# THE ZEROS OF THE DAUBECHIES POLYNOMIALS

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*Dedicated to Gabor Szegö on the 100th anniversary of his birth*

Abstract. To study wavelets and filter banks of high order, we begin with the zeros of $\mathbf{B}_p(y)$. This is the binomial series for $(1-y)^{-p}$, truncated after $p$ terms. Its zeros give the $p-1$ zeros of the Daubechies filter inside the unit circle, by $z+z^{-1}=2-4y$. The filter has $p$ additional zeros at $z=-1$, and this construction makes it orthogonal and maximally flat. The dilation equation leads to orthogonal wavelets with $p$ vanishing moments. Symmetric biorthogonal wavelets (generally better in image compression) come similarly from a subset of the zeros of $\mathbf{B}_p(y)$.

We study the asymptotic behavior of these zeros. Matlab shows a remarkable plot for $p=70$. The zeros approach a limiting curve $|4y(1-y)|=1$ in the complex plane, which is $|z-z^{-1}|=2$. All zeros have $|y|\le 1/2$, and the rightmost zeros approach $y=1/2$ (corresponding to $z=\pm i$ ) with speed $p^{-1/2}$. The curve $|4y(1-y)|=(4\pi p)^{1/2p}\,|1-2y|^{1/p}$ gives a very accurate approximation for finite $p$.

The wide dynamic range in the coefficients of $\mathbf{B}_p(y)$ makes the zeros difficult to compute for large $p$. Rescaling $y$ by 4 allows us to reach $p=80$ by standard codes. This is "spectral factorization" of high order. The zeros at $z=-1$ stabilize the iteration of the filter in a multiresolution.

## 1. Introduction.

Figure 1 shows the zeros of a particular polynomial of degree 69. The polynomial is the binomial series for $(1-y)^{-70}$, truncated after 70 terms. There is a close connection between those zeros and the 140 coefficients associated with the Daubechies wavelets $D_{140}$. Our first goal was to find the curve along which the zeros seem to lie.

This is the case $p=70$ of the truncated binomial series for $(1-y)^{-p}$

$$\mathbf{B}_p(y)=1+py+\frac{p(p+1)}{2}y^2+\ldots+\binom{2p-2}{p-1}y^{p-1} \tag{1}$$

The natural question is the behavior of the zeros as $p\to\infty$. The outstanding contribution to problems of this type was by Szegö [8] in 1924, who studied the truncation of the exponential series. His limiting curve was $|ze^{1-z}|=1$, when the zeros are divided by $p$.

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For the truncated binomial $\mathbf{B}_p(y)$, $p > 2$, we first prove that every zero satisfies $|Y| < 1/2$ and $|4Y(1-Y)| > 2^{1/p}$. All the zeros lie outside the limiting curve $|4y(1-y)| = 1$. Their convergence to this curve $C = C_\infty$ is slowest near the point $y = 1/2$, and we give a more exact expression $Y \approx 1/2 + W/2\sqrt{p}$ for the location of the rightmost zero. We also find a curve $C_p$ that gives the positions of the other zeros to extra accuracy. The curve $C_p$ lies slightly outside $C_\infty$.

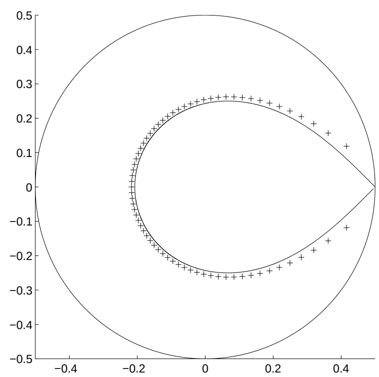


FIGURE 1. The zeros of $\mathbf{B}_{70}(y)$ are close to the curve $C_\infty$.

A note about the numerical computation of the zeros. Matlab creates the companion matrix whose characteristic polynomial is $\mathbf{B}_p(y)$. Then it finds the eigenvalues of that matrix. Without scaling, this breaks down at $p = 35$, because of the wide range in the coefficients of $\mathbf{B}_p(y)$. The first coefficient is 1, and by Stirling's formula, the coefficient of $y^{p-1}$ is

$$\binom{2p-2}{p-1} \approx \frac{\sqrt{2\pi(2p-2)}}{2\pi(p-1)} \frac{(2p-2)^{2p-2}}{(p-1)^{2p-2}} = \frac{4^{p-1}}{\sqrt{\pi(p-1)}} \tag{2}$$

