Pseudo-biorthogonal Multiwavelets and Finite Elements

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Abstract

This paper has two objectives. One is to propose a way to build perfect reconstruction multi-filters. This requires four $r \times r$ matrix polynomials H_0, H_1, F_0, F_1 such that

$$\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} F_0^*(z) & F_1^*(z) \\ F_0^*(-z) & F_1^*(-z) \end{bmatrix} = cI.$$

In the scalar case ($r = 1$) there are standard constructions of H_1, F_0, F_1 from a suitable H_0. A new procedure is needed for multi-filters, because matrices do not commute.

Our second purpose is to produce two specific pseudo-biorthogonal wavelet bases for L^2. We start with the piecewise cubic Hermite functions $\phi_0(t)$ and $\phi_1(t)$. These are "finite elements" supported on $[0, 2]$. They are the scaling functions for a particular multi-filter H_0 with $r = 2$. The connection between the filter coefficients $h_0(k)$ and the Hermite cubics is the matrix dilation equation

$$\phi(t) = \sum h_0(k)\phi(2t-k), \qquad \phi(t) = [\phi_0(t) \;\; \phi_1(t)]^T$$

The basis $\{\phi_0(t-k), \phi_1(t-k)\}$ for the subspace V_0 is not orthogonal. Our construction of the 2×2 matrix F_0 yields $\widetilde{\phi_0}(t)$ and $\widetilde{\phi_1}(t)$ supported on $[0, 4]$. The two bases are pseudo-biorthogonal. Then the construction of H_1 completes the finite elements $\{\phi_i(t-k)\}$ to a wavelet basis for L^2, and F_1 produces pseudo-biorthogonal wavelets. A similar construction completes the r finite element basis functions of degree $2r-1$ to a wavelet basis and also produces a pseudo-biorthogonal basis — all functions having short support and symmetry.

We also discuss the pseudo-biorthogonality itself.

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1 Introduction

We begin with the piecewise cubic finite elements $\phi_0(t)$ and $\phi_1(t)$. Those "Hermite cubics" are supported on two intervals (Figure 1). Their height and slope are specified at the ends of each interval — those four values determine the cubic. Their second derivatives jump at $t = 1$ (thus the functions are C^1). Their translates $\phi_0(t-k)$ and $\phi_1(t-k)$ are a basis for the space V_0 of all C^1 piecewise cubics on unit intervals.

This space is the beginning of a multiresolution $V_0 \subset V_1 \subset V_2 \subset \ldots$ in which V_j contains the C^1 piecewise cubics on intervals of length 2^{-j}. Our purpose is to construct a dual multiresolution $\widetilde{V}_0 \subset \widetilde{V}_1 \subset \widetilde{V}_2 \subset \ldots$ based on functions $\widetilde{\phi}_0(t)$ and $\widetilde{\phi}_1(t)$. We are looking for basis functions with good properties and satisfying the condition of perfect reconstruction. Those functions $\widetilde{\phi}_0(t-k)$ and $\widetilde{\phi}_1(t-k)$ will not be piecewise cubic, but they are smooth with short support.

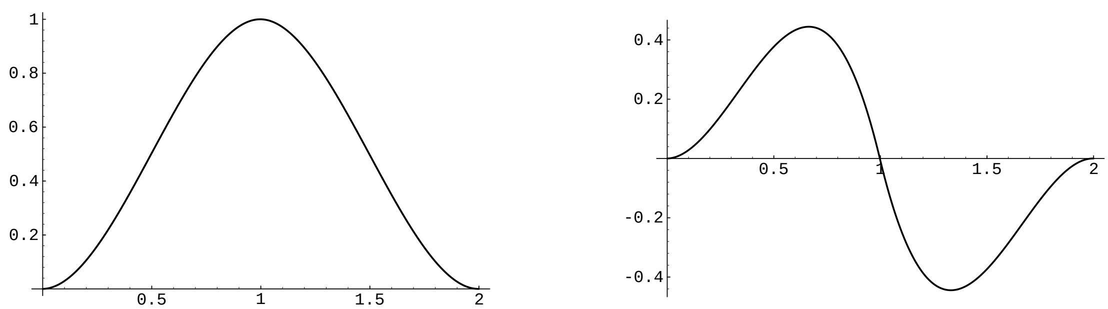


Figure 1: Hermite cubic scaling functions $\phi_0(t)$ and $\phi_1(t)$

It is useful to compare these Hermite cubics to splines. The cubic splines have C^2 continuity instead of C^1. The spline basis $\{\phi(t-k)\}$ comes from the translates of a single function instead of two functions. That B-spline $\phi(t)$ is supported on four intervals instead of two. The functions $\phi(2^j t - k)$ are a non-orthogonal basis for the space V_j of cubic splines on intervals of length 2^{-j}. This increasing family $V_0 \subset V_1 \subset V_2 \subset \ldots$ is completed to a multiresolution by wavelet subspaces W_j such that

$$V_{j+1} \text{ is the direct sum of } V_j \text{ and } W_j.$$

This construction can be *semiorthogonal* or *biorthogonal*, and we recall from [3] the essential difference:

Semiorthogonal: W_j is orthogonal to V_j. The spaces are orthogonal but their bases $\{\phi(2^j t - k)\}$ and $\{w(2^j t - k)\}$ are not self-orthogonal. In this case $V_j = \widetilde{V}_j$ and $W_j = \widetilde{W}_j$, but the biorthogonal basis functions $\widetilde{\phi}(2^j t - k)$ and $\widetilde{w}(2^j t - k)$ have infinite support.

