FINITE ELEMENT MULTIWAVELETS

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Abstract. Finite elements with support on two intervals span the space of piecewise polynomomials with degree $2n-1$ and $n-1$ continuous derivatives. Function values and $n-1$ derivatives at each meshpoint determine these "Hermite finite elements". The n basis functions satisfy a dilation equation with n by n matrix coefficients. Orthogonal to this scaling subspace is a wavelet subspace. It is spanned by the translates of n wavelets $W_i(t)$, each supported on *three* intervals. The wavelets are orthogonal to all rescalings $W_i(2^j t-k)$, but not to translates at the same level ($j=0$). These new multiwavelets achieve $2n$ vanishing moments and high regularity with symmetry and short support.

1. Splines and Finite Elements

Spline wavelets are semiorthogonal but they are not short. The B-spline plays the role of the scaling function. The box function on a unit interval is the first example ($p=1$), leading to the Haar wavelet. The hat function over two intervals is the second B-spline ($p=2$). The corresponding wavelet (piecewise linear) is in Figure 1. The convolution of p box functions gives the B-spline of degree $p-1$ with $p-2$ continuous derivatives. It is nonzero, in fact positive, over p intervals. For $p=4$ this is the cubic spline, and for $p>1$ the B-spline is not orthogonal to its translates. Nevertheless it satisfies a dilation equation and allows a multiresolution analysis.

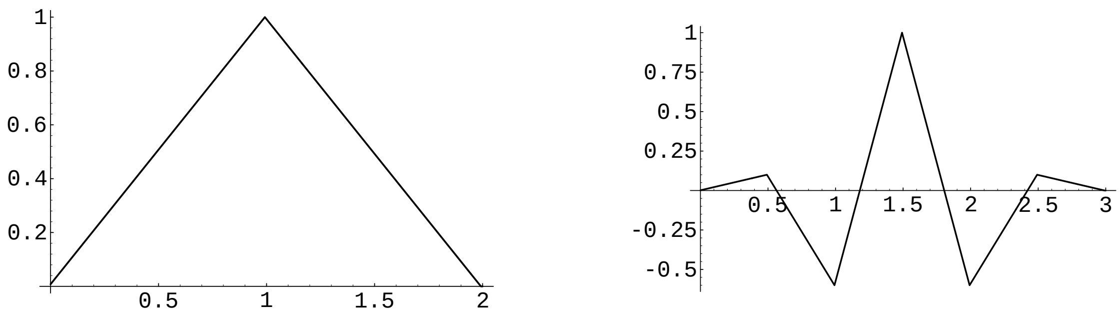


Figure 1. Hat function and the corresponding piecewise linear wavelet.

The space V_1 of splines with knots at the half-integers clearly contains the space V_0 of splines with knots at the integers. The orthogonal complement of V_0 is the wavelet subspace W_0:

$$\begin{array}{rcl} V_1 & = & V_0 \oplus W_0 \\ \text{splines at half integers} & = & \text{splines} \oplus \text{spline wavelets.} \end{array}$$

Where V_0 is spanned by translates of the B-spline, W_0 is spanned by translates of the spline wavelet. For each p this wavelet is nonzero over $2p-1$ intervals — certainly not short. Chui [1] gives a formula for the wavelets, and he observes that they are *semiorthogonal*: the spaces W_0, W_1, W_2, ... are orthogonal. This is immediate from the recursive multiresolution

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1.$$

Wavelets at different scaling levels are orthogonal, but the translates at the same scaling level are not. This semiorthogonality is more than biorthogonality but less than orthogonality.

V_0 contains all polynomials of degree less than p. The wavelets are orthogonal to these polynomials, so they have p vanishing moments. The lowpass filter coefficients, which enter the dilation equation, are the binomial coefficients in $(\frac{1}{2}+\frac{1}{2})^p$ The frequency response has a corresponding factor $(e^{-i\omega}+1)^p/2^p$, with a pth order zero at $\omega = \pi$. The iterated lowpass filter, with rescaling, converges quickly to the B-spline of degree $p-1$.