The leading term $4^{p-1}$ suggests that the variable $4y$ is preferable to $y$. With this scaling, the Matlab computation remains accurate to $p = 80$. For larger $p$, a bifurcation (see Figure 2 ) occurs from roundoff error. The coefficient $\binom{p-1+i}{i}4^{-i}$ of $(4y)^i$ is numbered $b(p-i)$ by Matlab. Then $b(p) = 1$ and the sequence of coefficients is created recursively;

$$\text{for } i = p-1 : -1 : 1 \quad b(i) = b(i+1) * (2p-i-1)/(4*(p-i)) \tag{3}$$

The command "$Y =$ roots$(b)/4$" produces the approximate zeros $Y(1), \ldots, Y(p-1)$.

Experiments with other root-finding algorithms were less successful, even though working with the companion matrix is *a priori* surprising. A polynomial with repeated roots leads to a defective matrix (not diagonalizable). Algorithms based on Newton's method had difficulty with the accurate evaluation of $\mathbf{B}_p$ and $\mathbf{B}_p'$. Lang's algorithm (Lang and Frenzel [4]) is comparable to Matlab 'roots', and probably faster.

We now explain how the zeros of $\mathbf{B}_p(y)$ are connected to the coefficients $h(n)$ that generate Daubechies wavelets. It is important to note that the same zeros also lead to biorthogonal filters and symmetric wavelets (cf. Daubechies [2], or Strang and Nguyen [7]). The Daubechies wavelets have orthogonality but not symmmetry. The translates and dilates $w(2^j t - k)$ are an orthogonal basis for $L^2(\mathbf{R})$. But the reconstruction of a compressed image is better using symmetric biorthogonal wavelets $w$ and $\tilde{w}$.

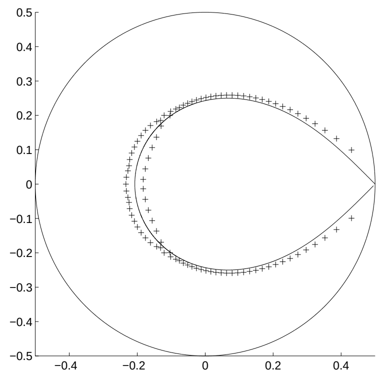

FIGURE 2. A bifurcation occurs from roundoff error, $p$=100.

## 2. The Daubechies Polynomials.

The wavelet coefficients or filter coefficients $h(n)$ are associated with the transfer function $H(z) = \sum_{n=0}^{N} h(n)z^{-n}$. The transpose filter with coefficients $h(-n)$ corresponds to $H(z^{-1})$. The product of the two filters yields a symmetric $P(z)$ that is nonnegative on the unit circle:

$$P(z) = H(z)H(z^{-1}) = \left(\sum_{n=0}^{N} h(n)z^{-n}\right)\left(\sum_{n=0}^{N} h(n)z^{n}\right) \tag{4}$$

The coefficients $h(n)$ of orthogonal filters and wavelets are chosen in two steps:

(1) Select $P(z)$ subject to $P(z) + P(-z) = 1$,
(2) Factor $P(z)$ into $H(z)H(z^{-1})$.

This " spectral factorization " is commonly done by computing the zeros of $P(z)$, which is the problem we study. The zeros come in pairs $Z$ and $Z^{-1}$. One member of each pair is assigned to $H(z)$—usually the one with $|Z| \leq 1$. The zeros on the unit circle have even multiplicity if and only if $P(z) \geq 0$ on the unit circle. Then this Fejér–Riesz factorization $P(z) = H(z)H(z^{-1})$ will succeed. The coefficients for biorthogonal wavelets come from other factorizations of the same polynomial. For symmetry, the roots $Z$ and $Z^{-1}$ go into the same factor. It is impossible to combine symmetry and orthogonality except in the special case

$$P(z) = \frac{1}{4}z^{-1} + \frac{1}{2} + \frac{1}{4}z = \left(\frac{1+z^{-1}}{2}\right)\left(\frac{1+z}{2}\right). \tag{5}$$

This has $P(z) + P(-z) = 1$. The coefficients $\frac{1}{2}, \frac{1}{2}$ in $H(z)$ lead to the Haar wavelet, which has the lowest possible accuracy $p = 1$.

