Laplacian Eigenvalues of Growing Trees

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Abstract

We study a family of trees with degree k at all interior nodes and degree 1 at boundary nodes. The eigenvalues of the Laplacian matrix have high multiplicities. As the trees grow, the graphs of those eigenvalues approach a piecewise-constant "Cantor function". The multiplicity of the most frequently repeated eigenvalue $\lambda = 1$ approaches a constant fraction of the number of nodes, and we compute that fraction.

1 Introduction

A tree is an attractive and deceptively simple graph. It has no loops, so the path connecting node i to node j is unique. A systematic construction can ensure that all interior nodes have the same degree k and all boundary nodes have degree 1. This finite tree is a subgraph of an infinite homogeneous tree. As the tree grows, it is natural to expect important properties (like the eigenvalues of the Laplacian matrix L or the adjacency matrix A) to approach the corresponding properties of the infinite tree. In our case this doesn't happen.

This small note computes the eigenvalues of L for a growing family of trees, and finds an entirely different limit. Repeated eigenvalues occur with astonishing multiplicities. The spectral distribution function looks like a singular Cantor function, constant almost everywhere.

In a companion paper [5] we give complete details for all the eigenvalues of the adjacency matrices of these trees. The limiting cantor function is computed. We were not sure that the Laplacian spectrum would also have this piecewise-constant structure, but it has. For the Laplacian matrices, the present paper shows that $\lambda = 1$ yields (asymptotically) a constant fraction of the eigenvalues.

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2 Construction

Choose any degree $k > 2$. The tree T_1 has a central node x_0 with k edges going out to nodes $x_1, x_2, \ldots, x_k$. The tree T_2 has $k-1$ new edges going out from each of those k nodes (previously boundary nodes, now interior nodes). There are $k(k-1)$ new boundary nodes. Figure 1 shows the first two trees for $k = 3$.

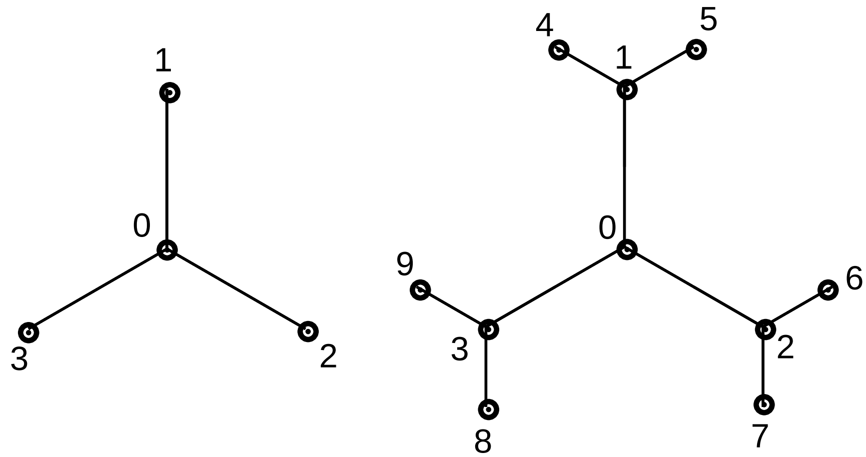

Figure 1: The trees T_1 and T_2 with $B_1 = 3$ and $B_2 = 6$ boundary nodes

After r steps, the tree T_r of radius r will have $B_r = k(k-1)^{r-1}$ boundary nodes. The number of interior nodes is:

$$1 + k + k(k-1) + \cdots + k(k-1)^{r-2} = \frac{k(k-1)^{r-1} - 2}{k-2}$$

The total number of nodes (boundary plus interior, so one more term in the sum) is given by the same expression with r in place of $r-1$:

$$N_r = N(k,r) = \frac{k(k-1)^r - 2}{k-2}$$

The number of interior nodes at stage r is the number N_{r-1} of all nodes at stage $r-1$. Boundary nodes outnumber interior nodes for large r by roughly $k:1$.

