq k \leq n$, at least one vertex from each connected component of $G[V - S]$ is contained in any maximum independent set. Since the size of the maximum independent set is bounded above by c, the number of connected components is bounded above by c, as well. Thus G is a totally c-bounded fragmentation graph. $\square$

In fact, we can generalize the approach used in Theorem 3 to other maximization problems.

The proof of the following lemma is trivial and hence omitted.

Lemma 4. *Let G be a graph with minimum degree $\delta(G) \geq k + h - 1$ for two positive integers k and h. Removing any set S of size at most k cannot produce any component with size less than h.*

Theorem 5. *Let P be a maximization problem which has a non-zero solution on every connected graph of size at least h, where h is a non-negative constant. We also assume P is additive on components. For any non-negative integer k, if P on a graph G has a maximum solution of constant size c and $\delta(G) \geq k + h - 1$ then G is a (k, c)-bounded fragmentation graph.*

Proof. By Lemma 4, we know that removing any set of size at most k cannot generate any connected component with size less than h. Using our assumption, P has a non-zero solution in each component. The number of connected components is at most c, since otherwise using the maximum solution of each component, we can construct a maximal solution of the whole graph which is of size greater than c. □

For example, the maximum matching problem is a problem which has a non-zero solution on every connected graph of at least two vertices.

Corollary 6. *For any non-negative integer k, if connected graph G has a maximum matching of constant size c and minimum degree at least $k+1$, i.e. $\delta(G) \geq k+1$, then it is a (k, c)-bounded fragmentation graph.*

The reader is referred to Garey and Johnson [3] and Yannakakis [8] to see more problems of this kind.

Example 7. A complete bipartite graph $K_{n-k-1,k+1}$, where $n \geq 2k+2$, has minimum degree $k+1$ and a maximum matching of size $k+1$. Hence it is a $(k, k+1)$-bounded fragmentation graph.

The result of Theorem 5 can be generalized to other problems which are not necessarily maximization problems.

Definition 8. *Covering a graph by at most m vertex-disjoint paths* means the vertices of a graph can be partitioned into m subsets such that for each set S, there exists a path in a graph that contains exactly the vertices in S.

Lemma 9. *Graphs whose vertices can be covered by at most c vertex-disjoint paths are totally $(k+c)$-bounded fragmentation graphs.*

Proof. The removal of a vertex from a path splits the path into at most two sub-paths and thus at most two connected components. Thus, removing any k vertices, $0 \leq k \leq n$, can add at most k connected components. Thus, we have at most $k+c$ connected components. □

Example 10. Consider a Hamiltonian graph F_n which is constructed from a path of length n by connecting one of its vertices to all its non-neighbors. Since vertices of every Hamiltonian graph can be covered by one path, F_n is a totally $(k+1)$-bounded fragmentation graph.

We can also relate bounded fragmentation to other properties of graphs.

Theorem 11. *A planar 3-connected graph is a totally $2k$-bounded fragmentation graph.*

Proof. Suppose we removed a set S of k vertices. Without loss of generality, we assume that no edge can be added to H connecting two vertices in S. Then each component of $H-S$ must occupy a distinct face in the planar embedding of S induced by a unique embedding of H. Since the number of faces of S is at most $2k$ by Euler's Formula [1], we obtain the desired result. □

Clearly, a complete graph K_n is a totally 1-bounded fragmentation graph. Intuitively, graphs with large minimum degree are bounded fragmentation graphs. In Theorem 13, we derive an exact bound on the minimum degree of a graph that guarantees the graph to be a bounded fragmentation graph.

Lemma 12 ([7]). *Let G be a simple n-vertex graph such that for two non-negative integers h and d, $n \geq h + d$ and $\delta(G) \geq \frac{n+d(h-2)}{d+1}$. If $G - S$ has more than d components, then $|S| \geq h$. The bound is tight: there exists a graph with $\delta(G) = \lfloor \frac{n+d(h-2)-1}{d+1} \rfloor$ such that $G - S$ with $|S| < h$ has more than d components.*

Theorem 13. *For each constant d, graphs with $\delta(G) \geq \frac{n+d(k-1)}{d+1}$ are (k, d)-bounded fragmentation graphs where $0 \leq k \leq n - d - 1$.*

Proof. By Lemma 12, for $h = k + 1$, after removing any set S with $|S| \leq h - 1 = k$ the graph G has at most d components where $n \geq h+d = k+1+d$. Thus it is a (k, d)-bounded fragmentation graph. $\square$

5. Numbers of edges of bounded fragmentation graphs

As discussed before, bounded fragmentation is a measure in reliability of a network. However, in network design, it is beneficial to have a linear number of communication lines. Thus, an interesting question is whether it is possible to have a linear number of edges and still a graph of bounded fragmentation. The answer to this question is affirmative. Clearly, graphs with constant maximum degree and planar graphs have linear numbers of edges. As shown in Examples 7 and 10, graphs with maximum matchings of constant size or graphs coverable by a constant number of vertex-disjoint paths can also have a linear number of edges.

However, the condition stated in Theorem 13 is valid only for graphs with quadratic numbers of edges. Graphs with constant maximum independent sets have quadratic numbers of edges. The proof follows from the fact that if a graph G has a constant maximum independent set c, its complement $\bar{G}$ has a constant maximum clique c. By Turán's theorem [6, 7], $\bar{G}$ has at most $(1 - 1/(c - 1))n^2/2$ edges. Thus G has a quadratic number of edges.

6. Conclusions and future work

In this paper, we introduced applications of bounded fragmentation graphs for networking and mentioned several instances of bounded fragmentation graphs. Here, we present some open problems that can be considered as possible extensions of this paper:

A naive algorithm for testing whether a graph G is (k, c)-bounded fragmentation, for constants k and c, is to check all subsets of vertices of size at most k and count the number of connected components. The running time of this algorithm is $O(n^{k+1})$. It might be possible to give an algorithm whose running time is $O(n^d)$, where d is a constant independent of k. A randomized approach might be another way to solve this problem.

In this paper, we introduced some properties which cause a graph to be bounded fragmentation. Finding other properties of this kind, especially those which impose a linear number of edges (if they exist), and finding an exact characterization of bounded fragmentation graphs are interesting questions. The relation between these properties and treewidth is also interesting, in particular when in solving subgraph isomorphism and minor containment, we search for graphs which are bounded fragmentation and have bounded treewidth (see [4, 5]). A path is a bounded fragmentation graph which has bounded treewidth. Graphs coverable with a constant number of vertex-disjoint paths and graphs with maximum constant degree are the only known classes of bounded fragmentation graphs which have bounded treewidth. Finding other classes with these properties is another possible extension of this paper.

Finally, all graphs introduced in this paper are $(k, O(k))$-bounded fragmentation. It would be instructive to determine whether there is any $(k, g(k))$-bounded fragmentation graph where $g(k)$ is not $O(k)$.

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