The Random Matrix Technique of Ghosts and Shadows

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Abstract

We propose to abandon the notion that a random matrix has to be sampled for it to exist. Much of today's applied finite random matrix theory concerns real or complex random matrices ($\beta = 1, 2$). The "threefold way" so named by Dyson in 1962 [2] adds quaternions ($\beta = 4$). While it is true there are only three real division algebras (β="dimension over the reals"), this mathematical fact while critical in some ways, in other ways is irrelevant and perhaps has been over interpreted over the decades.

We introduce the notion of a "ghost" random matrix quantity that exists for every beta, and a shadow quantity which may be real or complex which allows for computation. Any number of computations have successfully given reasonable answers to date though difficulties remain in some cases.

Though it may seem absurd to have a "three and a quarter" dimensional or "pi" dimensional algebra, that is exactly what we propose and what we compute with. In the end β becomes a noisiness parameter rather than a dimension.

1 Introduction

This conference article contains an "idea" which has become a "technique." Perhaps it might be labeled "a conjecture," but I think "idea" is the better label right now. The idea was discussed informally to a number of researchers and students at MIT for a number of years now, probably dating back to 2003 or so. It was also presented at a number of conferences [3] . As slides do not quite capture a talk, this seemed a good place to write down the ideas. The theory is not fully worked out yet. No apologies. I seem to know when the technique is working and whet it is not. Too much is lurking that seems important. The theory will come. No doubt some missteps will be caught. If this note encourages the evolution of the theory, it will have fulfilled its role.

Mathematics has many precedents, the number 0 was invented when we let go of the notion that a count requires objects to exist. Similarly negative numbers are more than the absence of existing objects, imaginary numbers can be squared to obtain negative numbers, and infinitesimals act like the "ghosts of departed quantities." Without belaboring the point, mathematics makes great strides by letting go of what at first seems so dear.

What we will obtain here is a rich algebra that acts in every way that we care about as a beta-dimensional real algebra for random matrix theory. Decades of random matrix theory have been somewhat misleading. The theory has focused on reals, complexes, and quaternions or $\beta = 1, 2, 4$. Statisticians would say the real theory is more than enough and those who study wireless antenna networks would say that the complexes are valuable, while physicists are an applied community that also find the quaternions of value.

Though it may seem absurd to have a "three and a quarter" dimensional algebra, as long as $\alpha = 2/\beta$ is associated with "randomness" rather than dimension, there is little mathematical difficulty. Thus we throw out two notions that are held very dear: 1) a random object has to be capable of being sampled to exist and 2) the three division algebras so important to non-random matrix theory must take such an absolute role in random matrix theory. One reference that captures some of this philosophy is [9].

The entire field of *free probability* introduced by Dan Voiculescu around 1986 is testament to the power of the first idea, that a random object need not be sampled in order to exist. Some good references are [7] or [8] . Here the entire theory is based on moments and generating functions rather than focusing on sampling. To be sure $\beta = 1, 2, 4$ will always be special, perhaps in the same way that as the factorial function melts away into the gamma function: permutations are no longer counted but analysis goes so very far.

We introduce the notion of a "ghost" in a straightforward manner in the next section. We propose that one can compute with "ghosts" through "shadow" quantities thereby making the notions concrete. Some of the goals that we wish to see are

1. The definition of a continuum of Haar measures on matrices that general the orthogonals, unitaries and symplectics
2. A mechanism to compute arbitrary moments of the above quantities
3. A mechanism to compute Jacobians of matrix factorizations over beta-dimensional objects
4. Various new definitions of the Jack polynomials that generalize the Zonal and Schur Polynomials
5. New proofs and insights on any number of aspects of random matrix theory

In a sense it has become increasingly clear that the large n limit of random matrix theory corresponds to a stochastic integral operator [4, 11, 10, 14, 15] where β inversely measures the amount of randomness. We believe that finite random matrix theory deserves an equal footing.

2 Ghost Random Variables

Definition 1. (Isotropic Ghost random variables*)* We define (without any further meaning!) an isotropic ghost random variable, i.e. a random spherically symmetric β-dimensional random variable, x in terms of a function $f_x(r)$ of $r \geq 0$ satisfying

$$\int_{r=0}^{\infty} f_x(r) r^{\beta-1} S_{\beta-1} dr = 1, \tag{1}$$

where the surface area of the (β dimensional) sphere for any positive real β (integer or not!) is

$$S_\beta = \frac{2\pi^{\beta/2}}{\Gamma(\beta/2)}.$$

We say that x is a ghost random variable or a β-ghost. We think of x as a random variable in beta dimensions, but once again there is no specific notion of sampling from this distribution unless $\beta = 1, 2, 4$. We do not allow this to bother us in the slightest. Though we could sample with any integer beta sensibly at this point.