There is another space of piecewise polynomials, the subject of this paper, with fewer continuous derivatives but shorter basis functions. We call this V_0 the *finite element space*, to distinguish it from the spline space. The finite elements have odd degree $2n-1$, with $n-1$ continuous derivatives. Where spline interpolation fits a set of function values by solving a coupled linear system, finite element interpolation is completely uncoupled. The function value and $n-1$ derivative values are given at each node. Between one node and the next, these $2n$ values produce a unique polynomial of degree $2n-1$. The finite elements are *locally* determined, while spline interpolation is global. This local property is particularly valuable at boundary points and jumps — we want wavelets that are as far as possible also local (= short).

The piecewise linear case $n = 1$ is included among splines (the hat function). What is important is that for all n the basis functions are nonzero only over *two intervals*. Also important: There are n basis functions. *In wavelet language this means n scaling functions $\phi_1(t), \ldots, \phi_n(t)$.* Their translates span V_0. The functions are alternately symmetric and antisymmetric about $t = 1$, if we place them on the interval $[0, 2]$. They are easy to construct.

Rescaled by $t \to 2t$, the space V_1 of fine-mesh elements clearly contains the space V_0 of coarse-mesh elements. Therefore each basis function $\phi(t)$ in V_0 is a combination of translates of the n fine-mesh basis functions $\phi(2t)$. This is the *dilation equation*. It is a vector equation

$$\begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_n(t) \end{bmatrix} = \sum_{k=0}^{2} C_k \begin{bmatrix} \phi_1(2t-k) \\ \vdots \\ \phi_n(2t-k) \end{bmatrix} \tag{1}$$

with matrix coefficients C_0, C_1, C_2. Our first step is to find these coefficients. They enter the *lowpass multifilter*, which has only three taps; the filter coefficients are n by n matrices.

The literature on multifilters and multiwavelets is small but growing [2–7]. An earlier paper of Auscher and thesis of Herve [10] have been mentioned to us. The corresponding filter bank has $2n$ channels, n low and n high. With matrix coefficients it becomes possible to combine orthogonality with symmetry (linear phase) and short support and simplicity. We believe that short support will be especially valuable near boundaries.

The wavelets are basis functions for the orthogonal complement W_0, in the multiresolution $V_1 = V_0 \oplus W_0$. The support of these finite element multiwavelets $W_1(t), \ldots, W_n(t)$ turns out to be *three intervals*. They are alternately symmetric and antisymmetric about $t = 3/2$, the center of $[0, 3]$. They lie in V_1 (half-integer nodes) and satisfy the wavelet equation

$$\begin{bmatrix} W_1(t) \\ \vdots \\ W_n(t) \end{bmatrix} = \sum_{k=0}^{4} D_k \begin{bmatrix} \phi_1(2t-k) \\ \vdots \\ \phi_n(2t-k) \end{bmatrix}. \tag{2}$$

We compute the D's from the C's, which have a particularly nice formula. Figure 2 shows the cubic scaling functions ϕ_1 and ϕ_2 — these are "Hermite cubics" in the finite element literature [8–9]. Figure 3 displays the cubic multiwavelets W_1 and W_2. We can provide a Mathematica code to construct and draw the finite element scaling functions and multiwavelets for each n. The wavelets have $p = 2n$ vanishing moments, because they are orthogonal to the space V_0 containing all polynomials of degree less than $2n$.

These multiwavelets were first constructed by Goodman and Lee [6]. Their method (not using the dilation equation) leads to spline wavelets of smoothness C^{r-1} for varying r, and they find an efficient recursive formula in the special finite element case. It is this case, with support on $[0,3]$, that we develop by an explicit formula for the C's and an implicit algorithm for the D's.

New piecewise polynomial multiwavelets have recently been constructed by Donovan, Geronimo, and Hardin, with orthogonality to translates. They have support $[0,2]$ with a larger multiplicity n. The translates of $n = 3$ piecewise linear wavelets are an orthogonal basis for their space W_0. It is remarkable that orthogonal piecewise polynomials are still waiting to be discovered.

