Orthogonal Multiwavelets with Vanishing Moments[1]

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Abstract

A scaling function is the solution to a dilation equation $\Phi(t) = \sum c_k \Phi(2t - k)$, in which the coefficients come from a low-pass filter. The coefficients in the wavelet $W(t) = \sum d_k \Phi(2t - k)$ come from a high-pass filter. When these coefficients are *matrices*, Φ and W are vectors: there are two or more scaling functions and an equal number of wavelets. By dilation and translation of the wavelets, we have an orthogonal basis $W_{ijk} = W_i(2^j t - k)$ for all functions of finite energy.

Those "*multiwavelets*" open new possibilities. They can be shorter, with more vanishing moments, than single wavelets. They can be symmetric, which is impossible for scalar wavelets (except for Haar's). We determine the conditions to impose on the matrix coefficients c_k in the design of multiwavelets, and we construct a new pair of piecewise linear orthogonal wavelets with two vanishing moments.

1. TWO SCALING FUNCTIONS AND TWO WAVELETS

The Haar wavelets are certainly the simplest. Among conventional orthogonal wavelets, no other is a piecewise polynomial and no other has an explicit elementary formula. But Haar's piecewise constants are poor at approximation and poor at frequency selection. They can represent a sudden change in the signal but they are hopeless at a smooth change. The error in approximating $\sin t$ is of order Δt. This is much too large.

The explosive growth in new wavelets has been fueled by the need to do better. Daubechies increased the number of coefficients in the dilation equation (and in the corresponding filter). With $2p$ coefficients, her scaling function D_{2p} is nonzero over $2p-1$ intervals. Its translates give an approximation error of order $(\Delta t)^p$. The piecewise linear hat function is not orthogonal to its translates, but the functions D_{2p} do have this property—although symmetry is lost. Our purpose is to construct wavelets with as many desirable properties as possible.

In parallel with wavelets, the analysis and construction of filter banks is going strong. Biorthogonal filter banks and wavelets do achieve symmetry (linear phase) together with perfect reconstruction. And also the possibility for improvement in a different direction has appeared, which is extended one step

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further in this note. It is the creation of "*multiwavelets*" and "*multifilters*", for which the coefficients are matrices.

In all cases the approximation error is connected to the degree of polynomials $1, t, \ldots, t^{p-1}$ that can be constructed from translates of the scaling functions. Since those are orthogonal to the wavelets, we may equivalently count the number p of *vanishing moments* of the wavelets. We speak of scaling functions and wavelets in the plural, because that is the novelty in the construction. Hardin, Geronimo, and Massopust[1] created *two* scaling functions with the following properties:

1. Combinations of translates yield the functions 1 and t. The order of approximation is $p = 2$ (*two vanishing moments in the wavelets*).

2. The first function vanishes outside $[0, 1]$ and the second vanishes outside $[0, 2]$ (*short support*).

3. The functions satisfy $\Phi_1(t) = \Phi_1(1-t)$ and $\Phi_2(t) = \Phi_2(2-t)$ (*symmetry*).

4. All translates $\Phi_1(t-k)$ and $\Phi_2(t-k)$ are *orthogonal.*

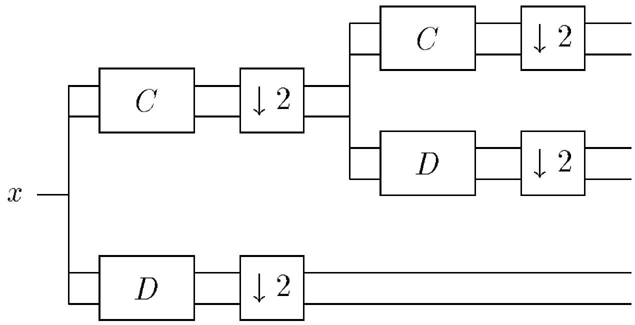
Those properties can be derived from the iterated interpolation that converges to Φ_1 and Φ_2, or from the matrix dilation equation

$$\begin{bmatrix} \Phi_1(t) \\ \Phi_2(t) \end{bmatrix} = \sum C_k \begin{bmatrix} \Phi_1(2t-k) \\ \Phi_2(2t-k) \end{bmatrix}$$

with matrix coefficients C_k. The analysis in [2,3] led from these scaling functions to two wavelets:

$$\begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix} = \sum D_k \begin{bmatrix} \Phi_1(2t-k) \\ \Phi_2(2t-k) \end{bmatrix}.$$