The N_r by N_r adjacency matrix A_r has $a_{ij} = 1$ if an edge connects node i to node j. In the absence of such an edge $a_{ij} = 0$ (in particular $a_{ii} = 0$). The N_r by N_r Laplacian matrix L_r satisfies the relationship $L_r = H_r - A_r$ where H_r is a diagonal matrix that has h_{ii} equal to the degree of node i. Then the sum along each row and of L is zero. With $k = 3$ the Laplacian matrices for the trees T_1 and T_2 have orders $N_1 = 4$ and $N_2 = 10$:

$$L_1 = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 3I - A_1 & -C_2 \\ -C_2^T & I \end{bmatrix}$$

The key to our analysis will be this recursive relationship between L_r and A_r, the Laplacian matrix and the adjacency matrix, so we go carefully. The identity block on the diagonal of L_r represents no edges between boundary nodes of the tree. The rectangular block C_r represents edges connecting interior nodes to boundary nodes. Thus C_r is an N_{r-1} by B_r matrix, but its only nonzeros will be in a submatrix D_r. This submatrix indicates the new edges connecting B_{r-1} previous boundary nodes to B_r new boundary nodes. In our example with $k = 3$, the matrix D_2 has $B_1 = 3$ rows (nodes 1,2,3) and $B_2 = 6$ columns (nodes 4,5,6,7,8,9):

$$C_2 = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The $k - 1$ ones in each row of D_r represent the $k - 1$ new edges going out from each of the earlier boundary points. The symmetry of the adjacency matrix ensures that its remaining block must be C_2^T.

For any k and r, the Laplacian matrices of the trees have this same recursive form. We need to indicate the shapes of all submatrices, so our counts of eigenvalues and eigenvectors are consistent. Recall that $B_r = k(k-1)^{r-1}$:

Laplacian matrix: $$A_r = \begin{bmatrix} kI - A_{r-1} & -C_r \\ -C_r^T & I \end{bmatrix}_{(N_r \times N_r)} \qquad N_{r-1} + B_r = N_r$$

Interior to boundary: $$C_r = \begin{bmatrix} 0 \\ D_r \end{bmatrix}_{(N_{r-1} \times B_r)} \qquad N_{r-2} + B_{r-1} = N_{r-1}$$

Old boundary nodes to new boundary nodes:

$$D_r = \begin{bmatrix} 1 \ \ldots \ 1 & & & \\ & 1 \ \ldots \ 1 & & \\ & & \ldots & \\ & & & 1 \ \ldots \ 1 \end{bmatrix}_{(B_{r-1} \times B_r)} \qquad k-1 \text{ ones in each row.}$$

3 The eigenvalues of the Laplacian matrix

Our paper began with a MATLAB computation of the eigenvalues of L_r. The result of a typical experiment $plot(sort(eig(A)))$ is shown in Figure 2. The eigenvalues are

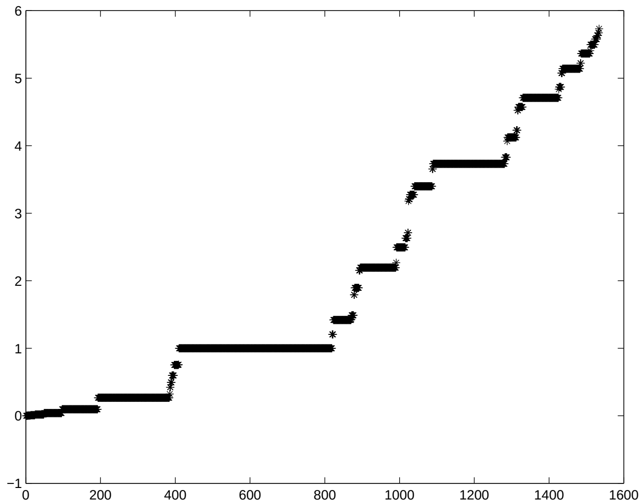

Figure 2: The eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N_r}$ of the Laplacian matrix for $k = 3$, $r = 9$, $N_r = 1534$. There are 410 ones.

plotted in increasing order, from λ_1 to λ_{1534}. The adjacency matrix also produces heavily repeated eigenvalues, and this feature of the graph caught our attention immediately. Actually Henrik Eriksson did the first experiment in a project [2] on models for "small-world" graphs. Those are partly structured and partly random graphs, following the experiments of Watts and Strogatz [7], [8]. They are not trees! And the eigenvalues do not look at all like those in Figure 2.