Example 2. (Ghost Gaussians) A β dimensional standard Gaussian may be defined by

$$\beta\text{−ghost Gaussian: } f_x(r) = (2\pi)^{-\beta/2} e^{-r^2/2}.$$

Definition 3. (Ghost norm) We define $\|x\|$ as the real random variable with probability density given in the integrand of Equation 1.

We recall that as long as the integrand integrates to 1, this is a perfectly valid probability density over the reals. As a random variable the ghost norm determines the ghost uniquely.

Example 4. (**Ghost Gaussian norm**) For a β dimensional Gaussian $\|x\|$ has the χ_β distribution, or equivalently $\|x\|^2$ has the χ^2_β distribution.

Definition 5. (Shadows) A shadow is a real or complex quantity derived from a ghost that we can sample and compute with.

We therefore have that the norm defined in Definition 3 is a shadow. Another shadow will be defined in Definition 8.

3 Algebra on Ghosts

We consider operations of addition and multiplication on independent ghosts. Dependence is a far more complicated issue which we will not treat here.

Definition 6. (Sum of Independent Ghosts): The sum of two independent ghosts z=x+y is the unique ghost satisfying

$$\|z\|^2 = \|x\|^2 + \|y\|^2 + 2c\|x\|\|y\|$$

as random variables where c is an independent random variable with beta distribution on $[-1,1]$with parameters $(n-1)/2$ and $(n-1)/2$. The probability density of c is

$$f(t) = \frac{\Gamma(n-1)}{2^{n-2}\left(\Gamma((n-1)/2)\right)^2}(1-t^2)^{(n-3)/2}.$$

For integer β, c is a random coordinate of a β dimensional hypersphere. The usual dot product of $x+y$ with itself requires the cosine of a random angle on the β dimensional sphere, which by isotropy is the same as a random coordinate on the sphere in β dimensions.

Definition 7. (Product of Independent Ghosts): The product of two independent ghosts $z = xy$ is the unique ghost satisfying

$$\|z\| = \|x\|\|y\|.$$

Definition 8. (Real parts of ghosts): The real part of a ghost is the random variable $\|x\|c$, where c is independent of $\|x\|$ and is defined as in Definition 6.

We can talk about conjugates $\bar{x}$ in the sense that $x + \bar{x} = 2$ real(x) and $x\bar{x} = \|x\|^2$. A ghost may be said to be real if $x = \bar{x}$. A matrix A of ghosts may be said to be symmetric if $A_{ij} = \bar{A}_{ji}$. The transposed matrix A^T has i,j element $\bar{A}_{ji}$ When $\beta = 2$, a symmetric matrix is known as Hermitian, and when $\beta = 4$, these are the quaternion self dual matrices.

We remark that we do not insist that ghosts be isotropic, for example we can add a real to a ghost, thus offsetting the center. What appears to be the key point is that the random objects be ultimately derived from isotropic quantities that we can keep track of.

4 Ghost Orthogonals

We reason by analogy with $\beta = 1, 2, 4$ and imagine a notion of orthogonals that generalizes the orthogonal, unitary, and symplectic groups. A matrix Q of ghosts may be said to be orthogonal if $Q^T Q = I$. The elements of course will not be independent.

We sketch an understanding based on the QR decomposition on general matrices of independent ghost Gaussians. We imagine using Householder transformations as is standard in numerical linear algebra software. We obtain immediately

Proposition 9. *Let A be an $n \times n$ matrix of standard β ghost Gaussians as in Definition 2. We may perform the QR decomposition into ghost orthogonal times ghost upper triangular. The resulting R matrix has independent entries in the upper triangle. They are standard ghost Gaussians above the diagonal, and the non-negative real quantity $R_{ii} = \chi_{\beta(n+1-i)}$ on the diagonal. The resulting Q matrix may be thought of as a β analogue of Haar measure. It may be thought of as the product of Householder matrices H_k obtained by reflecting on the uniform $k-$dimensional "β sphere."*

The Householder procedure may be thought of as an analog for the $O(n^2)$ algorithm for representing random real orthogonal matrices as described by Stewart [13].