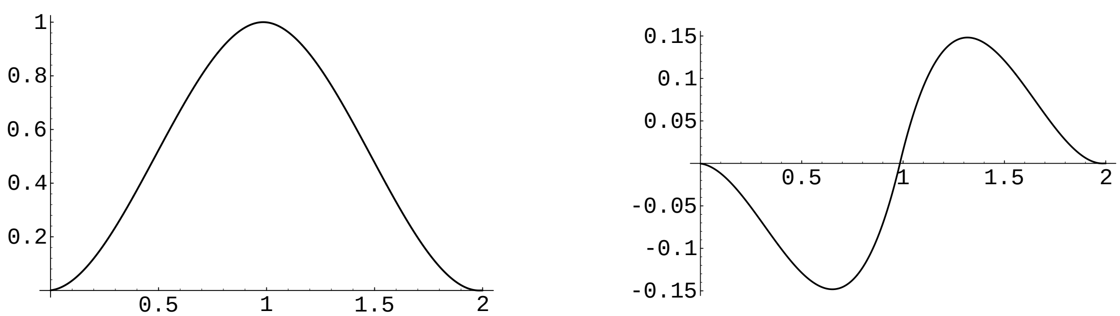


Figure 2. Cubic finite element scaling functions ϕ_1 and ϕ_2.

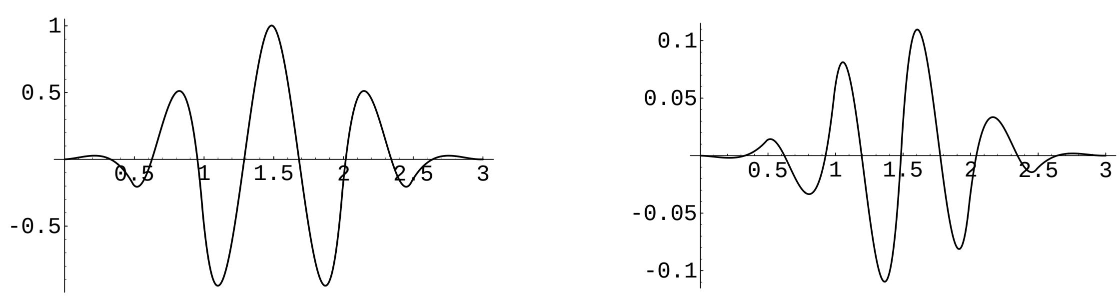


Figure 3. Cubic finite element wavelets w_1 and w_2.

2. Dilation Equation for the Finite Elements

We emphasize above all the distinction between splines and finite elements. The splines are smoother, with larger support. To determine an interpolating spline, we give *one* condition at each node, namely the value to be interpolated. There is *one* B-spline centered at that node (but neighboring B-splines are nonzero there, so we solve a linear system for the interpolating spline). The cubic B-spline is supported on $[0,4]$ and is nonzero at the nodes $t = 1,2,3$.

Finite elements are less smooth, with minimal support $[0,2]$. They are piecewise polynomials (two pieces) of degree $2n-1$. We can interpolate n values at each node, namely the function and its first $n-1$ derivatives. There are n different elements centered at this node, to match these n conditions. (Neighboring finite elements are *zero* at this node, so there is no linear system to be solved.)

Figure 2 shows the two "Hermite cubic" elements, which have $\phi_1(1) = 1$ and $\phi_2'(1) = 1$. To interpolate a value y and a velocity v at $t = 1$, we take $y\phi_1 + v\phi_2$. To see in another way how the nodal values determine the interpolating cubic: On a typical interval $[0,1]$, the cubic has four free coefficients and satisfies four conditions. The value and slope are given at 0 and 1. It is this totally local interpolation that distinguishes finite elements. They are popular as trial and test functions in Galerkin's method for differential equations (the finite element method).