For the Laplacian of the tree we have a piecewise-constant eigenvalue distribution that reminds us of a Cantor singular function. The eigenvalue 1 in Figure 2 is repeated 410 times out of $N_r = 1534$ eigenvalues, and this fraction approaches $\frac{4}{15}$ as $r \to \infty$.

We must emphasize that this piecewise-constant Cantor distribution is *not* the spectral distribution for the infinite homogeneous tree. The infinite case is linked to beautiful mathematics [3] of group representations, and there are no boundary nodes of degree one to produce a singular limit. The valuable book [1] by Fan Chung connects these eigenvalues to other properties of the graph.

For our trees, the diameter (maximum distance between nodes) is explicit:

$$\text{Diameter} \quad D = 2r, \qquad \text{so} \quad D \approx 2 \log_2 \frac{N_r}{3}$$

The *average* distance between nodes can also be computed (averaged over all pairs):

$$\begin{aligned}\text{average distance} &= \frac{2(N_r+2)^2}{N_r(N_r-1)}\log_2\frac{N_r+2}{3} - \frac{10N_r+14}{3N_r}\\ &\approx 2\log_2\frac{N_r}{3} - \frac{10}{3} \approx D - \frac{10}{3}\end{aligned}$$

This logarithmic growth is also seen for random graphs and small-world graphs, but with entirely different eigenvalues.

To find the eigenvalues of the Laplacian matrix L_r, we first study its characteristic polynomial $P_r(\lambda)$:

$$\begin{aligned}P_r(\lambda) &= \det(\lambda I - L_r)\\ &= \det\begin{bmatrix} A_{r-1}+(\lambda-k)I & C_r \\ C_r^T & (\lambda-1)I\end{bmatrix}\\ &= \det\begin{bmatrix} A_{r-1}+(\lambda-k)I-(\lambda-1)^{-1}C_rC_r^T & 0 \\ C_r^T & (\lambda-1)I\end{bmatrix}\\ &= (\lambda-1)^{B_r}\det(A_{r-1}+(\lambda-k)I-(\lambda-1)^{-1}C_rC_r^T)\end{aligned} \tag{3.1}$$

The size of I is N_r or N_{r-1} or B_r, indicated by its position. From the structure of C_r, we have:

$$C_rC_r^T = \begin{bmatrix}0\\ D_r\end{bmatrix}\begin{bmatrix}0 & D_r^T\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & D_rD_r^T\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & (k-1)I\end{bmatrix}$$

The $k-1$ ones in each row of D_r immediately give $D_rD_r^T = (k-1)I$ (of order B_{r-1}). So we have a recursive structure

$$\begin{aligned}P_r(\lambda) &= (\lambda-1)^{B_r}\det(A_{r-1}+(\lambda-k)I-(\lambda-1)^{-1}C_rC_r^T)\\ &= (\lambda-1)^{B_r}\det\begin{bmatrix} A_{r-2}+(\lambda-k)I & C_{r-1} \\ C_{r-1}^T & \lambda-k-(k-1)(\lambda-1)^{-1}I\end{bmatrix}\end{aligned} \tag{3.2}$$

This recursion is the key, if we can solve a more general problem: Find an expression for

$$f(r,\lambda,\omega) = \det\begin{bmatrix} A_{r-1}-\lambda I & C_r \\ C_r^T & -\omega I\end{bmatrix} \tag{3.3}$$

$P_r(\lambda)$ in equation (3.2) is the special case $f(r, k-\lambda, 1-\lambda)$ with $\omega = 1-\lambda$.. So an explicit expression for $f(r,k-\lambda,1-\lambda)$ will give us the characteristic polynomial of L_r.

To compute (3.3), we follow the same steps that led from (3.1) to (3.2). The backward recursive expression from r to $r-1$ becomes:

$$f(r, \lambda, \omega) = (-\omega)^{B_r} f(r-1, \lambda, \lambda - (k-1)\omega^{-1}) \tag{3.4}$$

Three things are worth noticing in the recursion (3.4):

1. $B_n = k(k-1)^{n-1}$ is an even number for $n \geq 2$. Thus $(\lambda - 1)^{B_n} = (1-\lambda)^{B_n}$.