Biorthogonal: W_j is orthogonal to a different subspace $\widetilde{V}_j$. At the same time V_j is orthogonal to $\widetilde{W}_j$. The direct sum of $\widetilde{V}_j$ and $\widetilde{W}_j$ is $\widetilde{V}_{j+1}$. In this case the basis functions $\widetilde{\phi}(2^j t - k)$ and $\widetilde{w}(2^j t - k)$ do have compact support.

There are advantages and disadvantages to both. A third possibility is to have fully orthogonal bases, always coming from the dilation to $2^j t$ and translation by k. But an orthogonal multiresolution cannot use splines. Its basis functions are not symmetric and their support is not so short. Therefore the spline wavelets and their variations are popular in applications. They are constructed by iteration of a filter bank. We will carry out a similar procedure for multiwavelets and multi-filters, starting from the Hermite cubics $\phi_0(t)$ and $\phi_1(t)$.

The semiorthogonal construction is already in the literature [2, 7]. It is parallel to the spline case (the dual basis functions again have infinite support). Bases presented in this paper are *pseudo-biorthogonal* in the sense that our multi-filter bank provides perfect reconstruction but the synthesis wavelets are not orthogonal to the analysis scaling functions. Such a construction is possible only in the case of multiwavelets. This phenomenon is explained in Section 8. Our dual wavelets $w(t)$ and $\widetilde{w}(t)$ will be supported on three intervals and the dual scaling function $\widetilde{\phi}(t)$ on four intervals. A similar construction applies for piecewise polynomials of degree $2n-1$ and smoothness C^{n-1}. The mathematical problem is to complete a block matrix and its inverse, *both to be polynomials in z*. These matrices come from filters, which we explain next.

2 Filters and Bases

A filter is a Toeplitz matrix. A multi-filter is a block Toeplitz matrix. In both cases the kth diagonal contains the kth filter coefficient $h(k)$. For multi-filters this is a block diagonal and $h(k)$ is an $r \times r$ matrix. The filter is a convolution $\sum h(k)x(n-k)$, when the input vector x comes in blocks (r components at a time). This convolution becomes multiplication in the frequency domain, and the filter multiplies the transform $X(\omega)$ by the frequency response $H(\omega)$:

$$H(\omega) = \sum_{0}^{N} h(k)e^{-ik\omega} = r \times r \text{ matrix polynomial.}$$

Iteration of a filter may or may not lead to a *scaling function* $\phi(t)$. At each step the time is rescaled to $2t$, so the frequencies are rescaled to $\omega/2$. This iteration $\phi^{(j+1)} = 2\sum h(k)\phi^{(j)}(2t-k)$ may approach a fixed point $\phi(t)$:

$$\phi(t) = 2\sum h(k)\phi(2t-k) \quad \text{and} \quad \widehat{\phi}(\omega) = H(\frac{\omega}{2})\widehat{\phi}(\frac{\omega}{2}).$$

This is the *dilation equation* or *refinement equation*. For multi-filters, each $h(k)$ is a matrix and ϕ is a vector $[\phi_0 \ \ldots \ \phi_{r-1}]^T$. In terms of the subspaces V_0 and V_1 spanned by $\{\phi_j(t-k)\}$ and $\{\phi_j(2t-k)\}$, the dilation equation shows how V_0 is contained in V_1.

For cubic splines, the coefficients $h(k)$ are $1, 4, 6, 4, 1$ (all divided by 16). Splines of higher degree also lead to binomial coefficients. In each case the space V_0 with nodes at the integers is clearly contained in V_1 (which also allows nodes at $t = n + \frac{1}{2}$). In the Hermite case, the same property $V_0 \subset V_1$ assures that some combination of the compressed functions $\phi_0(2t-k)$ and $\phi_1(2t-k)$ must produce $\phi_0(t)$ and $\phi_1(t)$:

$$\begin{bmatrix} \phi_0(t) \\ \phi_1(t) \end{bmatrix} = h_0(0) \begin{bmatrix} \phi_0(2t) \\ \phi_1(2t) \end{bmatrix} + h_0(1) \begin{bmatrix} \phi_0(2t-1) \\ \phi_1(2t-1) \end{bmatrix} + h_0(2) \begin{bmatrix} \phi_0(2t-2) \\ \phi_1(2t-2) \end{bmatrix}.$$

The 2×2 matrix coefficients are found to be

$$h_0(0) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix}, \qquad h_0(1) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad h_0(2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{1}{8} \end{bmatrix}.$$

The frequency response is $h_0(0)+h_0(1)e^{-i\omega}+h_0(2)e^{-2i\omega}$. If we replace $e^{-i\omega}$ by z (the correct notation is z^{-1}!) then this response becomes the transfer function

$$H_0(z) = h_0(0) + h_0(1)z + h_0(2)z^2 = \frac{1}{16} \begin{bmatrix} 4(1+z)^2 & -2(1-z)(1+z) \\ 3(1-z)(1+z) & -1+4z-z^2 \end{bmatrix}.$$