Like splines, finite elements satisfy a two-scale dilation equation:

$$\Phi(t) = \sum C_k \Phi(2t-k). \tag{3}$$

Since the support is $[0,2]$, the only coefficients are C_0, C_1, C_2. Since there are n basis functions at each node, those coefficients C_k are n by n matrices. $\Phi(t)$ represents the column vector of basis functions $[\phi_1(t)\ldots\phi_n(t)]^T$. These are polynomials of degree $2n-1$ on the pieces $[0,1]$ and $[1,2]$ determined by

$$(\frac{d}{dt})^{i-1}\phi_j(1) = \delta_{ij} \tag{4}$$

$$(\frac{d}{dt})^{i-1}\phi_j(0) = (\frac{d}{dt})^{i-1}\phi_j(2) = 0 \tag{5}$$

for $i,j = 1,\ldots,n$. There is symmetry for even j and antisymmetry for odd j:

$$\phi_j(2-t) = (-1)^j\phi_j(t). \tag{6}$$

Restricted to $[0,1]$, any polynomial $p(t)$ of degree less than $2n$ is a combination of $\phi_1(t)$, …, $\phi_n(t)$, $\phi_1(t+1)$, …, $\phi_n(t+1)$. These functions form a basis; the dimension is $2n$. Each function interpolates one of the $2n$ nodal values, given by $p(t)$ and its $n-1$ derivatives at the endpoints $t=0$ and $t=1$.

We come back to the dilation equation, to determine C_0, C_1, C_2:

$$\Phi(t) = C_0\Phi(2t) + C_1\Phi(2t-1) + C_2\Phi(2t-2). \tag{7}$$

At $t=1$, this equation immediately gives C_1 as a diagonal matrix with entries $1, \frac{1}{2}, \ldots, (\frac{1}{2})^{n-1}$. Since $C_0\Phi(2)$ and $C_2\Phi(0)$ give no contribution, by (5), the derivatives of (7) at $t=1$ are

$$(\frac{d}{dt})^{i-1}\Phi(1) = C_1 2^{i-1}(\frac{d}{dt})^{i-1}\Phi(1). \tag{8}$$

The left side is the ith column of the identity matrix, by (4). Dividing by 2^{i-1} therefore gives the ith column of C_1. We now know the diagonal matrix C_1.

The reader will not be surprised that there is a close relation between C_0 and C_2, coming from the symmetry–antisymmetry of the functions $\phi_j(t)$. Property (7) says that $\Phi(2-t) = S\Phi(t)$, where S is the "alternating sign matrix"

$$S = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & (-1)^p \end{bmatrix} = S^{-1}.$$

Compare the dilation equation (7) for $\Phi(t)$ with the same equation for $\Phi(2-t) = S\Phi(t)$ in each term. The conclusion (using linear independence of the polynomials) is that

$$C_0 = SC_2S^{-1} \quad \text{or} \quad [C_0]_{ij} = (-1)^{i+j}[C_2]_{ij}. \tag{9}$$

It remains to determine C_2.

There are two approaches to this main step. The direct approach is to compute the n functions $\phi_j(t)$ explicitly, and substitute into the dilation equation (7). Knowing the diagonal C_1 and knowing C_0 from C_2 (above), we could determine C_2. We preferred an indirect approach that gives more information about C_2 — its eigenvalues and eigenvectors.

Theorem The eigenvalues of C_2 are $(\frac{1}{2})^n, (\frac{1}{2})^{n+1}, \ldots, (\frac{1}{2})^{2n-1}$. They enter a diagonal matrix Λ. The left eigenvectors of C_2 are the rows of U:

$$[U]_{ij} = (-1)^{n+i-j}\frac{(n+i)!}{(n+i-j)!}. \tag{10}$$

Then $C_2 = U^{-1}\Lambda U$.