2. The third argument of $f(n, k-\lambda, q_{r+1-n})$ $(1 \leq n \leq r)$ follows a recursive relation $q_n = k - \lambda - (k-1)q_{n-1}^{-1}$, with $q_1 = 1 - \lambda$.

3. The backward recursion for f stops at radius $n = 1$, where

$$f(1, k-\lambda, \omega) = \det \begin{bmatrix} -(k-\lambda) & 1 & 1 & \dots & 1 \\ 1 & -\omega & 0 & \dots & 0 \\ & \dots & & \dots & \\ 1 & 0 & 0 & \dots & -\omega \end{bmatrix} = (-1)^{k-1}\omega^{k-1}((k-\lambda)\omega - k)$$

Now our characteristic polynomial $P_r(\lambda)$ is

$$\begin{aligned} f(r, k-\lambda, 1-\lambda) &= (1-\lambda)^{B_r} f(r-1, k-\lambda, q_2(\lambda)) \\ &= q_1^{B_r} q_2^{B_{r-1}} f(r-2, k-\lambda, q_3(\lambda)) \end{aligned}$$

Continuing the recursion we obtain

$$\begin{aligned} f(r, k-\lambda, 1-\lambda) &= q_1^{B_r} q_2^{B_{r-1}} \dots q_{r-1}^{B_2} f(1, k-\lambda, q_r) \\ &= (-1)^{k-1} q_1^{B_r} q_2^{B_{r-1}} \dots q_{r-1}^{B_2} q_r^{k-1}((k-\lambda)q_r - k) \end{aligned} \tag{3.5}$$

So if we could find an expression for q_n, then we have the characteristic polynomial.

To do this, let $p_0 = 1, p_1 = 1 - \lambda$, and $p_2 = q_2 p_1 = (k-\lambda)(1-\lambda) - (k-1)$. The relation satisfied by $p_n = q_n p_{n-1}$ is:

$$\begin{aligned} p_n &= (k - \lambda - (k-1)q_{n-1}^{-1})p_{n-1} \\ &= (k - \lambda - (k-1)\frac{p_{n-2}}{p_{n-1}})p_{n-1} \\ &= (k-\lambda)p_{n-1} - (k-1)p_{n-2} \end{aligned}$$

These polynomials $p_n(\lambda)$ of degree n are the coefficients in the generating function

$$g(t, \lambda) = \sum_{n=0}^{\infty} p_n(\lambda) t^n$$

From the recursive relation, we have

$$
\begin{aligned}
& p_{n+1}t^n = (k-\lambda)p_n t^n - (k-1)p_{n-1}t^n \\
\Rightarrow \quad & \frac{1}{t}\sum_{n=1}^{\infty} p_{n+1}t^{n+1} = (k-\lambda)\sum_{n=1}^{\infty} p_n t^n - t(k-1)\sum_{n=1}^{\infty} p_{n-1}t^{n-1} \\
\Rightarrow \quad & \frac{1}{t}(g(t,\lambda) - (1-\lambda)t - 1) = (k-\lambda)(g(t,\lambda)-1) - (k-1)tg(t,\lambda) \\
\Rightarrow \quad & g(t,\lambda) = \frac{1+(1-k)t}{1-(k-\lambda)t+(k-1)t^2}
\end{aligned}
$$

Fix λ, and solve $1-(k-\lambda)t+(k-1)t^2 = 0$ for the two roots

$$
\alpha = \frac{k-\lambda+\sqrt{(k-\lambda)^2-4(k-1)}}{2(k-1)} \quad \text{and} \quad \beta = \frac{k-\lambda-\sqrt{(k-\lambda)^2-4(k-1)}}{2(k-1)}
$$

So we have

$$
g(t,\lambda) = \frac{1+(1-k)t}{(k-1)(\alpha-\beta)}\left(\frac{1}{t-\alpha} - \frac{1}{t-\beta}\right) = \frac{1+(1-k)t}{(k-1)(\alpha-\beta)}\left(\frac{1}{\beta}\sum_{n=0}^{\infty}\frac{t^n}{\beta^n} - \frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{t^n}{\alpha^n}\right)
$$