Starting with this specific matrix polynomial, we can describe our task. It is to find three more matrix polynomials $F_0(z)$, $H_1(z)$, $F_1(z)$ such that

$$\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} F_0^*(z) & F_1^*(z) \\ F_0^*(-z) & F_1^*(-z) \end{bmatrix} = cI. \tag{2.1}$$

The matrix $F_0(z)$ with coefficients $f_0(k)$ will give the dual dilation equation

$$\tilde{\phi}(t) = 2\sum f_0(k)\tilde{\phi}(2t-k). \tag{2.2}$$

The wavelets $w(t)$ and $\tilde{w}(t)$ are defined by the matrix polynomials $H_1(z)$ and $F_1(z)$. The coefficients in these "highpass" polynomials appear in the wavelet equations

$$w(t) = 2\sum h_1(k)\phi(2t-k)$$

$$\tilde{w}(t) = 2\sum f_1(k)\tilde{\phi}(2t-k).$$

When $\tilde{\phi}(t) = [\tilde{\phi}_0(t) \;\; \tilde{\phi}_1(t)]^T$ is in L^2 and $c=1$, its translates are automatically biorthogonal to the Hermite cubics coming from H_0:

$$\int \phi_i(t)\tilde{\phi}_j(t-k)dt = \delta(i-j)\delta(k). \tag{2.3}$$

Furthermore, complete biorthogonality will hold between two dual bases bases for L^2:

$$\phi_i(t-k), w_i(t-k), \ldots, 2^{j/2}w_i(2^jt-k), \ldots \tag{2.4}$$

$$\tilde{\phi}_i(t-k), \tilde{w}_i(t-k), \ldots, 2^{j/2}\tilde{w}_i(2^jt-k), \ldots \tag{2.5}$$

We will consider the case of $c \neq 1$. This means that execution of one analysis step followed by one synthesis step will produce the initial signal multiplied by c. Strictly speaking it is not perfect reconstruction, but still the signal can be recovered exactly if one rescales by c at each synthesis step. However, when $c \neq 1$ the condition (2.3) does not hold and the corresponding wavelet bases (2.4), (2.5) are not biorthogonal. Explanation to this fact is given in Section 8.

Notice that $c \neq 1$ is not possible in the scalar case (one scaling function), because of the necessary condition $H_0(1) = F_0(1) = 1$.

3 Filter Banks: Analysis and Synthesis

The diagram below illustrates a filter bank. Each filter H_0, H_1, F_0, F_1 is a Toeplitz matrix (discrete convolution). The downsampling operator $\downarrow 2$ removes the odd-numbered components of H_0x and H_1x. Its transpose $\uparrow 2$ puts zeros in those positions. Thus the result from $\downarrow 2$ and $\uparrow 2$ is zero for odd n:

$$(\downarrow 2)y(n) = y(2n) \qquad (\uparrow 2)(\downarrow 2)y(n) = \frac{1}{2}(y_n + (-1)^n y_n).$$

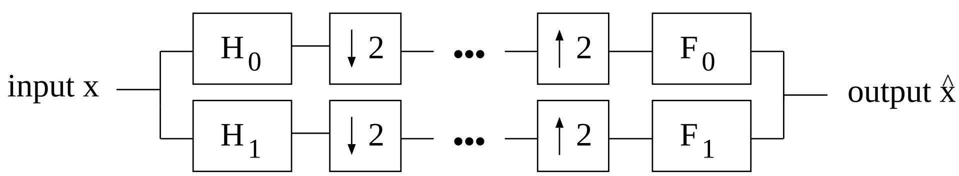
We are keeping the *even part* of the transform $Y(z) = \sum y(n)z^n$:

$$(\uparrow 2)(\downarrow 2)Y(z) = \frac{1}{2}(Y(z) + Y(-z)).$$

Following along the upper channel in the block diagram, the first step gives $Y(z) = H_0(z)Y(z)$. The output from that whole channel is

$$\frac{1}{2}F_0^*(z)[H_0(z)X(z) + H_0(-z)X(-z)].$$

Replace 0 by 1 for the second (highpass) channel, and add the outputs from the two channels. The sum agrees with $X(z)$ — *the reconstruction is perfect* — exactly when the matrices in (2.1) are inverses.



In words, the synthesis bank is the inverse of the analysis bank. The rows of a matrix A are biorthogonal to the columns of A^{-1}. Here A and A^{-1} are the two filter banks:

$$A = \begin{bmatrix} (\downarrow 2)H_0 \\ (\downarrow 2)H_1 \end{bmatrix} \qquad A^{-1} = [F_0^T(\uparrow 2) \;\; F_1^T(\uparrow 2)]$$

An orthogonal filter bank has $H_0 = F_0$ and $H_1 = F_1$ and $A^{-1} = A^T$. But the spline and finite element filters are not orthogonal. The ordinary case is the biorthogonal one, with four different filters. The vector Ax expresses the input x in a new basis, and $A^{-1}(Ax) = \sum$(new basis)(new coefficients) returns to the original x.