The accuracy $p$ is determined by the number of zeros at $z = -1$. Thus Daubechies considered polynomials of the particular form

$$P(z) = \left(\frac{1+z^{-1}}{2}\right)^p \left(\frac{1+z}{2}\right)^p Q_p(z) \tag{6}$$

She chose the unique $Q_p(z) = cz^{p-1} + \cdots + cz^{-p+1}$ that achieves, with the lowest degree, the condition that gives perfect reconstruction:

$$P(z) + P(-z) = 1 \tag{7}$$

We refer to Daubechies [2] or Strang and Nguyen [7] for the proof that orthogonality of the wavelets requires this condition. The wavelets are constructed from the scaling function that solves the dilation equation

$$\phi(t) = 2\sum_{n=0}^{N} h(n)\ \phi(2t-n). \tag{8}$$

The main point for this paper is the connection of $Q_p(z)$ to $\mathbf{B}_p(y)$.

**Theorem 1** [cf. Daubechies [2], page 168]**.** *The change of variables $z+z^{-1} = 2-4y$ yields $Q_p(z) = \mathbf{B}_p(y)$. These are the minimum degree polynomials that produce $P(z)+P(-z) = 1$ or equivalently*

$$(1-y)^p\, \mathbf{B}_p(y) + y^p\, \mathbf{B}_p(1-y) = 1. \tag{9}$$

*Proof.* First we connect $y$ to $z$. The factor $[(1+z^{-1})/2]\,[(1+z)/2]$ in $P(z)$ is exactly $1-y$. The factor $[(1-z^{-1})/2]\,[(1-z)/2]$ is $y$. On the unit circle $z = e^{i\omega}$, the symmetric $Q_p(z)$ reduces to a polynomial in $\cos\omega$, and therefore to some polynomial $\mathbf{B}(y)$ in $y = (1-\cos\omega)/2$. Then $-z$ corresponds to $e^{i(\omega+\pi)}$, thus to $(1-\cos(\omega+\pi))/2 = 1-y$.

With $P(z)$ as in (6), the orthogonality condition (7) is now reduced to

$$(1-y)^p\, \mathbf{B}(y) + y^p\, \mathbf{B}(1-y) = 1. \tag{10}$$

It remains to show that the polynomial $\mathbf{B}(y)$ is the truncated binomial $\mathbf{B}_p(y)$. At $y = 0$ and $y = 1$, equation (10) holds because $\mathbf{B}_p(0) = 1$. The first term has a $p$-fold zero at $y = 1$ and it is flat at $y = 0$ (with $p-1$ zero derivatives)

$$(1-y)^p\, \mathbf{B}_p(y) = (1-y)^p\, [(1-y)^{-p} + O(y^p)] = 1 + O(y^p). \tag{11}$$

The second term in (10) is the mirror image across $y = 1/2$ of the first, replacing $y$ by $1-y$. The sum has the correct value 1 with $p-1$ zero derivatives at each end. This uniquely determines a polynomial of degree $2p-1$. Therefore (10) is satisfied by $\mathbf{B}_p(y)$.

Note that $(1-y)^p\,\mathbf{B}_p(y)$ is the Hermite interpolating polynomial that has maximum flatness at $y=0$ and $y=1$ (where it equals 1 and 0). It is the response of a "maxflat lowpass halfband filter".

We prefer to work with $\mathbf{B}_p(y)$ instead of $Q_p(z)$ for two reasons. $\mathbf{B}_p(y)$ is an ordinary polynomial of degree $p-1$, with convenient coefficients, while $Q_p(z)$ is a Laurent polynomial of the same degree. Each zero of $\mathbf{B}_p(y)$ gives two zeros of $Q_p(z)$ from the rule $Z+Z^{-1}=2-4Y$. From that pair, we choose the zero $Z_n$ inside the unit circle to go into the Daubechies polynomial

$$H_p(z)=\left(\frac{1+z^{-1}}{2}\right)^p\prod_{n=1}^{p-1}(1-z^{-1}Z_n) \tag{12}$$

The $p$ zeros at $z=-1$ give high accuracy. For the wavelets, they give $p$ vanishing moments (cf. Daubechies [2], or Strang and Nguyen [7]). If the factor with the $Z$'s is omitted, the dilation equation produces spline functions —with accuracy $p$ but not orthogonal to their translates. It is these extra zeros $Z_1,\ldots,Z_{p-1}$ of $Q_p(z)$, coming from the zeros $Y_1,\ldots,Y_{p-1}$ of $\mathbf{B}_p(y)$, that achieve condition (7) and yield orthogonal wavelets.