The coefficient of t^n is

$$
p_n(\lambda) = \frac{1}{(k-1)(\alpha-\beta)}\left[\left(\frac{1}{\beta^{n+1}} - \frac{1}{\alpha^{n+1}}\right) + (1-k)\left(\frac{1}{\beta^n} - \frac{1}{\alpha^n}\right)\right]
$$

Returning to the characteristic polynomial,

$$
\begin{aligned}
P_r(\lambda) = f(r, k-\lambda, 1-\lambda) &= p_1^{B_r}\left(\frac{p_2}{p_1}\right)^{B_{r-1}}\left(\frac{p_3}{p_2}\right)^{B_{r-2}}\ldots\left(\frac{p_r}{p_{r-1}}\right)^k\left(k-\lambda-\frac{kp_{r-1}}{p_r}\right) \\
&= p_1^{B_r-B_{r-1}}p_2^{B_{r-1}-B_{r-2}}\ldots p_{r-1}^{k(k-2)}p_r^{k-1}((k-\lambda)p_r - kp_{r-1})
\end{aligned} \tag{3.6}
$$

So almost all the eigenvalues are roots of p_n ($1 \leq n \leq r$). A smaller set of eigenvalues, asymptotically a zero fraction, are roots of the extra factor $(k-\lambda)p_r - kp_{r-1}$.

We take a closer look at eigenvalue $\lambda = 1$ of the Laplacian matrix. By numerical experiment, 1 has the largest multiplicity among all the eigenvalues. Notice that when $\lambda = 1$, the recursion for $p_n(1)$ is $p_{n+1}(1) = (k-1)(p_n(1) - p_{n-1}(1))$ with initial conditions $p_0(1) = 1$ and $p_1(1) = 0$. The sequence $p_n(1)$ then proceeds as

$$
\begin{array}{l}
1 \quad 0 \quad -(k-1) \quad -(k-1)^2 \quad -(k-1)^2(k-2) \quad -(k-1)^3(k-3) \\
-(k-1)^3(k^2-5k+5) \quad -(k-1)^4(k-2)(k-4) \quad \ldots
\end{array}
$$

For degree $k = 3$, we will hit the first zero after $p_1(1)$ at $p_5(1)$. Then the sequence repeats itself with a different constant factor and we hit additional zeros at $p_9(1)$ and every p_{4n+1}. Similarly, for $k = 4$, we will get zeros at $p_{6n+1}(1)$. For $k > 4$, it is easy to prove from the recursion that the magnitude of the sequence $p_n(1)$ will increase geometrically and there are no more zeros after $p_1(1)$.

Thus, when the degree is $k = 3$, the multiplicity of $\lambda = 1$ is

$$\begin{aligned} h_1(r) = &\quad (B_r - B_{r-1}) + (B_{r-4} - B_{r-5}) + \cdots + (B_{r-4n} - B_{r-4n-1}) + \dots \\ = &\quad 3(2^{r-2} + 2^{r-6} + 2^{r-10} + \dots) \end{aligned} \tag{3.7}$$

As the trees grow, this is asymptotically

$$h_1(r) \quad \approx \quad \frac{4}{5}2^r \qquad \text{as} \quad r \to \infty$$

This accounts for a fraction $\frac{h_1(r)}{N_r} = \frac{4}{15}$ of all eigenvalues. Similarly, when $k = 4$, the multiplicity of $\lambda = 1$ is $\frac{81}{91}3^r$ asymptotically and this accounts for a fraction $\frac{81}{182}$ of all eigenvalues.

For $k > 4$, the multiplicity of $\lambda = 1$ is just $B_r - B_{r-1} = k(k-2)(k-1)^{r-2}$ and this accounts for a fraction $(\frac{k-2}{k-1})^2$ of all eigenvalues.

We turn now to the eigenvectors.

4 The eigenspace of the Laplacian matrix

We look first at the eigenvectors with eigenvalue $\lambda = 1$. Denote this space by $E_r(1)$. We solve $(I - L_r)x = 0$ to find the interior component x_i and boundary component x_b of these eigenvectors:

$$(I - L_r)x = \begin{bmatrix} A_{r-1} - (k-1)I & C_r \\ C_r^T & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There will be two orthogonal subspaces of eigenvectors, those concentrated entirely at the boundary (with $x_i = 0$) and those not concentrated at the boundary (with $x_i \neq 0$).

