Design and Analysis of Optimal Filters

Jianhong Shen Gilbert Strang

Department of Mathematics
Massachusetts Institute of Technology
jhshen and gs@math.mit.edu

The classical problem of lowpass filter design begins with a passband $|\omega| \le \omega_p$ and a stopband $\omega_s \le |\omega| \le \pi$. We look for the trigonometric polynomial of degree $2N+1$ that best approximates 1 on the passband and 0 on the stopband. If the maximum error is δ, then the error over these intervals oscillates with equal ripples of size δ. The Remez algorithm produces the optimal polynomial.

We study three basic questions about this polynomial:

1. The asymptotic value of δ has the form $c\,N^{-1/2}\exp(-N\beta)$

   We determine the value of β. This improves an empirical formula that is frequently used to choose the correct degree N for given δ.

2. We determine the leading term in the asymptotic form of the polynomial in the transition band $\omega_p < |\omega| < \omega_s$. It approaches an error function (confirmed numerically).

3. We propose a faster algorithm based on interpolation at rescaled Chebyshev points, to give an approximately optimal filter design.

The basic technique is analysis of the Green's function for a doubly connected region in the complex plane (the complement of two intervals). We rely on stationary phase for asymptotic estimates.