Starting with Two Matrices

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Imagine that you have never seen matrices. On the principle that examples are amazingly powerful, we study two matrices A and C. The reader is requested to be exceptionally patient, suspending all prior experience—and suspending also any hunger for precision and proof. Please allow a partial understanding to be established first.

The first sections of this paper represent an imaginary lecture, very near the beginning of a linear algebra course. That lecture shows by example where the course is going. The key ideas of linear algebra (and the key words) come very early, to point the way. My own course now includes this lecture, and Notes 1-6 below are addressed to teachers.

A first example Linear algebra can begin with three specific vectors a_1, a_2, a_3:

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The fundamental operation on vectors is to take *linear combinations*. Multiply these vectors a_1, a_2, a_3 by numbers x_1, x_2, x_3 and add. This produces the linear combination $x_1a_1 + x_2a_2 + x_3a_3 = b$:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{1}$$

Step 2 is to rewrite that vector equation as a matrix equation $Ax = b$. Put a_1, a_2, a_3 into the columns of a matrix and put x_1, x_2, x_3 into a vector:

$$\text{Matrix } A = \begin{bmatrix} & & \\ a_1 & a_2 & a_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \text{Vector } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Key point A times x is exactly $x_1a_1 + x_2a_2 + x_3a_3$, *a combination of the columns*. This definition of Ax brings a crucial change in viewpoint. At first, the xs were multiplying the as. Now, the matrix A is multiplying x. The matrix acts on the vector x to produce a vector b:

$$\boldsymbol{Ax = b} \qquad Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2}$$

When the xs are known, the matrix A takes their differences. We could imagine an unwritten $x_0 = 0$, and put in $x_1 - x_0$ to complete the pattern. A is a *difference matrix*.

Note 1 Multiplying a matrix times a vector is the crucial step. If students have seen Ax before, it was *row times column*. In examples they are free to compute that way (as I do). "Dot product with rows" gives

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the same answer as "combination of columns". When the combination $x_1 a_1 + x_2 a_2 + x_3 a_3$ is computed one component at a time, we are using the rows.

The example illustrates how the same Ax arrives both ways. Differences like $x_2 - x_1$ come from row times column. Combining the columns of A is probably new to the class: *good*. The relation of the rows to the columns is truly at the heart of linear algebra.

Note 2 Three basic questions in linear algebra, and their answers, show why the column description of Ax is so essential:

- When does a linear system $Ax = b$ have a solution?

  $Ax = b$ asks us to express b as a combination of the columns of A. So there is a solution exactly when b is in the *column space* of A.

- When are vectors $a_1, \ldots, a_n$ linearly independent?

  The combinations of a_1,...,a_n are the vectors Ax. For independence, $Ax = 0$ must have only the zero solution. The *nullspace* of A must contain only the vector $x = 0$.

- How do you express b as a combination of basis vectors?

  Put those basis vectors into the columns of A. Solve $Ax = b$.

Note 3 The reader may object that we have only answered questions by introducing new words. My response is, those ideas of column space and nullspace and basis are crucial definitions in this subject. The student moves to a higher level—a subspace level—by understanding these words. We are constantly putting vectors into the columns of a matrix, and then working with that matrix.

I don't accept that inevitably "The fog rolls in" when linear independence is defined [1]. The concrete way to dependence vs. independence is through $Ax = 0$: many solutions or only the solution $x = 0$. This comes immediately in returning to the example of specific a_1, a_2, a_3.

Suppose the numbers x_1, x_2, x_3 are not known but b_1, b_2, b_3 are known. Then $Ax = b$ becomes an equation for x, not an equation for b. We start with the differences (the bs) and ask which xs have those differences. This is a new viewpoint of $Ax = b$, and linear algebra is always interested first in $b = 0$:

$$\boldsymbol{Ax = 0} \qquad\qquad Ax = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{Then} \quad \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \tag{3}$$

For this matrix, the only solution to $Ax = 0$ is $x = 0$. That may seem automatic but it's not. A key word in linear algebra (we are foreshadowing its importance) describes this situation. These column vectors a_1, a_2, a_3 are *independent*. Their combination $x_1 a_1 + x_2 a_2 + x_3 a_3$ is $Ax = 0$ only when all the xs are zero.