1. Eigenvectors at the boundary: If the interior part $x_i = 0$, then we need $C_r x_b = 0$. The vector x_b has B_r components and the matrix C_r has rank B_{r-1}:

$$C_r = \begin{bmatrix} 0 \\ D_r \end{bmatrix}$$

So $B_r - B_{r-1}$ eigenvectors come from the equation $C_r x_b = 0$ which reduces to $D_r x_b = 0$:

$$D_r x_b = \begin{bmatrix} 1 & \cdots & 1 & & & & & \\ & & & \cdots & & & & \\ & & & & \cdots & & & \\ & & & & & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_{b,1} \\ \vdots \\ x_{b,k-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}$$

Each row of D_r corresponds to the $k-1$ boundary nodes that come out from an interior node. The one equation coming from a typical row has $k-1$ terms:

$$x_{b,1} + x_{b,2} + \cdots + x_{b,k-1} = 0$$

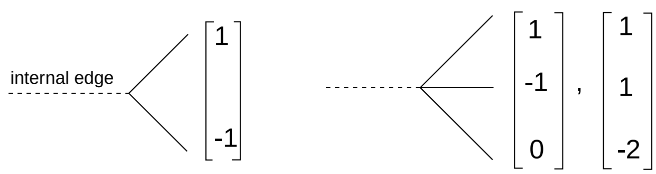

Figure 3: Boundary eigenvectors for $\lambda = 1$ in the case $k = 3$ and $k = 4$

This has $k-2$ independent solutions as illustrated in Figure 3. The boundary edges are "fluttering" and there is no movement in the interior. Again, the number of these eigenvectors is

$$B_r - B_{r-1} = k(k-1)^{r-1} - k(k-1)^{r-2} = k(k-2)(k-1)^{r-2}$$

For trees that have degree $k > 4$, the multiplicity of $\lambda = 1$ is $B_r - B_{r-1}$, and we have found all eigenvectors. For trees with degree $k = 3$ or 4, there is an additional set of eigenvectors.

2. If $x_i \neq 0$ then the interior components x_i separate into x_{ii} (the interior of the interior) and x_{ib} (the interior that is adjacent to the boundary):

$$(A_{r-1} - (k-1)I)x_i + C_r x_b = \begin{bmatrix} A_{r-2} - (k-1)I & C_{r-1} \\ C_{r-1}^T & -(k-1)I \end{bmatrix} \begin{bmatrix} x_{ii} \\ x_{ib} \end{bmatrix} + \begin{bmatrix} 0 \\ D_r x_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.1}$$

$$C_r^T x_i = \begin{bmatrix} 0 & D_r^T \end{bmatrix} \begin{bmatrix} x_{ii} \\ x_{ib} \end{bmatrix} = D_r^T x_{ib} = 0 \tag{4.2}$$

From the second equation, we have $x_{ib} = 0$:

$$D_r^T x_{ib} = \begin{bmatrix} 1 & & \\ \vdots & & \\ 1 & & \\ & \vdots & \\ & & 1 \\ & & \vdots \\ & & 1 \end{bmatrix} \begin{bmatrix} x_{ib,1} \\ \vdots \\ \vdots \\ x_{ib,B_{r-1}} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \Rightarrow x_{ib} = 0$$

The first equation in (4.1) now reduces to $(A_{r-2} - (k-1)I)x_{ii} = 0$ which means that x_{ii} is in the eigenspace of A_{r-2} with eigenvalue $k-1$. This eigenspace is described in [5]. Then for each such x_{ii}, the second part of (4.1) becomes $C_{r-1}^T x_{ii} + D_r x_b = 0$. This produces a unique x_b that is orthogonal to the boundary eigenvectors.

