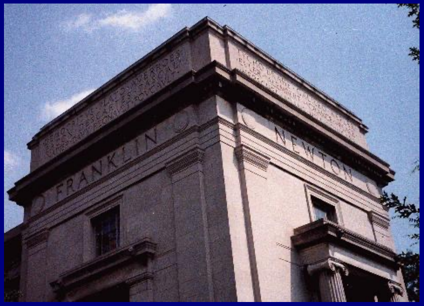
Stochastic Differential Equations and Random Matrices



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SIAM Applied Linear Algebra

July 18, 2003

Subject + Random = Wow!

Quantum Mechanics
Statistical Mechanics
Randomized Algorithms
Random Variation & Natural Selection
Option Pricing Model

Why not applied linear algebra????

Everyone's Favorite Tridiagonal

$$\frac{1}{n^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}$$

$$\boxed{\frac{d^2}{dx^2}}$$

Everyone's Favorite Tridiagonal

$$\frac{1}{n^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} + \frac{1}{(\beta n)^{1/2}}\begin{pmatrix} G & & & \\ & G & & \\ & & \ddots & \\ & & & G \end{pmatrix}$$

$$\boxed{\frac{d^2}{dx^2}} \quad + \quad \boxed{\frac{dW}{\beta^{1/2}}}$$

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O	O	O	O	O	O	χ_1

Same idea: sym matrix to tridiagonal form

G	χ_6					
χ_6	G	χ_5				
	χ_5	G	χ_4			
		χ_4	G	χ_3		
			χ_3	G	χ_2	
				χ_2	G	χ_1
					χ_1	G

Same idea: General beta

beta:
1: reals 2: complexes 4: quaternions

G	$\chi_{6\beta}$					
$\chi_{6\beta}$	G	$\chi_{5\beta}$				
	$\chi_{5\beta}$	G	$\chi_{4\beta}$			
		$\chi_{4\beta}$	G	$\chi_{3\beta}$		
			$\chi_{3\beta}$	G	$\chi_{2\beta}$	
				$\chi_{2\beta}$	G	χ_{β}
					χ_{β}	G