The coefficients C_k go into a low-pass multifilter. The matrices D_k go into an orthogonal high-pass multifilter. Together they yield an orthogonal (perfect reconstruction) filter bank. Iterating the low-pass filter produces the logarithmic tree structure associated with wavelet analysis, but with double channels:



The transpose filters after upsampling will recover x.

The downsampling operator $\downarrow 2$ removes all odd-numbered components. Upsampling puts zeros in their place.

The extra effort of multiple channels is compensated by the shortness of the filters. But the quality of these filters is still to be tested. Their construction with properties 1–4 seemed almost miraculous. To design other multifilters, we must know the correct orthogonality and approximation (vanishing moments) requirements on the matrices C_k. The object of this paper is to understand this problem in matrix algebra — in order to design new multifilters.

Our specific goals are

1. To determine the conditions on the C's which yield approximation of order p (polynomials from the scaling functions, p vanishing moments for the wavelets).

2. To create a multifilter in which $\Phi_1(t)$ is the Haar function and $\Phi_2(t)$ is piecewise linear — designed to produce second-order accuracy with orthogonality.

Our function $\Phi_2(t)$ has a fairly simple explicit formula. There are corresponding wavelets — also piecewise linear on $[0, 2]$, but less beautiful.

2. CONDITION FOR ACCURACY p

Our theorem will state the condition for p vanishing moments and any number of coefficients. We illustrate it for $p = 2$ and four coefficients. If those are scalars (lower case c_0, c_1, c_2, c_3 with $\sum c_k = 2$), then the condition for second-order accuracy has several equivalent forms:

A. $c_0 - c_1 + c_2 - c_3 = 0$ and $-c_1 + 2c_2 - 3c_3 = 0$.

B. $\sum c_k e^{ik\xi}$ has a factor $(1 + e^{i\xi})^2$.

C. The matrix $M = \begin{bmatrix} c_1 & c_0 \\ c_3 & c_2 \end{bmatrix}$ has eigenvalues 1 and 1/2 with $[1 \;\; 1]M = [1 \;\; 1]$.

The word "*regularity*" is frequently applied to this condition, but we avoid that word for this reason: It also refers to the smoothness of the associated solution $\Phi(t)$ of the dilation equation. The two meanings are related but very different. It may be too late to select a unique meaning, and better to say that p counts the *vanishing moments* of the wavelets.

Now suppose the coefficients are m by m matrices C_k. Statement **A**, taken literally to refer to the zero matrix, is too strong.

Example: The direct sum of the Daubechies filter and the Haar filter has $p > 1$ because of the Daubechies part. But **A** fails because of the Haar part, and statement **B** also fails. It is **C** that remains correct — after a suitable restatement to allow any number of matrix coefficients.

The matrix M in statement **C** arises as a finite section of the doubly infinite low-pass multifilter that we now call L:

$$L = \begin{bmatrix} \cdots & & & & & & & & & \\ & C_3 & C_2 & C_1 & C_0 & & & & & \\ & & & C_3 & C_2 & C_1 & C_0 & & & \\ & & & & & C_3 & C_2 & C_1 & C_0 & \\ & & & & & & & & & \cdots \end{bmatrix}.$$

The dilation equation $\sum C_k \Phi(2t-k) = \Phi(t)$ has a very convenient vector form (notice the double shift in L):

$$L \begin{bmatrix} \cdots \\ \Phi(2t-1) \\ \Phi(2t) \\ \Phi(2t+1) \\ \cdots \end{bmatrix} = \begin{bmatrix} \cdots \\ \Phi(t-1) \\ \Phi(t) \\ \Phi(t+1) \\ \cdots \end{bmatrix} \qquad \text{or} \qquad L\Theta(2t) = \Theta(t).$$

It is this infinite matrix L that must have the special eigenvalues 1 and $\frac{1}{2}$, in statement **C**. And it is the *left eigenvectors* of L that show how the translates of Φ produce the special polynomials 1 and t. That statement extends to any p and any number of coefficients. The theorem has a direct proof — we try to present it without technicalities.