For the eigenspace $E_r(\lambda)$, with eigenvalue $\lambda \neq 1$, we solve $(\lambda I - L_r)x = 0$ to find the eigenvectors:

$$(\lambda I - L_r)x = \begin{bmatrix} A_{r-1} + (\lambda - k)I & C_r \\ C_r^T & (\lambda - 1)I \end{bmatrix} \begin{bmatrix} x_i \\ x_b \end{bmatrix} = 0 \tag{4.3}$$

This gives us two equations:

$$(A_{r-1} + (\lambda - k)I)x_i + C_r x_b = 0 \tag{4.4}$$
$$C_r^T x_i + (\lambda - 1)x_b = 0 \tag{4.5}$$

Multiply (4.5) by C_r to find

$$\begin{aligned} & C_r C_r^T x_i + (\lambda - 1)C_r x_b = 0 \\ \Rightarrow \quad & C_r x_b = -(\lambda - 1)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & (k-1)I \end{bmatrix} x_i \end{aligned} \tag{4.6}$$

Substitute (4.6) into (4.4):

$$\begin{aligned} & (A_{r-1} + (\lambda - k)I + \begin{bmatrix} 0 & 0 \\ 0 & -(k-1)(\lambda - 1)^{-1}I \end{bmatrix})x_i \\ = & \begin{bmatrix} A_{r-2} + (\lambda - k)I & C_{r-1} \\ C_{r-1}^T & (\lambda - k) - (k-1)(\lambda - 1)^{-1}I \end{bmatrix} x_i = 0 \end{aligned} \tag{4.7}$$

So x_i is the solution of (4.7) while x_b is uniquely decided by x_i through (4.6).

Not surprisingly, we see that the matrix in (4.7) is actually the same as the matrix we get when calculating eigenvalues. This backward recursion can be carried on as long as the term in the lower right corner of the matrix is nonzero.

If the eigenvalue λ results from $p_n(\lambda) = 0$, we will hit a zero at the $(n-1)$th step of the backward recursion. At that point, the equation is

$$\begin{bmatrix} A_{r-n} + (\lambda - k)I & C_{r-n+1} \\ C_{r-n+1}^T & 0 \end{bmatrix} \begin{bmatrix} y_i \\ y_b \end{bmatrix} = 0$$

1. If $y_i = 0$, we have $C_{r-n+1} y_b = 0$. This produces $B_{r-n+1} - B_{r-n}$ boundary eigenvectors.

2. If $y_i \neq 0$, let $y_i = [y_{ii} \quad y_{ib}]^T$, Following the earlier steps in calculating the eigenspace $E_r(1)$, we have

$$y_{ib} = 0, \; (A_{r-n-1} + (\lambda - k)I) y_{ii} = 0 \text{ and } C_{r-n} y_{ii} + D_{r-n+1} y_b = 0$$

   If $k - \lambda$ is an eigenvalue of A_{r-n-1}, y_{ii} will be in the eigenspace of A_{r-n-1} (see [5]) with eigenvalue $k - \lambda$. For each such y_{ii}, we can uniquely solve for a y_b that is orthogonal to the boundary eigenvectors. If $k - \lambda$ is not an eigenvalue of A_{r-n-1}, we have $y_{ii} = 0$ which reduces back to the boundary eigenvector case.

Our conclusion flows directly from our introduction: The spectrum associated with a growing family of trees can be remarkable. The Laplacian matrix shares the property of high multiplicities with the adjacency matrix, which was worked out earlier in full detail. It remains an open problem to describe the (perhaps generalized) infinite graph whose spectrum agrees with the limit from the sequence of finite graphs.

References

[1] F. Chung, *Spectral Graph Theory*, CBMS Conference Series 92, American Mathematical Society, Providence, 1997.

[2] H. Eriksson, MIT-KTH student project on "small world" graphs, 1999.

[3] A. Figa-Talamanca, *Harmonic analysis and representation theory for groups acting on homogeneous trees.* Cambridge University Press, Cambridge, 1991.

[4] C.D. Godsil, *Algebraic combinatorics.* Chapman and Hall, New York and London, 1993.

[5] L. He, X. Liu and G. Strang, Trees with Cantor eigenvalue distribution. Submitted to Journal of Combinatorial Theory, 1999.

[6] B. McKay, On the spectral characterisation of trees. Combinatoria, **3**:219–232, 1977.

[7] D. Watts, *Small Worlds: The Dynamics of Networks Between Order and Randomness.* Princeton University Press, Princeton, 1999.

[8] D. Watts and S. Strogatz, Collective dynamics of 'small-world' networks. Nature, **393**:440–442, 1998.