The next section will show that the zeros approach the curve $|4y(1-y)|=1$ in the complex $y$-plane. In the $z$-plane, this curve becomes $|z-z^{-1}|=2$, and this figure looks like a moon (It consists of two circular arcs, $|z-1|=\sqrt{2}$ and $|z+1|=\sqrt{2}$, meeting at $z=\pm i$). By Theorem 2 below, the zeros $Z_n$ lie in the right halfplane $\mathrm{Re}(z)>0$. Figure 3 shows the 138 zeros of $Q_{70}(z)$; each pair $Z$ and $Z^{-1}$ corresponds to one point $Y$ in Figure 1. The zeros are outside the limiting curve, by Theorem 3. They approach the curve most slowly near $z=\pm i$ (which corresponds to $y=1/2$ ). The limiting curve retains the special property of each $Q_p(z)$, that the zeros come in pairs $Z$ and $Z^{-1}$.

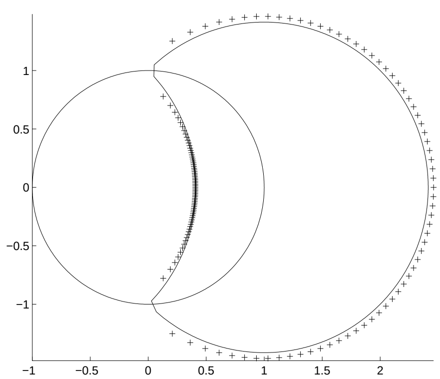


FIGURE 3. The 138 zeros of $Q_{70}(z)$ are close to the limiting curve.

## 3. The Position of the Zeros of $\mathbf{B}_p(y)$.

The first step is to prove that $|Y|<1/2$ (Figure 4) and that $|4Y(1-Y)|>2^{1/p}$ (Figure 5). The former is easy, and the latter begins with Szegö's key idea — to represent the remainder between $(1-y)^{-p}$ and $\mathbf{B}_p(y)$ by Taylor's integral formula.

**Theorem 2.** *For $p = 2$, the only zero is $Y = -1/2$. For $p > 2$ all the zeros satisfy $|Y| < 1/2$. Therefore each $Z$ has $Re(Z) > 0$.*

Before proving it, we need a theorem due to Eneström and Kakeya (cf. Marden [5]):

**EK Theorem.** *Let $p(y)$ be a polynomial of degree $n$ with all coefficients $a_i$ real and positive. Define $r_i = a_i/a_{i+1}$, $0 \le i \le n-1$. Then all zeros of $p(y)$ must lie in the closed annulus:* $\min_i r_i \le |y| \le \max_i r_i$.

The details about when and how the zeros can really lie on the border of the prescribed annulus is discussed by Anderson, Saff, and Varga [1]. Their sharpened form gives the strict inequality $|Y| < 1/2$ for $p > 2$.

*Proof of Theorem 2.* By (1), $\mathbf{B}_p(y)$ satisfies the condition of the EK Theorem. And in this case, $r_i = (i+1)/(p+i)$ for $0 \le i \le p-2$. Thus $\min_i r_i = r_0 = 1/p$, and $\max_i r_i = r_{p-2} = 1/2$. Then the truth of the statement on $Y$ follows immediately from the EK Theorem. Therefore $Z + Z^{-1} = 2 - 4Y$ lies in the right halfplane, which implies that $\mathrm{Re}(Z) > 0$.

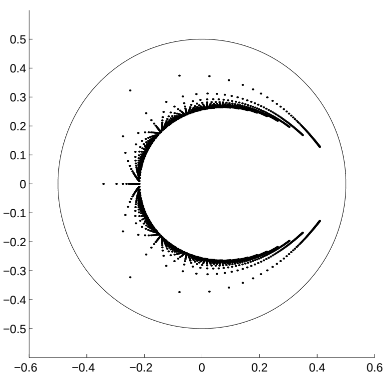


FIGURE 4. All zeros lie inside the circle $|y| = 1/2$, $p = 3 : 1 : 60$.