Signal processing includes *compression* of Ax. That nonlinear step is represented by dots in the block diagram — we don't deal with it here. The reconstruction is quick and close but not perfect; the coefficients Ax are quantized (rounded off), and partly lost. Our goal in constructing A and A^{-1} for Hermite cubics is the hope that they will give a good basis for compression — this is still to be tested in experiment. We are free to reverse the H's with the F's. In fact we will do so in the experiments, because it is better to have the short smooth Hermite cubics in synthesis.

The key point is that dual basis functions come from dual filters. We iterate the filters to reach $\phi(t)$, $w(t)$ and $\tilde{\phi}(t)$, $\tilde{w}(t)$.

4 Construction of F_0 when $r = 2$

Starting from a given $H_0(z)$, the first entry in the matrix multiplication (2.1) gives the requirement on F_0 (or its complex conjugate F_0^*):

$$H_0(z)F_0^*(z) + H_0(-z)F_0^*(-z) = cI. \tag{4.6}$$

In words, the product filter $H_0F_0^*$ is a "halfband filter". *Its even part is a constant.* When $H_0(z)$ is a *scalar* polynomial (from a filter instead of a multi-filter) there is a solution $F_0(z)$ provided $H_0(z)$ and $H_0(-z)$ have no common roots. One important example of a halfband product filter $H_0(z)F_0^*(z)$ is

$$z^{-3}(\frac{1+z}{2})^4(-1+4z-z^2) = \frac{1}{16}(-z^{-3}+9z^{-1}+16+9z-z^3). \tag{4.7}$$

Actually the Daubechies construction begins with this choice. Each factorization into H_0 times F_0^* gives a biorthogonal filter bank. (H_1 and F_1 come from an alternation of signs in this scalar case.) The particular factorization with $H_0 = (\frac{1+z}{2})^4$ gives the cubic splines as basis functions. The filter coefficients in $(\frac{1+z}{2})^4$ are $1, 4, 6, 4, 1$ divided by 16. But the other factor $-1 + 4z - z^2$ is not an acceptable $F_0^*(z)$; it fails to satisfy Condition E. The good filters F_0 in this spline case have higher degree, because Condition E for stability of iteration needs enough powers of $1 + z$.

In the matrix case, when H_0 and F_0 are multi-filters, we return to equation (4.6). Now $H_0(z)$ is an $r \times r$ matrix polynomial; so is the unknown $F_0(z)$. For the 2×2 case we propose the following construction:

$$\text{Given} \quad H_0(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} \quad \text{choose} \quad F_0^*(z) = z^{-k}e_0(z)\begin{bmatrix} d(z) & -b(z) \\ -c(z) & a(z) \end{bmatrix}. \tag{4.8}$$

The product $H_0(z)F_0^*(z)$ is a multiple of the identity matrix:

$$H_0(z)F_0^*(z) = z^{-k}e_0(z)(a(z)d(z) - b(z)c(z))I. \tag{4.9}$$

The matrix will be halfband when this scalar polynomial is halfband. Thus we are back to the scalar case: *for a given determinant $a(z)d(z) - b(z)c(z)$ choose $z^{-k}e_0(z)$ so that the product has even part = constant.* Note that $F_0(z)$ involves a, b, c, d as well as e_0! If it happened that $e_0(z) = 1$, which is not true, then the dilation equation for F_0 would be a simple reversal of the equation for H_0:

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{has scaling functions} \quad \begin{bmatrix} \phi_1(-t) \\ -\phi_0(-t) \end{bmatrix}.$$

The presence of $e_0(z)$ means that the actual $\tilde{\phi}_i(t)$ are the convolutions of this pair with the scaling function $\phi_e(t)$ for the filter E_0.

Important: The given basis $\{\phi_0(t-k), \phi_1(t-k)\}$ has approximation order m if all the polynomials $1, t, \ldots, t^{m-1}$ are linear combinations of the basis functions. The Hermite cubics clearly have approximation order $m = 4$. All cubics can be produced from those basis

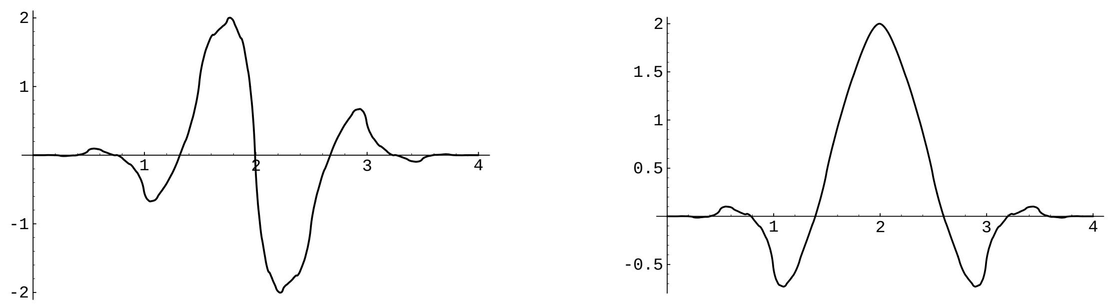


Figure 2: The scaling functions $\tilde{\phi}_0(t)$ and $\tilde{\phi}_1(t)$ pseudo-biorthogonal to Hermite cubics

functions. The translates of $\phi_1(-t), -\phi_0(-t)$ always give the same approximation order. We will show that the presence of $e_0(z)$ does not change this order if $e_0(-1) \neq 0$. The dual basis $\{\tilde{\phi}_0(t-k), \tilde{\phi}_1(t-k)\}$ in Figure 2 also has $m = 4$.