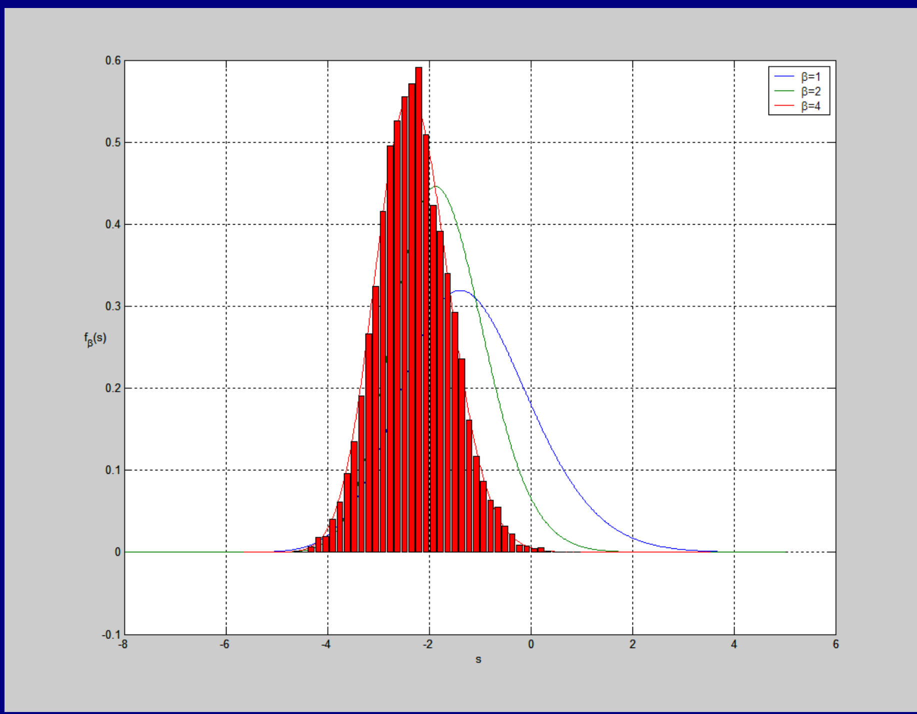
Stochastic Operator Limit

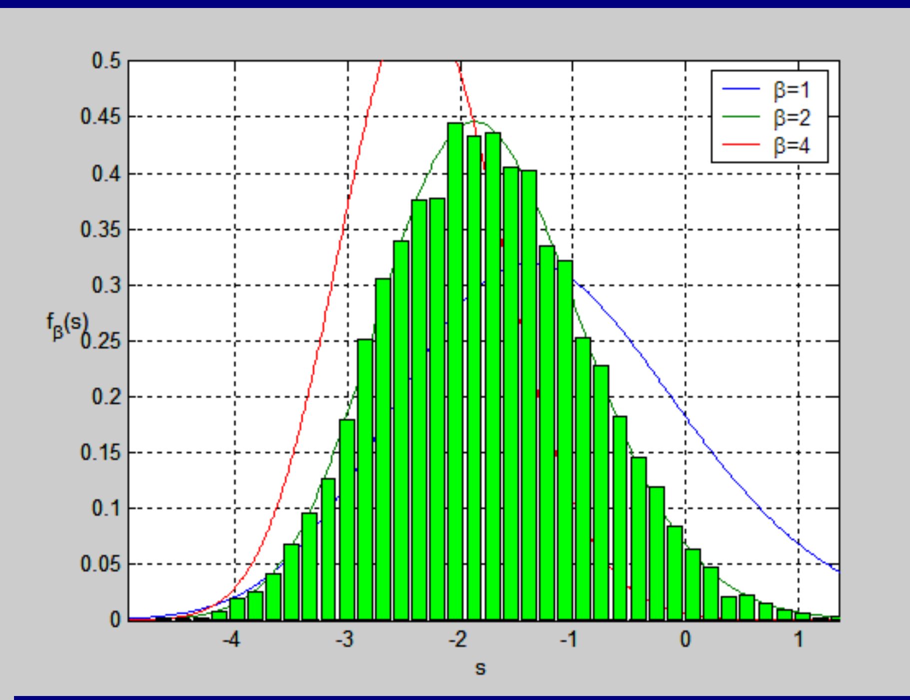
$$\frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}} dW ,$$

$$H_n^\beta \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix},$$

$$H_n^\beta \approx H_n^\infty + \frac{2}{\sqrt{\beta}} G_n ,$$

Largest Eigenvalue Plots





MATLAB

```
beta=1; n=1e9; opts.disp=0;opts.issym=1;
alpha=10; k=round(alpha*n^(1/3));  % cutoff parameters
d=sqrt(chi2rnd( beta*(n:-1:(n-k-1))))';
H=spdiags( d,1,k,k)+spdiags(randn(k,1),0,k,k);
H=(H+H')/sqrt(4*n*beta);
eigs(H,1,1,opts)
```

Tricks to get $O(n^9)$ speedup

- Sparse matrix storage (Only O(n) storage is used)
- Tridiagonal Ensemble Formulas (Any beta is available due to the tridiagonal ensemble)
- The Lanczos Algorithm for Eigenvalue Computation (This allows the computation of the extreme eigenvalue faster than typical general purpose eigensolvers.)
- The shift-and-invert accelerator to Lanczos and Arnoldi (Since we know the eigenvalues are near 1, we can accelerate the convergence of the largest eigenvalue)
- The ARPACK software package as made available seamlessly in MATLAB (The Arnoldi package contains state of the art data structures and numerical choices.)
- The observation that if $k = 10n^{1/3}$, then the largest eigenvalue is determined numerically by the top $k \times k$ segment of n. (This is an interesting mathematical statement related to the decay of the Airy function.)

Open Problems

The distribution for general beta
Seems to be governed by a convection-diffusion equation