Example The piecewise cubic case $n = 2$ leads to

$$\Lambda = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{8} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \frac{3!}{2!} & -\frac{3!}{1!} \\ \frac{4!}{3!} & \frac{4!}{2!} \end{bmatrix}.$$

Then the matrices $C_2 = U^{-1}\Lambda U$ and C_1 and C_0 are

$$C_2 = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad C_0 = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}.$$

These are the filter coefficients in the "lowpass filter with downsampling", which lies behind the scaling functions and wavelets. They are matrix coefficients because we have a multifilter. In the time domain, a filter is represented by a constant-diagonal matrix (a Toeplitz matrix) containing the filter coefficients. Downsampling removes every other row of this matrix to leave the key operator in this theory, the downsampled lowpass filter matrix L:

$$L = \begin{bmatrix} \cdots & & & & \\ C_2 & C_1 & C_0 & & \\ & & C_2 & C_1 & C_0 \\ & & & & \cdots \end{bmatrix}.$$

We proved elsewhere [5] that the eigenvalues of L determine the degree $p-1$ of the polynomials that lie in the scaling space V_0, and therefore the number p of vanishing moments for the wavelets in the orthogonal complement W_0. For the finite element multiwavelets we know *a priori* that $p = 2n$. Then our condition in [5] says that L must have eigenvalues $1, \frac{1}{2}, (\frac{1}{2})^2, \ldots, (\frac{1}{2})^{2n-1}$. The first n of these eigenvalues are accounted for by C_1. The last n must be produced by C_0 and C_2. This is our indirect approach: We determine the eigenvalues in $\Lambda = \text{diag}[(\frac{1}{2})^n, \ldots, (\frac{1}{2})^{2n-1}]$ by knowing that they are needed as eigenvalues of L.

It is the eigenvectors of L that will lead us to U. Then from Λ and U we have found C_2. The full argument will be given in [9]; here we continue with the cubic example $n = 2$.

The key to [5] was the role of the left eigenvectors $y^{(0)}, y^{(1)}, \ldots$ of the infinite matrix L. The components of $y^{(m)}$ give the combinations of the basis functions and their translates that produce the polynomial t^m:

$$t^m = \sum_{m=-\infty}^{\infty} y_k^{(m)} \Phi(t-k+1), \qquad m < p. \tag{11}$$

(For proof, substitute for Φ using the dilation equation. The result simplifies eventually to $y^{(m)}L = y^{(m)}/2^m$.)

The components $y_k^{(m)}$ are themselves vectors with n components, conforming to the length of Φ and the block form of L. The simplest example is $m = 0$, where the constant polynomial 1 is produced by adding the finite elements ϕ_1 with weight 1 at each node. The sum equals 1 with slope zero at each node, so it must be identically 1. The corresponding eigenvector is $y^{(0)} = [\ldots(1\ \ 0)(1\ \ 0)\ldots]$:

$$[\ldots(1\ \ 0)(1\ \ 0)\ldots]\begin{bmatrix} \cdots & & & & \\ C_2 & C_1 & C_0 & & \\ & & C_2 & C_1 & C_0 \\ & & & & \cdots \end{bmatrix} = [\ldots(1\ \ 0)(1\ \ 0)\ldots].$$

This requires $[1\ \ 0]C_1 = [1\ \ 0]$ and also $[1\ \ 0][C_0 + C_2] = [1\ \ 0]$. The first is clear for $C_1 = \mathrm{diag}(1, \frac{1}{2})$. The second produces the upper left entry $\frac{1}{2}$ in C_0 and C_2 for our cubic example.

The other entries of C_0 and C_2 come from the way in which the polynomials t, t^2, t^3 are produced from the cubics. At a typical node $t = k$, we know the function values k, k^2, k^3 and the slopes 1, $2k$, $3k^2$. These multiply $\phi_1(t-k+1)$ and $\phi_2(t-k+1)$, respectively. Therefore the eigenvectors containing these values and slopes must be

$$\begin{aligned} y^{(1)} &= [\ldots(0\ \ 1)(-1\ \ 1)(-2\ \ 1)\ldots] \\ y^{(2)} &= [\ldots(0\ \ 0)(1\ \ -2)(4\ \ -4)\ldots] \\ y^{(3)} &= [\ldots(0\ \ 0)(-1\ \ 3)(-8\ \ 12)\ldots]. \end{aligned}$$

The equations $y^{(m)}L = y^{(m)}/2^m$ complete our knowledge of C_0 and C_2.