The assumed form of F_0 has reduced the matrix equation (4.6) to a scalar problem when the matrices are 2×2. This includes the Hermite cubic example, which has a very particular determinant:

$$\det H_0(z) = \frac{1}{16^2} \begin{vmatrix} 4(1+z)^2 & -2(1-z^2) \\ 3(1-z^2) & -1+4z-z^2 \end{vmatrix} = \frac{(1+z)^4}{128}.$$

This is the factor $ad - bc$ in the product $H_0(z)F_0^*(z)$ of equation (4.9). The other factor $z^{-k}e_0(z)$ is chosen to make the whole product halfband. But this is exactly the Daubechies construction in equation (4.7). Her choice $z^{-3}(-1+4z-z^2)$ multiplies $(1+z)^4$ from the determinant to produce the halfband filter with coefficients $-1, 0, 9, 16, 9, 0, -1$. It is those zero coefficients multiplying z^{-2} and z^2 that make the even part constant.

We summarize our new construction in Theorem 4.1. It applies to the general case with $r = 2$, and Figure 2 is for the specific case of Hermite cubics.

Theorem 4.1 *Suppose $\Delta(z) = a(z)d(z) - b(z)c(z)$ is the determinant of a 2×2 matrix $H_0(z)$. If $\Delta(z)$ and $\Delta(-z)$ have no common zeros, there are halfband polynomials (even part = constant) of the form $z^{-k}e_0(z)\Delta(z)$. Then*

$$F_0^*(z) = z^{-k}e_0(z) \begin{bmatrix} d(z) & -b(z) \\ -c(z) & a(z) \end{bmatrix}$$

solves equation (4.9). The scaling functions $\tilde{\phi}_0(t-k), \tilde{\phi}_1(t-k)$ from the dilation equation for F_0 provide the same approximation order m if $e_0(-1) \neq 0$.

But the functions $\tilde{\phi}_0$, $\tilde{\phi}_1$ will be longer and less smooth than ϕ_0, ϕ_1 because of that factor e_0. The Hermite cubics ϕ_0, ϕ_1 have jumps in the second derivative; they belong to the Sobolev space H^s for every $s < 1.50$. The dual scaling functions in Figure 2 belong to H^s for $s < 1.44$.

5 Construction of F_0 when $r > 2$

The entries $d(z), -b(z), -c(z), a(z)$ in this 2×2 case were the *cofactors* of $H_0(z)$. This is the key to our construction in all cases. Any $r \times r$ matrix times its cofactor matrix is a multiple of the identity. The multiple is the determinant $\Delta(z)$. Therefore we choose

$$F_0^*(z) = z^{-k} e_0(z) H_{cof}(z) \text{ to find } F_0^*(z) H_0(z) = z^{-k} e_0(z) \Delta(z) I. \tag{5.10}$$

Then we select $z^{-k} e_0(z)$ to make this product halfband. There is an important normalization which determines the constant $c \neq 1$ (in the even part of $F_0^*(z) H_0(z)$) and the approximation order for $\tilde{\phi}_0(t), \ldots, \tilde{\phi}_{r-1}(t)$.

Algorithm 5.1 (Cofactor Method) *Suppose $\Delta(z) = \det H_0(z)$ is not divisible by $z^2 - \lambda^2$ for any $\lambda \neq 0$, and the eigenvalue of $H_0(1)$ with the smallest absolute value is simple. Then $F_0(z)$ is constructed in three steps:*

1. *Compute the cofactor matrix $H_{cof}(z)$*
2. *Find a polynomial $e_0(z)$ such that $e_0(1) = 1$ and $z^{-k} e_0(z) \Delta(z)$ is halfband*
3. *Compute the dual symbol $F_0(z) = \frac{1}{\lambda_0^*} e_0^*(z) H_{cof}^*(z)$, where λ_0 is the eigenvalue of $H_{cof}(1)$ with the largest absolute value.*

This cofactor method ensures nice properties of $F_0(z)$ such as approximation and symmetry.

Theorem 5.2 *If $H_0(z)$ has approximation order m, the dual symbol $F_0(z) = \frac{1}{\lambda_0^*} e_0^*(z) H_{cof}^*(z)$ provides approximation order not less than m.*

Remark. Theorem 5.2 does not hold in the scalar case, when $H_{cof}(z) \equiv 1$ and $F_0(z) = e_0(z)$ does not give any approximation at all.

Theorem 5.3 *Suppose all components of $\phi(t)$ are either symmetric or antisymmetric. Then $F_0(z)$ also yields symmetric or antisymmetric scaling functions, which are the dual basis functions $\tilde{\phi}_j(t-k)$.*

Proofs of Theorems 5.2 and 5.3 can be found in [5].