Move now to nonzero differences $b_1 = 1$, $b_2 = 3$, $b_3 = 5$. Is there a choice of x_1, x_2, x_3 that produces those differences $1, 3, 5$? Solving the three equations in forward order, the xs are $1, 4, 9$:

$$\boldsymbol{Ax = b} \qquad\qquad \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{leads to} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}. \tag{4}$$

This case $x = 1, 4, 9$ has special interest. When the bs are the odd numbers in order, the xs are the perfect squares in order. But linear algebra is not number theory—forget that special case! For any b_1, b_2, b_3 there is a neat formula for x_1, x_2, x_3:

$$\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{leads to} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}. \tag{5}$$

This general solution includes the examples with $b = 0, 0, 0$ (when $x = 0, 0, 0$) and $b = 1, 3, 5$ (when $x = 1, 4, 9$). One more insight will complete the example.

We started with a linear combination of a_1, a_2, a_3 to get b. Now b is given and equation (5) goes backward to find x. Write that solution with three new vectors whose combination gives x:

$$x = b_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = Sb. \tag{6}$$

This is beautiful, to see a *sum matrix* S in the formula for x. The equation $Ax = b$ is solved by $x = Sb$. The matrix S is the "inverse" of the matrix A. The difference matrix is inverted by the sum matrix. *Where A took differences of x_1, x_2, x_3, the new matrix S takes sums of b_1, b_2, b_3.*

Note 4 I believe there is value in *naming* these matrices. The words "difference matrix" and "sum matrix" tell how they act. It is the action of matrices, when we form Ax and Cx and Sb, that makes linear algebra such a dynamic and beautiful subject.

The linear algebra symbol for the inverse matrix is A^{-1} (not $1/A$). Thus $S = A^{-1}$ finds x from b. This example shows how linear algebra goes in parallel with calculus. Sums are the inverse of differences, and integration is the inverse of differentiation:

$$\boldsymbol{S = A^{-1}} \qquad Ax = \frac{dx}{dt} = b(t) \quad \text{is solved by} \quad x(t) = Sb = \int_0^t b. \tag{7}$$

The integral starts at $x(0) = 0$, exactly as the sum started at $x_0 = 0$.

The second example This example begins with almost the same three vectors—only one component is changed:

$$c_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad c_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad c_3 = \begin{bmatrix} \mathbf{-1} \\ 0 \\ 1 \end{bmatrix}.$$

The combination $x_1c_1 + x_2c_2 + x_3c_3$ is again a matrix multiplication Cx:

$$Cx = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{8}$$

With the new vector in the third column, C is a "cyclic" difference matrix. Instead of $x_1 - 0$ we have $x_1 - x_3$. The differences of xs "wrap around" to give the new bs. The inverse direction begins with b_1, b_2, b_3 and asks for x_1, x_2, x_3.

We always start with $0, 0, 0$ as the bs. You will see the change: nonzero xs can have zero differences. As long as the xs are equal, all their differences will be zero:

$$\boldsymbol{Cx = 0} \qquad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{is solved by} \quad x = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{9}$$

The zero solution $x = 0$ is included (when $x_1 = 0$). But $1, 1, 1$ and $2, 2, 2$ and π, π, π are also solutions—all these constant vectors have zero differences and solve $Cx = 0$. The columns c_1, c_2, c_3 are *dependent* and not independent.

In the row-column description of Ax, we have found a vector $x = (1, 1, 1)$ that is perpendicular to every row of A. The columns combine to give $Ax = 0$ when x is perpendicular to every row.