We can state the whole argument briefly, omitting details: The eigenvalue equation $y^{(m)}L = y^{(m)}/2^m$ leads us to

$$y_1^{(m)}C_2 = y_1^{(m)}/2^m, \qquad n \le m < 2n. \tag{12}$$

This gives the eigenvalue matrix for C_2. The fact that L is a downsampled Toeplitz matrix leads to a special form (seen above) for its left eigenvectors $y^{(m)}$. This form gives the matrix U in our theorem.

3. The Coefficients D_k and the Wavelets

The wavelets $w_1(t), \ldots, w_n(t)$ are orthogonal to $\phi_1(t), \ldots, \phi_n(t)$. At the same time the wavelets are combinations of $\phi_1(2t-k), \ldots, \phi_n(2t-k)$. This is the *wavelet equation* with matrix coefficients D_k:

$$\begin{bmatrix} w_1(t) \\ \vdots \\ w_n(t) \end{bmatrix} = \sum_{k=0}^{4} D_k \begin{bmatrix} \phi_1(2t-k) \\ \vdots \\ \phi_n(2t-k) \end{bmatrix}. \tag{13}$$

The first question is, why five D's?

The support of $\Phi(2t-k)$ is $[\frac{k}{2}, \frac{k}{2}+1]$. With coefficients $D_0, \ldots, D_4$, the support of the wavelets in (13) will be $[0,3]$. Orthogonality against $\Phi(t)$ and its translates will give four equations for the D's:

$$\begin{aligned} W(t) \cdot \Phi^T(t+1) &= 0 \\ W(t) \cdot \Phi^T(t) &= 0 \\ W(t) \cdot \Phi^T(t-1) &= 0 \\ W(t) \cdot \Phi^T(t-2) &= 0. \end{aligned} \tag{14}$$

There is a nontrivial solution which we want. If we try for a shorter support with fewer D's the homogeneous system becomes square and leads to $D_k = 0$.

The equations (14) involve integrals of $w_j(t)\phi_j(t-k)$. Use the wavelet equation (13) to substitute for the w's. Use the dilation equation (7) to substitute for the ϕ's. Then we are integrating $\phi_i(2t-k)\phi_j(2t-m)$. A change of variables brings these inner product integrals back to

$$\int \phi_i(2t-k)\phi_j(2t-m)dt = \frac{1}{2}\int \phi_i(t)\phi_j(t-m+k)dt. \tag{15}$$

The ϕ_i are supported on $[0,2]$, so the only inner products we need are in the two matrices

$$X = \Phi(t) \cdot \Phi^T(t) \quad \text{and} \quad Y = \Phi(t) \cdot \Phi^T(t-1) = \Phi(t+1) \cdot \Phi^T(t). \tag{16}$$

As a point of honor, we want to avoid an explicit computation of the polynomials $\phi_i(t)$ and their inner products. Therefore we substitute the dilation equation into (16), and use (15) to bring all arguments involving $2t$ back to t. The result is two matrix equations for the matrices X and Y:

$$\begin{aligned} 2X &= C_0XC_0^T + C_1Y^TC_0^T + C_0YC_1^T + C_1XC_1^T + C_2Y^TC_1^T + C_1YC_2^T + C_2XC_2^T \\ 2Y &= C_1YC_0^T + C_2XC_0^T + C_2YC_1^T. \end{aligned} \tag{17}$$

This determines X and Y up to a scalar factor. Then X and Y enter the orthogonality equation (14). Remember that substituting in (14) for $W(t)$ introduced the unknown D's, and substituting for $\Phi(t)$ introduced the known C's. After these substitutions, the four orthogonality equations (14) become

$$\begin{aligned} &D_0(Y^TC_1^T + XC_2^T) + D_1Y^TC_2^T = 0 \\ &D_0(XC_0^T + YC_1^T) + D_1(Y^TC_0^T + XC_1^T + YC_2^T) + D_2(Y^TC_1^T + XC_2^T) + D_3Y^TC_2^T = 0 \\ &D_1YC_0^T + D_2(XC_0^T + YC_1^T) + D_3(Y^TC_0^T + XC_1^T + YC_2^T) + D_4(Y^TC_1^T + XC_2^T) = 0 \\ &D_3YC_0^T + D_4(XC_0^T + YC_1^T) = 0. \end{aligned} \tag{18}$$

It is this system of $4n^2$ homogeneous equations that we solve for $5n^2$ entries in $D_0, \dots, D_4$. We pick out the solution that has $D_2 = I$. Then the symmetry-antisymmetry of the ϕ's is true also for the wavelets.