6 Construction of $F_1(z)$

The next step is to satisfy the $r \times r$ matrix equation

$$H_0(z) F_1^*(z) + H_0(-z) F_1^*(-z) = 0. \tag{6.11}$$

This is a linear homogeneous system in the coefficients of F_1^*. The odd part of $H_0 F_1^*$ automatically satisfies this equation. Suppose that $H_0(z)$ has degree N. We also choose $F_1(z)$ to be an $r \times r$ matrix polynomial of degree N — multiplied by an extra factor z when N is even. (Then F_1^* has an extra factor z^{-1}.) The number of unknowns is $(N+1)r^2$.

The number of equations from (6.11) is Nr^2, so the system has nonzero solutions F_1^*. There are r^2 equations from each even power of z:

$$\text{odd } N : z^{N-1}, \ldots, z^0, \ldots, z^{1-N} \qquad \text{even } N : z^{N-2}, \ldots, z^0, \ldots, z^{-N}.$$

Writing $\widetilde{\Delta}(z)$ for the determinant of $F_1(z)$, equation (6.11) gives

$$\Delta(z)\widetilde{\Delta}^*(z) = -\Delta(-z)\widetilde{\Delta}^*(-z). \tag{6.12}$$

We assume as earlier that the determinant $\Delta(z)$ of $H(z)$ shares no zeros with $\Delta(-z)$. Then there must be a polynomial $p(z)$ such that $\widetilde{\Delta}(z) = -p(z)\Delta(-z)$. In most cases one can identify $p(z)$ as 1 or z^r:

Lemma 6.1 *If the last coefficient $h_0(N)$ in the given filter is nonsingular, then $\Delta(z)$ has highest power z^{rN}. From the degree of $F_1(z)$, the highest power in $\widetilde{\Delta}(z)$ is z^{rN} for odd N and $z^{r(N+1)}$ for even N. Thus up to a constant factor, $p(z) = 1$ for odd N and $p(z) = z^r$ for even N.*

The determinants satisfy precisely the same alternating sign condition that applies to the scalar case $r = 1$. There the highpass filters are $F_1(z) = -H_0(-z)$ and $H_1(z) = F_0(-z)$. In [3] we assume N to be odd, by introducing a trivial coefficient $h_0(N) = 0$ if necessary. That produces (when N is originally even) the same zero constant term that we created above with the extra factor z in F_1.

In short, the scalar conditions $F_1(z) = -H_0(-z)$ and $H_1(z) = F_0(-z)$ generally extend to the *determinants* in the matrix case. They do not extend to the matrix polynomials themselves, because (6.11) would not hold when matrices fail to commute. In the specific case of Hermite cubics, the convenient form of $H_0(z)$ gives an equally convenient $F_1(z)$:

$$F_1(z) = \frac{z}{16}\begin{bmatrix} 4(1-z)^2 & -6(1-z^2) \\ 1-z^2 & -1-4z-z^2 \end{bmatrix}.$$

The 2×2 matrix coefficients $f_1(k)$ of z, z^2, z^3 produce the highpass filter F_1. Then the wavelet $\widetilde{w}(t)$ is given by $\sum f_1(k)\widetilde{\phi}(2t-k)$. Its two components $\widetilde{w}_0(t)$ and $\widetilde{w}_1(t)$ are drawn in Figure 3.

The construction of $F_1(z)$ is also straightforward [5] for finite elements of higher degree $2r-1$. We emphasize that the r wavelets and dual wavelets obtained by this cofactor method are always symmetric or antisymmetric, and all filter coefficients are integers divided by powers of 2. Thus all filter operations can be computed by adds and shifts.

The preprint [4] solves a related problem of constructing biorthogonal highpass filters. Starting from two $n \times r$ matrices $Q(z)$ and $P(z)$ with $Q^T(z)P(z) = I_r$, Shen constructs $n \times n$ matrices $Y(z)$ and $X(z)$ with $Y^T(z)X(z) = I_n$.

7 Construction of $H_1(z)$

The perfect reconstruction condition (2.1) imposes two matrix equations on $H_1(z)$:

$$H_1(z)F_0^*(z) + H_1(-z)F_0^*(-z) = 0 \tag{7.13}$$

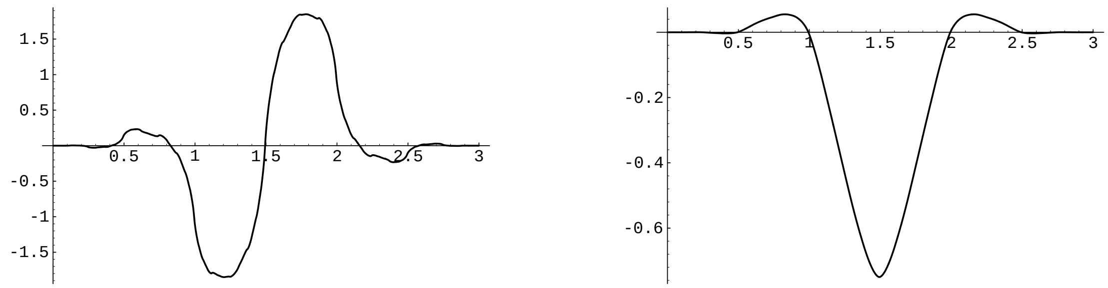


Figure 3: Wavelets $\tilde{w}_0(t)$ and $\tilde{w}_1(t)$ pseudo-biorthogonal to Hermit cubics.

