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APPROXIMATE EDGE SPLITTING*

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Abstract. We show that, in any undirected graph, splitting-off can be performed while preserving all cuts of value at most 4/3 times the minimum value, and this is the best possible. This generalizes a classical splitting-off result of Lovász.

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1. Introduction. In an undirected graph, splitting off two edges incident to a vertex s, say (s,u) and (s,v), means deleting them and adding the edge (u,v). Classical splitting-off theorems, such as those of Lovász [5] (exercise 6.53) and Mader [6], show that splitting off can be performed while preserving certain connectivity properties of the graph. Edge splitting is an important operation for connectivity problems. For example, suppose we would like to make a graph $G=(V,E)$ k-edge-connected by adding the minimum number of edges. A beautiful result of Frank [2] shows that it is sufficient to add a vertex s to the graph, add the minimum even number of edges between s and V to make it k-edge-connected (and this is an easy task), and finally perform splitting off while preserving k-edge-connectivity between the vertices in V (using Lovász's splitting-off result). For extensions of this result, see [2] and the survey [3]. As another (less algorithmic) illustration of the use of edge splitting, Nagamochi, Nishimura, and Ibaraki [7] have shown inductively using edge splitting that there are at most $\binom{n}{2}$ cuts of value strictly less than 4/3 times the minimum cut value in any undirected graph on n vertices. (See [4] for a sketch of a more direct proof.)

To describe the result, we need the following notation. Let $G=(V,E)$ be an undirected graph, possibly with multiple edges. For any set $S\subset V$, let $\delta(S)$ be the set of edges with exactly one endpoint in S, and let $d(S)=|\delta(S)|$ be the value of the corresponding cut. For simplicity, we write $d(s)$ for $d(\{s\})$ for any vertex s. Also, we let $d(s,A)$ denote $|\{(s,u)\in E : u\in A\}|$ (with repetitions counted if there are multiple edges).

When splitting off two edges (s,u) and (s,v), observe that the value of any cut $\delta(S)$ does not increase, and decreases precisely if $u,v\in S$ and $s\notin S$ (or similarly with S replaced by its complement $\bar{S}$). Let λ' denote the minimum edge-connectivity between any two vertices distinct from s, i.e., $\lambda'=\min_{\emptyset\neq S\subset V'} d(S)$, where $V'=V-s$, and let N be the neighbor set of s, i.e., $N=\{u\in V' : (s,u)\in E\}$.

The classical splitting-off result of Lovász [5] (exercise 6.53) shows that if $\lambda'\geq 2$ and $d(s)$ is even, then for any $u\in N$, there exists an edge (s,v) such that splitting off (s,u) and (s,v) does not reduce λ'. Since splitting off changes the value of cuts

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by an even number, Lovász's result can be interpreted as saying that the cuts of minimum value and minimum value plus one can all be preserved while performing splitting off. By repeated applications of Lovász's result, one can isolate any vertex while maintaining the connectivity between the other vertices.

Recently, Benczúr [1] introduced the notion of approximate splitting off in which the goal is to preserve all cuts of value less than α times the minimum, for some value of α. Since the values of the cuts $\delta(S)$ and $\delta(V'-S)$ become identical once s is completely isolated in the graph, we should not always be able to preserve both $d(S)$ and $d(V'-S)$. As a result, we say that (s,u) and (s,v) are *admissible* for k-splitting off if $\min(d(S), d(V'-S))$ (for $\emptyset \neq S \neq V'$) is preserved whenever this quantity is less than k. Using the polygon representation of cuts of value less than $\frac{6}{5}\lambda'$, Benczúr has shown the existence of an admissible pair of edges for $\frac{6}{5}\lambda'$-splitting off when $d(s)$ is even.

In this short note, we show that if s is even, then for any edge (s,u), there exists an edge (s,v) such that this pair of edges is admissible for $\frac{4\lambda'+2}{3}$-splitting off. By repeated applications of this result, one can isolate s while maintaining all cuts of value less than $\frac{4\lambda'+2}{3}$. Observe that for $\lambda' \geq 2$, the values of cuts of value λ' and $\lambda'+1$ are maintained by $(4\lambda'+2)/3$-splitting off since $\lambda'+1 < \frac{4\lambda'+2}{3}$. Thus, our result generalizes Lovász's result.

THEOREM 1.1. *For any vertex s with $d(s)$ even and for any edge (s,u), there exists an edge (s,v) such that the pair (s,u) and (s,v) is admissible for $(4\lambda'+2)/3$-splitting off.*

Observe that, for $\lambda' = 0$, the statement is vacuous. Similarly, for $\lambda' = 1$, the only cuts we need to preserve are of value 1 and this is done by any choice of v (since the cut values do not change or decrease by 2). Thus the only interesting cases are when $\lambda' \geq 2$.

This approximate splitting-off theorem is the best possible. Consider indeed K_5 in which the edges nonadjacent to a specific vertex s are duplicated M times. Then $\lambda' = 3M+1$, but if we split off (s,u) and (s,v), then $d(\{u,v\})$ decreases while $d(\{u,v\}) = 4M+2 = \frac{4\lambda'+2}{3}$. This shows that there is no admissible pair of edges.

By repeatedly using Theorem 1.1, we derive that vertex s can be isolated in the graph, as follows.