**Theorem 3.** *The zeros of $\mathbf{B}_p(y)$ satisfy $|4Y(1-Y)| > 2^{1/p}$.*

*Proof.* $\mathbf{B}_p(y)$ is the truncated Taylor series for the function $(1-y)^{-p}$. The $p$th derivative of this function is $p(p+1)\dots(2p-1)(1-y)^{-2p}$. Then Taylor's integral formula for the remainder $\mathbf{R}_p(y) = (1-y)^{-p} - \mathbf{B}_p(y)$ is

$$\mathbf{R}_p(y) = (2p-1)\binom{2(p-1)}{p-1} \int_0^y (y-s)^{p-1}\,(1-s)^{-2p}\,ds \tag{13}$$

$$= (2p-1)\binom{2(p-1)}{p-1} \cdot y^p \cdot \int_0^1 (1-t)^{p-1}\,(1-yt)^{-2p}\,dt \tag{14}$$

Call this last integral $\mathbf{I}_p(y)$. Since each zero has $|Y| < 1/2$, we have $|1-Yt|^{-1} < (1-t/2)^{-1}$, for any $t \in (0\,,\,1]$. Then

$$|\mathbf{I}_p(Y)| < \int_0^1 (1-t)^{p-1}\,(1-t/2)^{-2p}\,dt = \mathbf{I}_p(\frac{1}{2}) \tag{15}$$

At $y = 1/2$, equation (9) gives $\mathbf{B}_p(1/2) = 2^{p-1}$. Thus the remainder is

$$\mathbf{R}_p(\frac{1}{2}) = (1-\frac{1}{2})^{-p} - 2^{p-1} = 2^{p-1} \tag{16}$$

At each zero of $\mathbf{B}_p$, we know that $\mathbf{R}_p(Y) = (1-Y)^{-p}$. Now (14)–(16) combine into

$$|4Y(1-Y)|^{-p} = |4^{-p}\,Y^{-p}\,\mathbf{R}_p(Y)| < |4^{-p}\,(\frac{1}{2})^{-p}\,\mathbf{R}_p(\frac{1}{2})| = \frac{1}{2}.$$

This is the bound $|4Y(1-Y)| > 2^{1/p}$ that puts $Y$ outside the limiting curve, and completes Theorem 3.

Now we describe more precisely the location of the zeros of $\mathbf{B}_p(y)$. As in Szegö's problem for the exponential series (see the new methods and additional results in Varga [9]), there are two regions to consider: near $y = 1/2$ and away from $y = 1/2$. Suppose $D$ is a circle around $y = 1/2$, with fixed small radius $\delta$. Theorem 5 studies the zeros inside $D$, and Theorem 4 studies the zeros outside. Together they prove that the zeros approach the limiting curve $|4y(1-y)| = 1$.

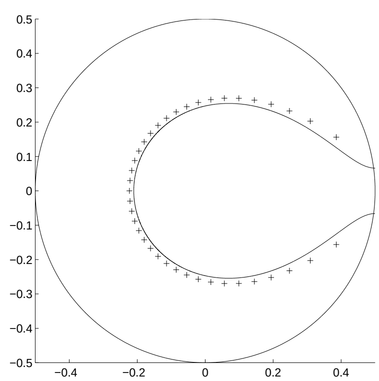


FIGURE 5. All zeros lie outside the curve $|4y(1-y)| = 2^{1/p}$, $p$=40.

**Lemma.** *At any point with* $|y| < 1/2$ *and* $|y-1/2| > \delta$,

$$\mathbf{I}_p(y) = \frac{1}{p(1-2y)} + O(p^{-2}). \tag{17}$$

*Proof.* In the integral $\mathbf{I}_p$, change variables from $t$ to $w = (1-t)/(1-yt)^2$. Then $w$ goes from 1 to 0 and the derivative is $dw/dt = (2y - yt - 1)/(1-yt)^3$. We leave part of the integral in terms of $t$

$$\mathbf{I}_p(y) = -\int_0^1 w^{p-1} \cdot \frac{1-yt}{2y-1-yt} \cdot dw. \tag{18}$$

As $p \to \infty$ the power $w^{p-1}$ is concentrated near $w = 1$. Around that endpoint the leading term of the expression in parentheses is $(2y-1)^{-1}$. The integration of $w^{p-1}$ gives (17) and proves the lemma.