$$H_1(z)F_1^*(z) + H_1(-z)F_1^*(-z) = cI. \tag{7.14}$$

We use the same construction to determine H_1 from the cofactors of F_1 that earlier determined F_0 from H_0. The choice

$$H_1(z) = z^{-K}e_1(z)F_{cof}(z) \text{ leads to } H_1(z)F_1^*(z) = z^{-K}e_1(z)\tilde{\Delta}(z)I. \tag{7.15}$$

Now select the scalar polynomial $e_1(z)$ to make this halfband, and (7.14) is satisfied.

In the nondegenerate case, when $h_0(N)$ is nonsingular, we noted above that $\tilde{\Delta}(z)$ is equal to $-\Delta(-z)$ or $-z^r\Delta(-z)$. In this case we already know the successful $e_1(z)$. It is exactly $e_0(-z)$! The scalar polynomials in (5.10) and (7.15) are identical apart from powers of z and a reflection from z to $-z$. Thus they are both halfband. The Hermite cubic example yields

$$H_1(z) = e_0(-z)F_{cof}(z) = \frac{1}{32}\begin{bmatrix} 1+8z+18z^2+8z^3+z^4 & -1-4z+4z^3+z^4 \\ 6+24z-24z^3-6z^4 & -4-8z+24z^2-8z^3-4z^4 \end{bmatrix}.$$

From these matrix coefficients $h_1(0), \ldots, h_1(4)$ we find the wavelets. They are the piecewise cubics $w(t) = \sum h_1(k)\phi(2t-k)$ drawn in Figure 4. Note that they are supported on $[0,3]$ and they are cubics on half-intervals.

It remains to verify equation (7.13). The constructions (7.15) and (5.10) have

$$\begin{aligned} H_1(z)F_0^*(z) &= z^{-K}e_1(z)F_{cof}(z)(z^{-k}e_0(z)H_{cof}(z))^* \\ &= z^{k-K}e_1(z)e_0(z)\Delta(z)\tilde{\Delta}^*(z)F^{-1}(z)H_0^{-1}(z)^*. \end{aligned} \tag{7.16}$$

Equation (6.11) gives $H_0(z)F_1^*(z) = -H_0(-z)F_1^*(-z)$. Invert and substitute into (7.16). Use (6.12) to change z to $-z$ in the product $\Delta(z)\tilde{\Delta}^*(z)$. The result is (7.13).

Note that $w_0(t)$ and $w_1(t)$ are not orthogonal to each other or to Hermite cubics $\phi(t)$. This is not the semiorthogonal construction or the biorthogonal construction, because $w_0(t)$ is not orthogonal to the dual multi-scaling function $\tilde{\phi}(t)$. (The integral of $w_0(t)$ is not zero, thus w_0 is not orthogonal to the constant function, which is a linear combination of the translates $\tilde{\phi}_0(t-k), \tilde{\phi}_1(t-k)$.) Nevertheless perfect reconstruction holds. We call these bases *pseudo-biorthogonal*.

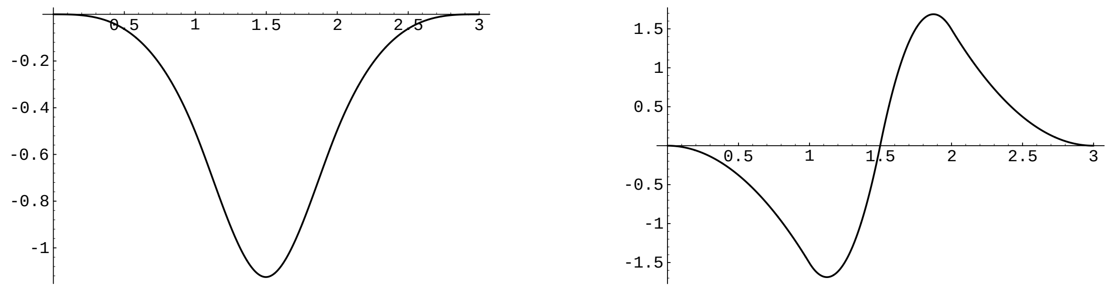


Figure 4: Piecewise cubic wavelets $w_0(t)$ and $w_1(t)$.