COROLLARY 1.2. *If vertex s has even degree, then the edges incident to s can be partitioned into $d(s)/2$ admissible pairs for $(4\lambda'+2)/3$-splitting off.*

2. The proof. In order to prove Theorem 1.1, we first need the following simple lemma [1].

LEMMA 2.1. *Let $d(s) > 0$ be even. Then (s,u) and (s,v) is not an admissible pair for k-splitting off if and only if there exists a set $S \subset V'$ with $u, v \in S$ such that* (i) *$d(S) < k$ and* (ii) *$d(s,S) \leq \frac{1}{2}d(s)$.*

Proof. After splitting off, only sets containing both u and v see their $d(.)$ value change (by 2 units). Thus, for $\min(d(S), d(V'-S))$ not to be preserved, we need either $u, v \in S$ or $u, v \in V - S'$. By complementing S (in V') if needed, we can restrict our attention to sets S with $u, v \in S$. As a result, $\min(d(S), d(V'-S))$ will decrease if and only if $d(S) - 2 < d(V'-S)$. Since $d(S) - d(s,S) = d(V'-S) - d(s, V'-S)$ and $d(s,S) + d(s,V'-S) = d(s)$, the condition $d(S) - 2 < d(V'-S)$ is equivalent to $2d(s,S) - 2 < d(s)$. Since $d(s)$ is even, this is equivalent to $d(s,S) \leq d(s)/2$. This can also be written as $d(S) \leq d(V'-S)$, and the condition $\min(d(S), d(V'-S)) < k$ is therefore equivalent to $d(S) < k$. This proves the lemma. □

Theorem 1.1 follows from Lemma 2.1 and the following result.

LEMMA 2.2. *Let $d(s) > 0$ be even and let $u \in N$. There exists $v \in N$ such that there is no set $S \subset V'$ with* (i) $u, v \in S$, (ii) $d(s, S) \leq \frac{1}{2}d(s)$, *and* (iii) $d(S) < \frac{4\lambda'+2}{3}$.

For the proof of this lemma, we need 3-set submodularity (see [5, exercise 6.48(c)]).

LEMMA 2.3 (3-set submodularity; see [5, ex. 6.48(c)]). *For any 3 sets A, B, C, we have*

$$d(A) + d(B) + d(C) \geq d(A - B - C) + d(B - C - A) + d(C - A - B) \\ + d(A \cap B \cap C) + 2d(V - A - B - C, A \cap B \cap C).$$

3-set submodularity simply follows from evaluating the contribution of any edge to both the left-hand side and the right-hand side.

Proof of Lemma 2.2. We can assume that $\lambda' \geq 2$ and $d(s) \geq 4$ since otherwise the statement is trivial (just take for v the endpoint of an edge (s, v) distinct from (s, u)).

Assume that for every $v \in N$, there exists S_v such that (i) $u, v \in S_v$, (ii) $d(s, S_v) \leq \frac{1}{2}d(s)$, and (iii) $d(S_v) < \frac{4\lambda'+2}{3}$. For a given $v \in N$, we can furthermore assume that S_v is chosen to maximize $d(s, S_v)$ among the sets satisfying (i)–(iii).

First choose $i, j \in N$ such that $d(s, S_i \cup S_j)$ is maximum. Because of (ii), we have that

$$d(s, S_i \cup S_j) = d(s, S_i) + d(s, S_j) - d(s, S_i \cap S_j) \leq \frac{1}{2}d(s) + \frac{1}{2}d(s) - d(s, S_i \cap S_j).$$

Since $u \in S_i \cap S_j$, we have that $d(s, S_i \cap S_j) > 0$, implying that $d(s, S_i \cup S_j) < d(s)$. Thus, there exists $k \in N - (S_i \cup S_j)$.

Since k was not chosen instead of i or j, we have that $(S_i - S_j - S_k) \cap N \neq \emptyset$ and $(S_j - S_i - S_k) \cap N \neq \emptyset$.

By 3-set submodularity, we have that

$$\begin{aligned} 4\lambda' + 2 &> d(S_i) + d(S_j) + d(S_k) \\ &\geq d(S_i - S_j - S_k) + d(S_j - S_i - S_k) + d(S_k - S_i - S_j) \\ &\quad + d(S_i \cap S_j \cap S_k) + 2, \end{aligned}$$

where we have used the fact that the edge (s, u) contributes 1 unit to $d(V - S_i - S_j - S_k, S_i \cap S_j \cap S_k)$. Hence,

$$d(S_i - S_j - S_k) + d(S_j - S_i - S_k) + d(S_k - S_i - S_j) + d(S_i \cap S_j \cap S_k) < 4\lambda',$$

which is a contradiction since all these sets contain an element of N ($k \in S_k - S_i - S_j$ and $u \in S_i \cap S_j \cap S_k$) . ☐

Although the proof of Theorem 1.1 is essentially existential, one can find pairs of edges incident to v which are admissible for $k\lambda'$-splitting in polynomial time (for k fixed). Indeed, using [7], one can enumerate all cuts of value less than $k\lambda'$ in time $O(nm^2 + n^{2k}m)$, where n is the number of vertices and m is the number of edges, and then check which pairs are admissible. For $k = 4/3$, Nagamochi, Nishimura, and Ibaraki [7] show that all these cuts can in fact be enumerated in time $O(m^2n + mn^2 \log n)$.

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