The property $C_0 = SC_2S$ of the C's extends to the D's with the sign matrix $S = \mathrm{diag}(1, -1, 1, \dots, (-1)^n)$:

$$D_0 = SD_4S \qquad \text{and} \qquad D_1 = SD_3S. \tag{19}$$

The first two equations (18) become identical to the last two, using this pattern for the D's and verifying the corresponding pattern $X = SXS$ and $Y = SY^TS$:

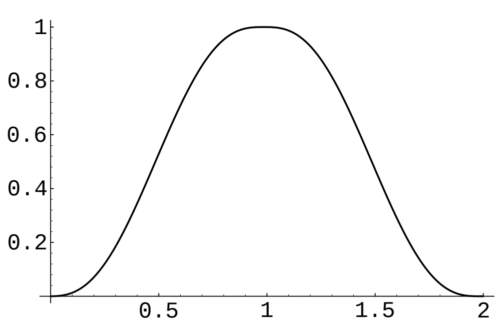
$$[X]_{ij} = [X]_{ji} = \int \phi_i(t)\phi_j(t)dt = \int \phi_i(2-t)\phi_j(2-t)dt =$$

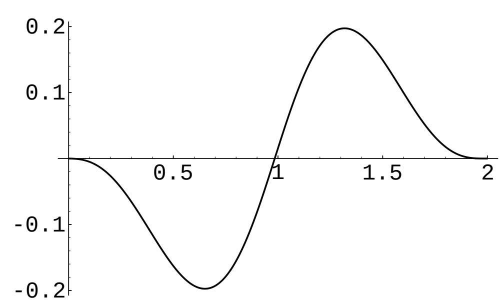
$$(-1)^{i+j} \int \phi_i(t)\phi_j(t)dt = (-1)^{i+j}[X]_{ij}$$

$$[Y]_{ij} = \int \phi_i(t)\phi_j(t-1)dt = \int \phi_i(3-t)\phi_j(2-t)dt =$$

$$(-1)^{i+j} \int \phi_i(t-1)\phi_j(t)dt = (-1)^{i+j}[Y]_{ji}$$

Thus (14) reduces to two matrix equations for two unknowns D_3 and D_4. Our Mathematica subroutine solves these equations, applies the wavelet equations (13), and draws the wavelets. Figure 4 shows the quintic finite elements ϕ_1, ϕ_2, ϕ_3 and Figure 5 shows the corresponding wavelets w_1, w_2, w_3.





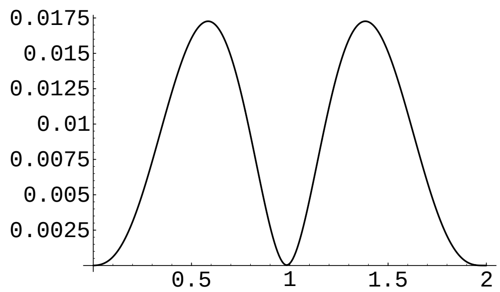


Figure 4. Quintic finite element scaling functions ϕ_1, ϕ_2, ϕ_3 .