THEOREM If the accuracy is p then L has eigenvalues $1, \frac{1}{2}, \ldots, (\frac{1}{2})^{p-1}$.

Proof The fact that the accuracy is of order p, means that polynomials $1, t, \ldots, t^{p-1}$ are the scaling subspace spanned by the Φ's, or in other words

$$t^j = G_j(t) = \sum_{-\infty}^{\infty} y_k^{(j)} \Phi(t+k) = y^{(j)} \Theta(t)$$

where $y^{(j)} = [\ldots \; y_0^{(j)} \; y_1^{(j)} \; y_2^{(j)} \; \ldots]$. Each piece $y_k^{(j)}$ is a row vector with m components, to match the vectors $\Phi(t+k)$. Substitute for $\Theta(t)$ by using the dilation equation above:

$$G_j(t) = y^{(j)} \Theta(t) = y^{(j)} L \Theta(2t).$$

On the other hand,

$$G_j(t) = t^j = 2^{-j}(2t)^j = 2^{-j}G_j(2t) = 2^{-j}y^{(j)}\Theta(2t)$$

and

$$y^{(j)}L\Theta(2t) = 2^{-j}y^{(j)}\Theta(2t).$$

The linear independence of the translates $\Phi(2t+k)$ (the components of $\Theta(2t)$) gives

$$y^{(j)}L = 2^{-j}y^{(j)}$$

which means that L has eigenvalue 2^{-j} with eigenvector $y^{(j)}$.

Let us mention that the argument is almost reversible, so if L has eigenvalues $1, \frac{1}{2}, \ldots, (\frac{1}{2})^{p-1}$ with eigenvectors $y^{(j)}$, then $G_j(t) = \sum_{-\infty}^{\infty} y_k^{(j)}\Phi(t+k) = 2^{-j}G_j(2t)$.

The theorem extends to biorthogonal and multidimensional wavelets. We note the special forms imposed on the left eigenvectors by translation invariance — starting with the constant function:

$$\sum_{-\infty}^{\infty} y_k^{(0)}\Phi(t+k) \equiv 1.$$

Replacing k by $k-1$ establishes that the components $y_k^{(0)}$ are all equal. This explains the repeated 1 in statement **C** above. (We are assuming linear independence of the translates.) The eigenvectors $y^{(0)}$, $y^{(1)}$, and $y^{(2)}$ that produce 1, t, and t^2 have forms built from fixed vectors u, v, w with m components:

$$y^{(0)} = [\ldots\ u\ u\ \ldots\ u\ \ldots]$$

$$y^{(1)} = [\ldots\ v\ \ v-u\ \ldots\ v-ku\ \ldots]$$

$$y^{(2)} = [\ldots\ w\ \ w-2v+u\ \ldots\ w-2kv+k^2u\ \ldots].$$

With this knowledge the eigenproblem for L reduces to a finite system involving the matrices T_0 and T_1 in [4,5], or simply M. The eigenvalue requirement can be used to design filter coefficients C_k, as we now show. These accuracy conditions are supplemented by orthogonality, just as Daubechies added $c_0^2+c_1^2+c_2^2+c_3^2=2$ and $c_0c_2+c_1c_3=0$ to statements **A** — **C**.

The orthogonality condition (scalar or matrix) is $LL^T = 2I$.

3. PIECEWISE LINEAR ORTHOGONAL WAVELETS

The simplest C's for minimum accuracy $p=1$ are scalars $c_0 = c_1 = 1$:

$$L = \begin{bmatrix} \cdots & & & & & & & \\ & 1 & 1 & & & & & \\ & & & 1 & 1 & & & \\ & & & & & 1 & 1 & \\ & & & & & & & \cdots \end{bmatrix} \quad \text{has left eigenvector } [\ldots\ 1\ 1\ 1\ \ldots].$$

The dilation equation is $\Phi(t) = \Phi(2t) + \Phi(2t-1)$. Its solution is the Haar function $\Phi = H$: $H(t) = 1$ for $0 < t \le 1$ and otherwise $H \equiv 0$.