We illustrate the procedure when $n = 3$. We use G_β to denote independent standard ghost Gaussians as distributions. They are not meant in any way to indicate common values or that even there is a meaning to having values at all.

$$\begin{pmatrix} G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \end{pmatrix} = H_3^T \begin{pmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ 0 & G_\beta & G_\beta \\ 0 & G_\beta & G_\beta \end{pmatrix} = H_2^T H_3^T \begin{pmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ 0 & \chi_{2\beta} & G_\beta \\ 0 & 0 & G_\beta \end{pmatrix} = H_1^T H_2^T H_3^T \begin{pmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ 0 & \chi_{2\beta} & G_\beta \\ 0 & 0 & \chi_\beta \end{pmatrix}.$$

The H_i are reflectors that do nothing on the first $n-i$ elements and reflect uniformly on the remaining i elements. The absolute values of the elements on the sphere behave like i independent $\chi_\beta$random variables divided by their root mean square. The Q is the product of the Householder reflectors.

We expect that the Jack Polynomial formula gives consistent moments for Q through what might be seen as a generating function. Let A and B be diagonal matrices of indeterminants. The formula

$$E_Q J_\kappa(AQBQ^T) = J_\kappa(A) J_\kappa(B) / J_\kappa(I),$$

provides expressions for moments in Q. Here the J_κ are the Jack Polynomials with parameter $\alpha = 2/\beta$ [5, 12]. This formula is an analog of Theorem 7.2.5 of page 243 of [6]. It must be understood though that the formula is a generating function involving the moments of Q and Q^T. This is formally true whether or not one thinks that Q exists, or whether the formula is consistent or complete.

5 Ghost Gaussian Ensembles and Ghost Wishart Matrices

It is very interesting that if we tridiagonalize a complex Hermitian matrix (or a quaternion self-dual matrix), as is done with software for computing eigenvalues, the result is a real tridiagonal matrix. Equally interesting, and perhaps even easier to say, is that the bidigonalization procedure used in software for computing singular values, takes general rectangular complex (or quaternion) matrices into real bidiagonal matrices.

The point of view is that the Hermite and Laguerre models introduced in [1] are not artificial constructions, but they are shadows of symmetric or general rectangular ghost matrices respectively. If we perform the traditional Householder reductions on the ghosts the answers are the tridiagonal and bidiagonal models. The tridiagonal reduction of a normalized symmetric Gaussian ("The Gaussian $\beta - orthogonal$ Ensemble") is

$$H_n^\beta \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N & \chi_\beta \\ & & & \chi_\beta & N \end{pmatrix},$$

where the N on the diagonal are each independent Gaussians with mean 0 and variance 2. The $X's$ on the super and subdiagonal are equal giving a symmetric tridiagonal.

The bidiagonal for a the singular values of a general ghost is similar with chi's running on the diagonal and off-diagonal respectively. See [1] for details.

We repeat the key point that these "shadow" matrices are real and can therefore be used to compute the eigenvalues or the singular values very efficiently. The notion is that they are not artificial constructions, but what we must get when we apply the ghost Householder transformations.

6 Ghost Jacobian Computations

We propose that a β dimensional volume in what in retrospect must seem a straightforward manner. We define $(dx)^\wedge$ as the volume element. It satisfies the key scaling relationship $(rdx)^\wedge = r^\beta (dx)^\wedge$, that at least makes us want to look a little into "fractal theory," but at the moment we are suspecting this is not really the key direction to look. Nonetheless we keep an open mind. The important relationship must be $\int_{a<\|x\|<b} (dx)^\wedge = \int_a^b S_{\beta-1} r^{\beta-1} dr = S_{\beta-1}(b^\beta - a^\beta)/\beta$. We use the wedge notation to indicate the wedge product of the independent quantities in the vector or matrix of differentials.

This allows the computations of Jacobians of matrix factorizations for general β. As an example relevant to the tridiagonalization above, we can compute the usual jacobian for the symmetric eigenvalue problem obtaining $(dA)^\wedge = \prod_{i<j}(|\lambda_i - \lambda_j|^\beta)(Q^T dQ)^\wedge (d\Lambda)^\wedge$. The derivation feels almost straightforward from the differential of $A = Q\Lambda Q^T$ or $Q^T dAQ = (Q^T dQ)\Lambda - \Lambda(Q^T dQ) + d\Lambda$. The reason it is straightforward is that the quantity in the (i,j) position that multiplies $(q_i^T dq_j)$ is exactly $\lambda_i - \lambda_j$ and in a β dimensional space this must be scaled with a power of β respecting the dimensionality scaling $(rdx)^\wedge = r^\beta (dx)^\wedge$.

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