This misfortune produces a new difficulty, when we try to solve $Cx = b$:

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{cannot be solved unless} \quad b_1 + b_2 + b_3 = 0.$$

The three left sides add to zero, because x_3 is now cancelled by $-x_3$. So the bs on the right side must add to zero. There is no solution like equation (5) for every b_1, b_2, b_3. There is no inverse matrix like S to give $x = Sb$. The cyclic matrix C is *not invertible*.

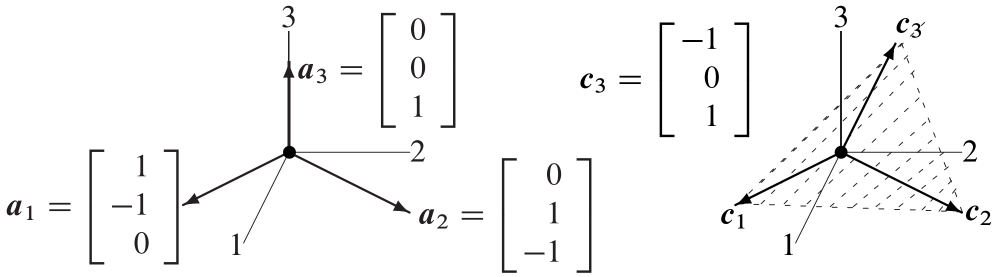
Summary Both examples began by putting vectors into the columns of a matrix. Combinations of the columns (with multipliers x) became Ax and Cx. Difference matrices A and C (noncyclic and cyclic) multiplied x—that was an important switch in thinking. The details of those column vectors made $Ax = b$ solvable for all b, while $Cx = b$ is not always solvable. The words that express the contrast between A and C are a crucial part of the language of linear algebra:

The vectors a_1, a_2, a_3 are independent.
The nullspace for $Ax = 0$ contains only $x = 0$.
The equation $Ax = b$ is solved by $x = Sb$.
The square matrix A has the inverse matrix $S = A^{-1}$.

The vectors c_1, c_2, c_3 are dependent.
The nullspace for $Cx = 0$ contains every "constant vector" x_1, x_1, x_1.
The equation $Cx = b$ cannot be solved unless $b_1 + b_2 + b_3 = 0$.
C has no inverse matrix.

A picture of the three vectors, a_1, a_2, a_3 on the left and c_1, c_2, c_3 on the right, explains the difference in a useful way. On the left, the three directions are *independent*. The three arrows don't lie in a plane. The combinations $x_1a_1 + x_2a_2 + x_3a_3$ produce every three-dimensional vector b. The good multipliers x_1, x_2, x_3 are given by $x = Sb$.

On the right, the three arrows do lie in a plane. The vectors c_1, c_2, c_3 are *dependent*. Each vector has components adding to $1 - 1 = 0$, so all combinations of these vectors will have $b_1 + b_2 + b_3 = 0$ (this is the equation for the plane). The differences $x_1 - x_3$ and $x_2 - x_1$ and $x_3 - x_2$ can never be $1, 1, 1$ because those differences add to zero.



Note 5 Almost unconsciously, one way of teaching a new subject is illustrated by these examples. *The ideas and the words are used before they are fully defined.* I believe we learn our own language this way—by hearing words, trying to use them, making mistakes, and eventually getting it right. A proper definition is certainly needed, it is not at all an afterthought. But maybe it is an afterword.

Note 6 Allow me to close these lecture ideas by returning to Note 1: Ax is a combination of the columns of A. Extend that matrix-vector multiplication to *matrix-matrix*: If the columns of B are b_1, b_2, b_3 then the columns of AB are Ab_1, Ab_2, Ab_3.