It is natural to look for the simplest coefficients that yield $p = 2$. We need three 2 by 2 matrices (or four scalars, found by Daubechies). With three matrices the scaling function will vanish outside $[0, 2]$. If the first rows of the matrices copy Haar's coefficients, the first scaling function will be $\Phi_1 = H$. Our choice for second rows must introduce $1/2$ as a second eigenvalue of L:

$$C_0 = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$L = \begin{bmatrix} \cdots & & & & & \\ & C_2 & C_1 & C_0 & & \\ & & C_2 & C_1 & C_0 & \\ & & & & & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & & & & & & & & \\ & 0 & 0 & 1 & 0 & 1 & 0 & & \\ & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & & \\ & & & & & 0 & 0 & \cdots & \\ & & & & & 0 & \frac{1}{2} & \cdots & \\ & & & & & & & \cdots & \end{bmatrix}.$$

The left eigenvector for $\lambda = 1$ is $y^{(0)} = [\ldots\ u\ u\ u\ \ldots]$ with $u = [1\ \ 0]$. The left eigenvector for $\lambda = \frac{1}{2}$ is $y^{(1)} = [\ldots\ v\ v-u\ v-2u\ \ldots]$ with $v = [-\frac{1}{2}\quad -\frac{1}{2\sqrt{3}}]$. The orthogonality condition $LL^T = 2I$ reduces to two finite equations that are easily checked:

$$C_0 C_2^T = 0 \quad \text{and} \quad C_2 C_2^T + C_1 C_1^T + C_0 C_0^T = 2I.$$

Note that our eigenvectors have infinite length ($yy^T = \infty$), since otherwise $yLL^Ty^T = y(2I)y^T$ would be impossible. The matrix L is doubly infinite but not really "square". The true unitary matrix includes also the high-pass filter H found below.

Figure 1 shows the scaling functions (Φ_1 is Haar's box function, Φ_2 is piecewise linear). They solve the dilation equation with the matrix coefficients C_0, C_1, C_2. The left and right limits of Φ_2 at jumps lie on straight lines and the explicit formula is

$$\Phi_2(t) = \frac{\sqrt{3}}{2} \begin{cases} -2 + 3 \cdot 2^{-n} & \text{for } 2^{-n-1} < 1 - t < 2^{-n} \\ 8 - 4t - 3 \cdot 2^{-n} & \text{for } 2^{-n-1} < 2 - t < 2^{-n} \end{cases}.$$

Notice that the two halves of Φ_2 add to $\frac{\sqrt{3}}{2}(6 - 4t)$. This linear function (with no jumps!) is exactly reproduced by the scaling functions. The two halves of Φ_2 are orthogonal.

We turn to the wavelets $W(t) = \sum D_k \Phi(2t - k)$. In the scalar case, the D's come directly from the C's: $d_k = (-1)^k c_{1-k}$. The row c_2, c_1, c_0 in L is automatically perpendicular to the row

0, $-c_0$, c_1 and also the row c_1, $-c_2$, 0 in the high-pass filter. But matrices do not commute! Orthogonality fails when $C_1 C_0 \ne C_0 C_1$. More work is needed to choose the matrices D_k in the high-pass filter

$$H = \begin{bmatrix} \cdots & & & & & \\ & D_2 & D_1 & D_0 & & \\ & & D_2 & D_1 & D_0 & \\ & & & & & \cdots \end{bmatrix}.$$

Orthogonality requires $LH^T = 0$ and $HH^T = 2I$. Then the filter bank $\frac{1}{\sqrt{2}}\begin{bmatrix} L \\ H \end{bmatrix}$ is a unitary matrix. These two conditions on H give five matrix equations:

$$\begin{array}{ll} LH^T = 0 \text{ requires} & C_2D_2^T + C_1D_1^T + C_0D_0^T = 0, \quad C_0D_2^T = 0,\ C_2D_0^T = 0, \\ HH^T = 2I \text{ requires} & D_2D_2^T + D_1D_1^T + D_0D_0^T = 2I \quad \text{and} \quad D_0D_2^T = 0. \end{array}$$