Suppose that $\mathbf{B}_p(Y) = 0$ and thus $\mathbf{R}_p(Y) = (1-Y)^{-p}$. By (14) and the lemma,

$$\begin{aligned} 4Y(1-Y)^{-p} &= 4^{-p}(2p-1)\binom{2(p-1)}{p-1}\mathbf{I}_p(Y) \\ &= 4^{-p}\binom{2p-2}{p-1}\frac{2}{1-2Y}(1+O(p^{-1})) \qquad \text{from (17)} \\ &= \frac{1}{(1-2Y)\sqrt{4\pi p}}(1+O(p^{-1})) \qquad \text{from (2)} \end{aligned} \tag{19}$$

The $p$th root displays the equation of the approximate curve $C_p$ and the error term

$$|4Y(1-Y)| = |1-2Y|^{\frac{1}{p}}\,(4\pi p)^{\frac{1}{2p}}\,(1+O(p^{-2})). \tag{20}$$

**Theorem 4.** *All zeros outside the circle $|y - 1/2| = \delta$ are not farther than $c(\delta)p^{-2}$ from the curve $C_p$:*

$$|4y(1-y)| = |1-2y|^{\frac{1}{p}} \cdot (4\pi p)^{\frac{1}{2p}}. \tag{21}$$

*Proof.* Let $y$ be the point on $C_p$ nearest to $Y$ and $\epsilon = Y - y$. We must show that $\epsilon$ is $O(p^{-2})$. Since $|1+\epsilon|^{1/p} = 1 + O(|\epsilon|/p)$ for complex $\epsilon$, one has

$$\begin{aligned} |1-2Y|^{\frac{1}{p}} &= |1-2y|^{\frac{1}{p}} \cdot \left|1 + \frac{\epsilon}{1-2y}\right|^{\frac{1}{p}} \\ &= |1-2y|^{\frac{1}{p}} \cdot (1 + O(\frac{|\epsilon|}{p})) \\ |4Y(1-Y)| &= |4y(1-y)| \cdot \left|1 + \frac{1-2y}{y(1-y)} \cdot \epsilon + O(\epsilon^2)\right| \\ &= |4y(1-y)| \cdot |1 + E\epsilon + O(\epsilon^2)| \end{aligned}$$

where $E = (1-2y)/(y(1-y))$. Since $y$ is on the curve $C_p$, division yields

$$\begin{aligned} \frac{|4Y(1-Y)|}{|1-2Y|^{\frac{1}{p}}(4\pi p)^{\frac{1}{2p}}} &= \frac{|1+E\epsilon+O(\epsilon^2)|}{1+O(\frac{|\epsilon|}{p})} \\ &= |1 + E\epsilon + o(|\epsilon|)| \\ &= 1 + O(p^{-2}) \qquad \text{using (20)} \end{aligned}$$

Since $\delta$ is fixed, $E = O(1)$. Therefore $\epsilon = O(p^{-2})$.

**Corollary.** *All zeros outside the circle $|y-1/2|=\delta$ are not farther than $c'(\delta)p^{-1}$ from the curve $D_p$ drawn in Figure 6:*

$$|4y(1-y)| = r_p, \quad \textit{where } r_p = \frac{\log(4\pi p)}{2p}.$$

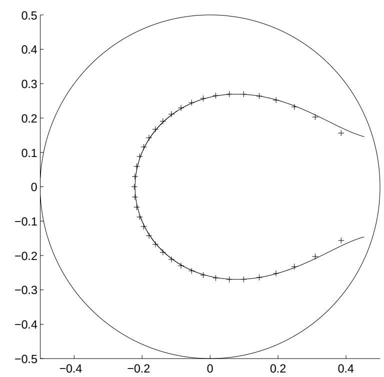


FIGURE 6. $D_p$ is a first order approximation curve for 'regular' zeros, $p$=40.

The value of $y = 1/2$ is in every respect a singular point for this problem. It corresponds to points $z = i$ and $z = -i$ on the unit circle. We now prove that the zeros $Y$ approach $1/2$ at speed $p^{-1/2}$, as Moler discovered by Matlab experiment. Surprisingly, the coefficient of $p^{-1/2}$ comes from a zero $W$ of the complementary error function

$$\operatorname{erfc}(w) = 1 - \operatorname{erf}(w) = \frac{2}{\sqrt{\pi}} \int_w^\infty e^{-s^2}\, ds$$

The corollary will improve slightly a known result for the location of these zeros.