The crucial fact about matrix multiplication is that $\boldsymbol{(AB)C = A(BC)}$. By the previous sentence we may prove this fact by considering one column vector c.

$$\textit{Left side} \qquad (AB)c = [Ab_1 \ \ Ab_2 \ \ Ab_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1\, Ab_1 + c_2\, Ab_2 + c_3\, Ab_3 \tag{10}$$

$$\textit{Right side} \qquad A(Bc) = A(c_1 b_1 + c_2 b_2 + c_3 b_3). \tag{11}$$

In this way, $(AB)C = A(BC)$ brings out the even more fundamental fact that matrix multiplication is linear: (10) = (11).

Expressed differently, the multiplication AB has been defined to produce the composition rule: AB acting on c is equal to A acting on B acting on c.

Time after time, this associative law is the heart of short proofs. I will admit that these “proofs by parenthesis” are almost the only ones I present in class. Here are examples of $(AB)C = A(BC)$ at three key points in the course. (I don’t always use the ominous word *proof* in the video lectures [2] on ocw.mit.edu, but the reader will see through this loss of courage.)

- If $AB = I$ and $BC = I$ then $C = A$.

  Right inverse = Left inverse $\qquad C = (AB)C = A(BC) = A$

- If $y^{\mathrm{T}}A = 0$ then y is perpendicular to every Ax in the column space.

  Nullspace of $A^{\mathrm{T}} \perp$ *column space of* A $\qquad y^{\mathrm{T}}(Ax) = (y^{\mathrm{T}}A)x = 0$

- If an invertible B contains eigenvectors b_1, b_2, b_3 of A, then $B^{-1}AB$ is diagonal.

  Multiply AB by columns $\qquad A[b_1 \ \ b_2 \ \ b_3] = [Ab_1 \ \ Ab_2 \ \ Ab_3] = [\lambda_1 b_1 \ \ \lambda_2 b_2 \ \ \lambda_3 b_3]$

Then separate this AB into B times the eigenvalue matrix Λ:

$$AB = [\lambda_1 b_1 \;\; \lambda_2 b_2 \;\; \lambda_3 b_3] = [b_1 \;\; b_2 \;\; b_3] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \qquad \text{(again by columns!)}$$

$AB = B\Lambda$ gives the diagonalization $B^{-1}AB = \Lambda$. Equivalently it produces the factorization $A = B\Lambda B^{-1}$. Parentheses are not necessary in any of these triple factorizations:

Spectral theorem for a symmetric matrix	$A = Q\Lambda Q^{\mathrm{T}}$
Elimination on a symmetric matrix	$A = LDL^{\mathrm{T}}$
Singular Value Decomposition of any matrix	$A = U\Sigma V^{\mathrm{T}}$

One final comment: Factorizations express the central ideas of linear algebra in a very effective way. The eigenvectors of a symmetric matrix can be chosen orthonormal: $Q^{\mathrm{T}}Q = I$ in the spectral theorem $A = Q\Lambda Q^{\mathrm{T}}$. For all matrices, eigenvectors of AA^{T} and $A^{\mathrm{T}}A$ are the columns of U and V in the Singular Value Decomposition. And our favorite rule $(AA^{\mathrm{T}})A = A(A^{\mathrm{T}}A)$ is the key step in establishing that SVD, long after this early lecture...

These orthonormal vectors $u_1, ..., u_m$ and $v_1, ..., v_n$ are perfect bases for the *Four Fundamental Subspaces*: the column space and nullspace of A and A^{T}. Those subspaces become the organizing principle of the course [2]. The Fundamental Theorem connects their dimensions to the rank of A. The flow of ideas is from numbers to vectors to subspaces. Each level comes naturally, and everyone can get it—by seeing examples.

References

[1] David Carlson, Teaching Linear Algebra: Must the Fog Always Roll in?, *College Mathematics Journal*, **24** (1993) 29-40.

[2] Gilbert Strang, *Introduction to Linear Algebra*, Fourth edition, Wellesley-Cambridge Press (2009). Video lectures on web.mit.edu/18.06 and on ocw.mit.edu.