These conditions are satisfied by a one-parameter family of D's:

$$D_0 = \begin{bmatrix} -b & 0 \\ \sqrt{\frac{1}{4} - b^2} & 0 \end{bmatrix} \qquad D_1 = \begin{bmatrix} b & \sqrt{3}b + \sqrt{1 - 4b^2} \\ -\sqrt{\frac{1}{4} - b^2} & 2b - \sqrt{\frac{3}{4} - 3b^2} \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0 & \sqrt{3}b - \sqrt{1 - 4b^2} \\ 0 & -2b - \sqrt{\frac{3}{4} - 3b^2} \end{bmatrix}.$$

To reduce the support of one wavelet to $[0, \frac{3}{2}]$ we put $\sqrt{3}b = \sqrt{1 - 4b^2}$. This value $b = \frac{1}{\sqrt{7}}$ produces an extra zero in D_2:

$$D_0 = \frac{1}{\sqrt{7}}\begin{bmatrix} -1 & 0 \\ \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad D_1 = \frac{1}{\sqrt{7}}\begin{bmatrix} 1 & 2\sqrt{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad D_2 = \frac{1}{\sqrt{7}}\begin{bmatrix} 0 & 0 \\ 0 & -\frac{7}{2} \end{bmatrix}.$$

Figure 2 shows the wavelets $W(t) = \sum D_k \Phi(2t - k)$.

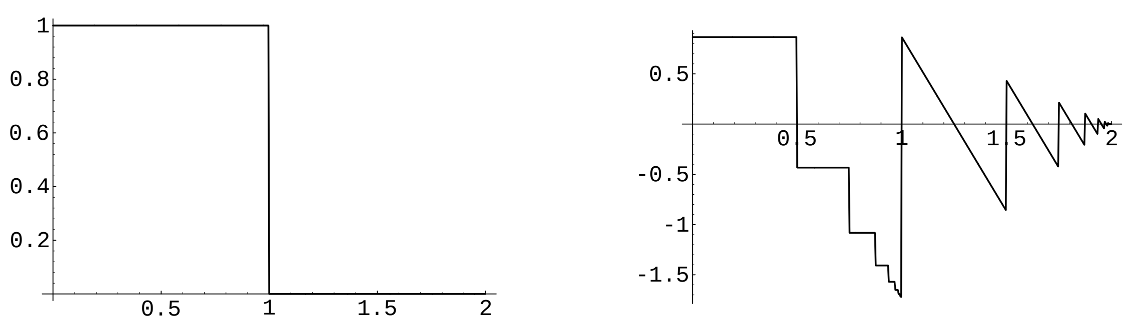


Figure 1. Scaling functions Φ_1 and Φ_2.

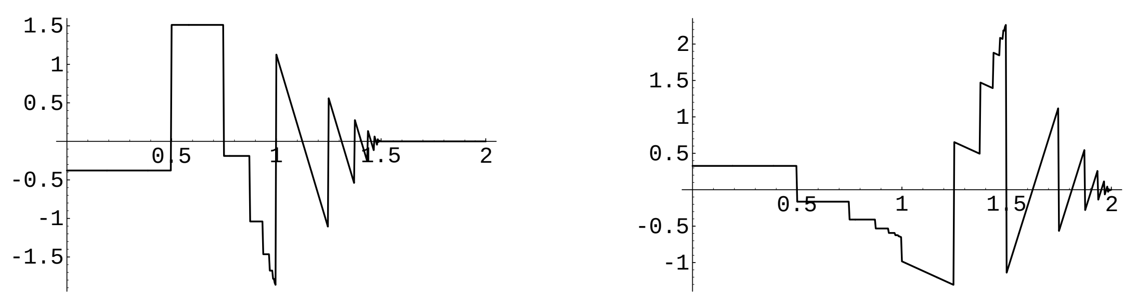


Figure 2. Wavelet functions W_1 and W_2.

4. ACKNOWLEDGEMENT

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