**Theorem 5.** *If $W$ is a zero of erfc$(w)$, there is a zero $Y$ of $\mathbf{B}_p(y)$ and a zero $Z$ of $Q_p(z)$ such that*

$$Y = \frac{1}{2} + \frac{W}{2\sqrt{p}} + O(p^{-\frac{3}{2}}) \tag{22}$$

$$Z = i - \frac{W}{\sqrt{p}} - \frac{iW^2}{2p} + O(p^{-\frac{3}{2}}) \tag{23}$$

*Proof.* We introduce a new expression for $P(y) = (1-y)^p \mathbf{B}_p(y)$, which is exactly $P(z)$ defined in (6) with $z + z^{-1} = 2 - 4y$. As a function of $y$, this is a polynomial of degree $2p-1$ whose derivative has $p-1$ zeros both at $y = 0$ and $y = 1$ (see (11)). Therefore the derivative is a multiple of $y^{p-1}(1-y)^{p-1}$, and we have an *incomplete beta function*

$$P(y) = (1-y)^p\, \mathbf{B}_p(y) = 1 - c_p^{-1}\, 2^{2p-1} \int_0^y t^{p-1}\,(1-t)^{p-1} dt. \tag{24}$$

The number $c_p$ is determined by setting $y = 1$:

$$c_p = 2^{2p-1} \int_0^1 t^{p-1} (1-t)^{p-1}\, dt = 2^{2p-1} \frac{\Gamma(p)^2}{\Gamma(2p)} = 2^{2p-1} \left( (2p-1) \binom{2p-2}{p-1} \right)^{-1}$$

By Stirling's formula or using the result of (2), we have

$$c_p = \sqrt{\frac{\pi}{p}}\,(1 + O(p^{-1})). \tag{25}$$

By symmetry, the value of the integral in (24) at $y = 1/2$ should be $2^{1-2p} c_p/2$. Therefore $P(1/2) = 1/2$. In order to see the detail of the zeros of $\mathbf{B}_p(y)$ near $y = 1/2$, we introduce a new variable by $y - 1/2 = w/(2\sqrt{p})$. Then

$$\begin{aligned} P(y) = P(\frac{1}{2} + \frac{w}{2\sqrt{p}}) &= P(\frac{1}{2}) - c_p^{-1}\, 2^{2p-1} \int_0^{\frac{w}{2\sqrt{p}}} (\frac{1}{2} + t)^{p-1} (\frac{1}{2} - t)^{p-1}\, dt \\ &= \frac{1}{2} - 2\, c_p^{-1} \int_0^{\frac{w}{2\sqrt{p}}} (1 - 4t^2)^{p-1}\, dt \\ &= \frac{1}{2} - \frac{2\sqrt{p}}{\sqrt{\pi}} \int_0^{\frac{w}{2\sqrt{p}}} e^{-4pt^2}\, dt\, (1 + O(p^{-1})) \qquad (26) \\ &= \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{w} e^{-s^2} ds\, (1 + O(p^{-1})) \qquad (27) \\ &= \frac{1}{2} \mathrm{erfc}(w) + O(p^{-1}) \end{aligned}$$

The third step (26) used (25) and $e^{-4t^2} = 1 - 4t^2 + O(t^4)$, and in (27) $s = 2\sqrt{p}\,t$.

Let $W$ be a zero of $\mathrm{erfc}(w)$. All zeros are simple, because the derivative $e^{-w^2}$ is never zero. The fundamental theorem of complex analysis says that as $p \to \infty$, $P(1/2 + w/(2\sqrt{p})\,)$ is zero at some point $w = W + O(p^{-1})$. In terms of $y$, $Y = 1/2 + W/(2\sqrt{p}) + O(p^{-3/2})$, which is (22), because $\mathbf{B}_p(y)$ shares every zero with $P(y)$ except $y = 1$.

**Corollary.** *Every zero of* $\mathrm{erfc}(w)$ *has* $|\arg\ W| < 3\pi/4$.

*Proof.* The corresponding $Y$ lies outside the limiting curve $|4y(1-y)| = 1$, which intersects itself at $y = 1/2$ with slopes $\pm 1$. In the limit, $W = (Y - 1/2)/\sqrt{p} + O(p^{-1})$ must have $|\arg\ W| \le 3\pi/4$. If equality held, $W^2$ would be purely imaginary. Then Theorem 5 would give

$$|4Y(1-Y)| = |1 - W^2 p^{-1} + O(p^{-2})| = 1 + O(p^{-2})$$

This contradicts the inequality $|4Y(1-Y)| > 2^{1/p}$ in Theorem 2, proving the corollary.

Fettis, Caslin, and Cramer [3] computed the zeros of $\mathrm{erfc}(w)$ to very high accuracy. They also proved an asymptotic form of the statement $|\arg\ W| \le 3\pi/4$. It is interesting to see the complete statement (which their numerical table confirms ) proved by such an indirect argument involving the zeros of $\mathbf{B}_p(y)$.