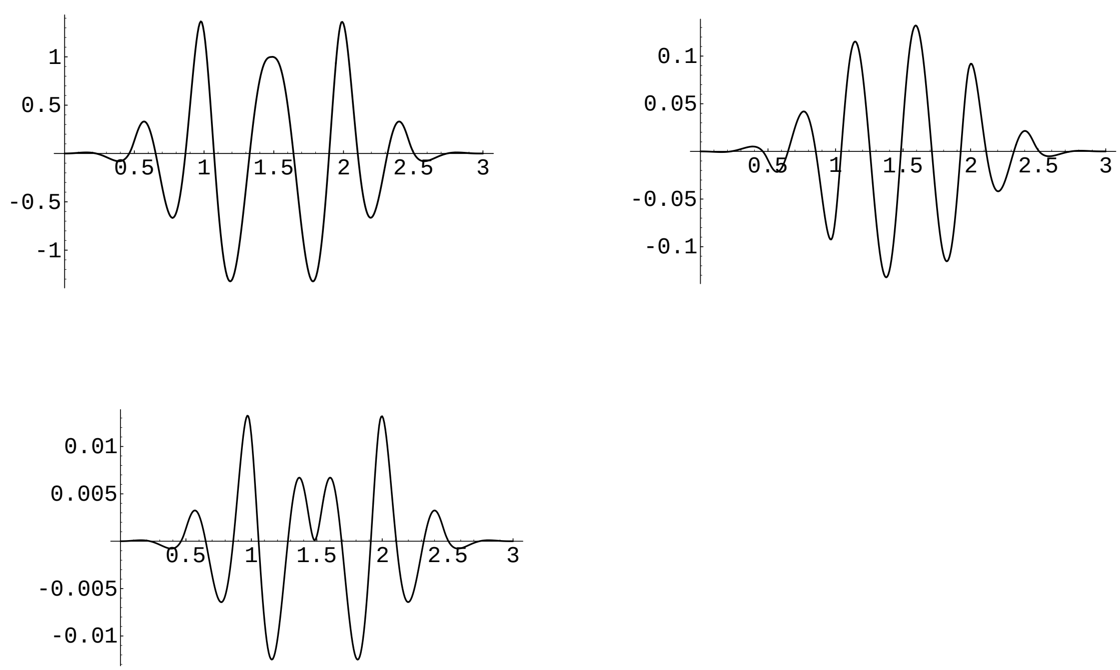


Figure 5. Quintic finite element wavelets w_1, w_2, w_3.

By construction, the finite elements are orthogonal to the wavelets and their translates. The two spaces are orthogonal complements in $V_1 = V_0 \oplus W_0$. This is invariant under dilation so $V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$. Therefore the wavelets in W_0 are orthogonal to their dilates and translates in W_1. This is true at all scaling levels, $W_i \perp W_j$, except at the same level $i = j$. The wavelets are therefore called "semiorthogonal", or "pre-wavelets". They have $2n$ vanishing moments and $n-1$ continuous derivatives, with symmetry and support $[0,3]$.

Acknowledgements

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References

1. Chui, C.K. (1992) *An Introduction to Wavelets*, Academic Press.
2. Geronimo, J.S., Hardin, D.P., and Massopust, P.R. (in press) Fractal functions and wavelet expansions based on several scaling functions, *J. Approximation Theory.*
3. Donovan, G., Geronimo, J.S., Hardin, D.P., and Massopust, P.R. (1994) Construction of orthogonal wavelets using fractal interpolation functions, preprint.
4. Strang, G. and Strela, V. (in press) Short wavelets and matrix dilation equations, IEEE *Transactions on Signal Processing.*
5. Strang, G. and Strela, V. (in press) Orthogonal multiwavelets with vanishing moments, SPIE Proceedings, Orlando (April 1994); *Optical Engineering.*
6. Goodman, T.N.T. and Lee, S.L. (1994) Wavelets of multiplicity r, *Trans. Amer. Math. Soc.*
7. Vetterli, M. and Strang, G. (in press) Time-varying filter banks and multiwavelets, Sixth Digital Signal Processing Workshop, Yosemite, October 1994.
8. Strang, G. and Fix, G. (1973) *An Analysis of the Finite Element Method*, Wellesley–Cambridge Press, Wellesley MA.
9. Strela, V. (in press) Ph.D. Thesis, Massachusetts Institute of Technology.
10. Herve, L. (1992) Analyses multiresolutions de multiplicite d. Applications a l'interpolation dyadique, preprint.