8 Pseudo-biorthogonality

This section identifies the condition for pseudo-biorthogonality. First let us define the joint transition operator T. Assume that two r by r matrix trigonometric polynomials $H_0(\omega)$ and $F_0(\omega)$ are given, $H_0(0)$ and $F_0(0)$ have a simple eigenvalue 1, and the dilation equations hold for ϕ and $\tilde{\phi}$:

$$\widehat{\phi}(\omega) = H_0(\omega/2)\widehat{\phi}(\omega/2), \quad \widehat{\tilde{\phi}}(\omega) = F_0(\omega/2)\widehat{\tilde{\phi}}(\omega/2).$$

Consider the operator T acting on an r by r matrix function $A(\omega)$:

$$T(A)(\omega) = H_0(\omega/2)A(\omega/2)F_0^*(\omega/2) + H_0(\omega/2+\pi)A(\omega/2+\pi)F_0^*(\omega/2+\pi).$$

It is easy to see that T always has an eigenvalue 1 with eigenmatrix G:

$$G(\omega) = \sum_l \widehat{\phi}(\omega + 2\pi l)\widehat{\tilde{\phi}}(\omega + 2\pi l).$$

$\phi(t)$ and $\tilde{\phi}(t)$ form a pair of biorthogonal bases if and only if $G = I$, or in other words, perfect reconstruction holds with $c = 1$:

$$H_0(\omega/2)F_0^*(\omega/2) + H_0(\omega/2+\pi)F_0^*(\omega/2+\pi) = I.$$

Nevertheless, it could happen that $\phi(t)$ and $\tilde{\phi}(t)$ are not biorthogonal, $G \neq I$, but T has the eigenmatrix I corresponding to an eigenvalue $c \neq 1$:

$$H_0(\omega/2)F_0^*(\omega/2) + H_0(\omega/2+\pi)F_0^*(\omega/2+\pi) = cI \tag{8.17}$$

This is exactly the case which we considered.

The cofactor method constructs F_0 such that the the eigenvalue is $c \neq 1$. Then the generalized condition of perfect reconstruction (8.17) is satisfied with c equal to the smallest eigenvalue of $H_0(0)$, but $G \neq I$ and the bases are not biorthogonal.

Our Hermite cubic example has $c = 1/8$. This means that cubic finite elements are actually biorthogonal to the third derivative of the multi-scaling function presented in the paper.

One can go further and relax the perfect reconstruction condition to

$$H_0(z)F_0^*(z) + H_0(-z)F_0^*(-z) = Q(z),$$

where $Q(z)$ is a matrix polynomial such that $\det Q(z) = z^k$ for some integer k. This will require multiplication by the matrix polynomial $Q^{-1}(z)$ after each synthesis step. This generalization is non-trivial only for multi-filters, because in the scalar case $\det Q(z) = z^k$ means $Q(z) = z^k$.

More on biorthogonality of multiwavelets and Hermite cubics can be found in [1, 6, 8].

9 Summary

The underlying problem is to complete the lower half of a matrix in such a way that the determinant is a power of z. This is the the $2r \times 2r$ "modulation matrix"

$$H_m(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}$$

Then the inverse $F_m(z)$ of $H_m(z)$ is also a polynomial matrix. The analysis filters H_0, H_1 and the synthesis filters F_0, F_1 are all FIR (the matrices are banded and the convolutions are finite).

Our construction went directly to the inverse matrix. We found $F_0(z)$ from the cofactors of $H_0(z)$, multiplied by an extra factor $e_0(z)$. Then $F_1(z)$ was the solution of a homogeneous linear system. Finally $H_1(z)$ came from the cofactors of $F_1(z)$. In the non degenerate case, the determinants follow the same rule of alternating signs that succeeds in the scalar case.

The Hermite cubic example (with $r = 2$) allows a particularly simple construction. The factor $e_0(z) = -1 + 4z - z^2$ is the one that Daubechies used for the same purpose, to make $z^{-3}e_0(z)(1+z)^4$ a halfband polynomial (even part = constant). The accuracy $m = 4$ of the Hermite cubics from $H_0(z)$ is maintained by the dual functions $\tilde{\phi}_0(t)$, $\tilde{\phi}_1(t)$ from $F_0(z)$. The filters are as short as possible with this accuracy and with symmetry. The support intervals are $[0, 2]$ for $\phi(t)$ and $[0, 4]$ for $\tilde{\phi}(t)$ and $[0, 3]$ for $w(t)$ and $\tilde{w}(t)$.

The Hermite cubics are important in the finite element method. We hope that this pseudo-biorthogonal completion will make them useful in the "wavelet method". The cubics are the trial functions and the dual construction gives the test functions. All the Galerkin integrals of products can be computed quickly ([3], Section 11.6) from the filter coefficients. This wavelet method deserves a try.

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References

[1] A. Aldroubi, Oblique and biorthogonal multiwavelet bases, *preprint* (1996).

[2] T. N. T. Goodman and S. L. Lee, Wavelets of multiplicity r, *Trans. Amer. Math. Soc.* **342** (1994) 307–324.

[3] G. Strang and T. Nguyen, Wavelets and Filter Banks, Wellesley-Cambridge Press (1996).

[4] Z. Shen, Extension of matrices with Laurent polynomial entries, *preprint* (1996).

[5] V. Strela, Multiwavelets: Theory and applications, *Ph. D. thesis, MIT* (1996).

[6] V. Strela, A note on construction of biorthogonal multi-scaling functions, *preprint* (1997).

[7] V. Strela and G. Strang, Finite element multiwavelets, *Proc. Maratea NATO Conference*, Kluwer (1995).

[8] R. Warming and R. Beam, Discrete multiresolution analysis using Hermite interpolation: Biorthogonal multiwavelets, *preprint* (1997).