These zeros approach 1/2 at order $p^{-1/2}$, close to the line $Y - 1/2 = W/2\sqrt{p}$. By the corollary, the slope of this line is not $\pm 1$. Therefore the distance from $Y_p$ to the limiting curve $C$ is of strict order $p^{-1/2}$ near $y = 1/2$. In this region, the error order in equation (20) rises to $p^{-1}$. This applies in particular to the rightmost zero, which comes from the first $W$ tabulated in [3], $Y \approx 1/2 + (-1.3548\ldots + i1.9914\ldots)/2\sqrt{p}$.

## 5. Steepness at $\omega = \frac{\pi}{2}$.

A change of variables $t = (1 - \cos\theta)/2$ in (24) produces the integral of $\sin^{2p-1}\theta$. The limits of intergration are related by $y = (1-\cos\theta)/2$, which is exactly the change associated with $z = e^{i\omega}$ in the proof of Theorem 1. Thus (24) is Meyer's form (cf. Meyer [6], page 43) of the halfband filter $P(z)$ in equation (6)

$$P(e^{i\omega}) = 1 - c_p^{-1} \int_0^{\omega} \sin^{2p-1}\theta \, d\theta. \tag{28}$$

The zero at $y = 1$ becomes the celebrated "zero at $\pi$" for the frequency response $P(e^{i\omega})$. This zero at $\omega = \pi$ is of order $2p$, from the power of $\sin\theta$ in (28) and the form of $P(z)$ in (6). Factorization gives $p$th order zeros for the Daubechies polynomials in $P(z) = H(z)H(z^{-1})$. That zero at $\omega = \pi$ and $z = -1$ is responsible for the $p$ vanishing moments in the wavelets.

The trigonometric polynomial $P(e^{i\omega})$ drops monotonically from one to zero on $0 \le \omega \le \pi$ (see (28)). The first $2p-1$ derivatives are zero at $\omega = 0$, and $\omega = \pi$, from the vanishing of $\sin^{2p-1}\theta$. Furthermore this integral of $(1 - \cos\theta)^{p-1}\sin\theta$ involves only odd powers of $\cos\theta$, and the only even power is the constant term. $P(e^{i\omega})$ is odd around its value 1/2 at $\omega = \frac{\pi}{2}$, and it is called "halfband".

An important question for such a filter is the slope at $\omega = \frac{\pi}{2}$. This slope determines the width of the frequency band, in which $P$ drops from 1 to 0. An ideal filter has a jump; its graph is a brick wall (however, this ideal is not a polynomial). An optimally designed polynomial of order $N$ has slope nearly $O(N^{-1})$. There will be ripples in the graph of $P(e^{i\omega})$—a monotonic polynomial cannot provide such a sharp cutoff. The Daubechies filters are necessarily less sharp: $O(N)$ becomes $O(\sqrt{N})$.

**Theorem 6.** *The slope of $P(e^{i\omega})$ in (28) is approximately $\sqrt{p/\pi}$ at $\omega = \pi/2$. The transition from nearly 1 to nearly 0 is over an interval (i.e. transition band ) of with $2\sqrt{2/p}$.*

*Proof.* The integral in (28) has derivative $\sin^{2p-1}(\pi/2) = 1$ at $\omega = \pi/2$. The slope of $P(e^{i\omega})$ is exactly the constant $-c_p^{-1}$. By (25) this is $-\sqrt{p/\pi} + O(p^{-\frac{3}{2}})$. To measure the drop in $P(e^{i\omega})$ around $\omega = \pi/2$, we integrate from $\pi/2 - \sigma/\sqrt{p}$ to $\pi/2 + \sigma/\sqrt{p}$. Shifting by $\pi/2$ to center the integral, and scaling by $\theta = \tau/\sqrt{p}$, the drop is

$$c_p^{-1} \int_{-\sigma/\sqrt{p}}^{\sigma/\sqrt{p}} \sin^{2p-1} \theta d\theta \approx \frac{1}{c_p\sqrt{p}} \int_{-\sigma}^{\sigma} (1 - \frac{\tau^2}{2p})^{2p-1} d\tau$$
$$\approx \frac{1}{\sqrt{\pi}} \int_{-\sigma}^{\sigma} e^{-\tau^2}\, d\tau. \tag{29}$$

Thus 95% of the drop comes for $\sigma = \sqrt{2}$ (within two standard derivatives of the mean, for the normal distribution). This transition interval has width $\Delta\omega = 2\sqrt{2/p}$, as the theorem predicts. That rule was found experimentally by Kaiser and Reed at the beginning of the triumph of digital filters.

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