RANDOM HYPERPLANES, GENERALIZED SINGULAR VALUES & "WHAT'S MY β?"

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ABSTRACT

We streamline the treatment of the Jacobi ensemble from random matrix theory by providing a succinct geometric characterization which may be used directly to compute the Jacobi ensemble distribution without unnecessary matrix baggage traditionally seen in the MANOVA formulation. Algebraically the Jacobi ensemble naturally corresponds to the Generalized Singular Value Decomposition from the field of Numerical Linear Algebra.

We further provide a clear geometric interpretation for the Selberg constant in front of the distribution which may sensibly be defined even beyond the reals, complexes, and quaternions. On the application side, we propose a new learning problem where one estimates a β that best fits the sample eigenvalues from the Jacobi ensemble.

Index Terms— Random matrix theory, Jacobi ensemble, generalized singular value decomposition (GSVD), β-ensemble, parameter estimation

1. INTRODUCTION

Suppose we have two Gaussian random matrices A $(m_1 \times n)$ and B $(m_2 \times n)$ with $m_1 \geqslant n$ and $m_2 \geqslant n$. For example, `A=randn(m1,n)` and `B=randn(m2,n)` using common technical computing notation. The so-called *MANOVA matrix* (Multivariate Analysis of Variance) is defined to be

$$(A'A + B'B)^{-1}A'A \tag{1}$$

or in the symmetric form $(A'A + B'B)^{-1/2}A'A(A'A + B'B)^{-1/2}$. The eigenvalues are jointly distributed as [1]

$$c \cdot \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{n} \lambda_i^{a_1-p}(1-\lambda_i)^{a_2-p}, \tag{2}$$

where $a_1 = \frac{\beta}{2}m_1, a_2 = \frac{\beta}{2}m_2$ and $p = 1 + \frac{\beta}{2}(n-1)$,

$$c = \prod_{j=1}^{n} \frac{\Gamma(1+\frac{\beta}{2})\Gamma(a_1+a_2-\frac{\beta}{2}(n-j))}{\Gamma(1+\frac{\beta}{2}j)\Gamma(a_1-\frac{\beta}{2}(n-j))\Gamma(a_2-\frac{\beta}{2}(n-j))},$$

where $\beta = 1$ for real matrices, $\beta = 2$ for complex matrices, $\beta = 4$ for quaternion matrices, and general β is worth considering, as in [2] and also Section 4 of this paper. The eigenvalue distribution is known as the *Jacobi ensemble*, which was first referred by name in [3]. However, the development of Jacobi ensemble dates back to 1930's, we encourage interested readers to consult the wonderful piece of historical note by Anderson [4].

In the current paper we streamline the derivation of the Jacobi ensemble from random matrix theory by providing a succinct geometric characterization which may be used directly to compute the Jacobi ensemble distribution without unnecessary matrix baggage traditionally seen in the MANOVA formulation. Intimately linked to this formulation is the generalized singular value decomposition (the GSVD) [5] from the field of numerical linear algebra.

This allows us to directly view the Jacobi density as a Jacobi Jacobian. There are three key ideas that facilitate the derivation. The first idea is the representation of a point π on the Grassmann manifold $G(m,n)$ relative to $m = m_1 + m_2$ by an $m \times n$ matrix of the form

$$W_\pi = \left[\begin{array}{c|c} U & \\ \hline & V \end{array}\right]\left[\begin{array}{c} C \\ \hline S \end{array}\right] = \left[\begin{array}{c} UC \\ VS \end{array}\right], \tag{3}$$

where $U \in \mathbb{R}^{m_1\times n}$ and $V \in \mathbb{R}^{m_2\times n}$ have orthonormal columns, and we have the cosine matrix $C \in \mathbb{R}^{n\times n}$ and sine matrix $S \in \mathbb{R}^{n\times n}$ such that $C'C + S'S = \mathbf{I}$.

The second key idea is that the Jacobian computation is facilitated by constructing the completion of W_π to an orthogonal matrix (with a sign change for convenience):

$$W_\pi^\perp = \begin{bmatrix} -US & U^\perp & \\ VC & & V^\perp \end{bmatrix},$$

and recognizing that the wedge product of the differentials in $(W_\pi^\perp)'\mathrm{d}W_\pi$ is exactly the Jacobian we need.

The last one, which is related to the complicated looking constants in Selberg's integral [6, 7, 8], is the recognition of the geometric volumes underlying their existence. Ultimately these volumes are recognizable as (potentially non-integer dimensional) Stiefel and Grassmann manifolds, which

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all trace back to the formula for a (potentially non-integer dimensional) hypersphere.

We begin with a formulation of the Jacobi ensemble

> **Random Hyperplane Jacobi formulation**: Choose an n dimensional hyperplane in $\mathbb{R}^m$ $(m > n)$ uniformly at random. Take a fixed reference hyperplane of any dimension $m_1 \geq n$, the orthogonal projection of the unit ball in the random hyperplane onto the reference hyperplane is a random ellipsoid with semi-axes of length $c_1 \geq c_2 \geq \ldots \geq c_n$. These n non-negative numbers are the Jacobi ensemble in *cosine format.*

Even more succinct are commands for the numerical computation of the Jacobi ensemble in technical computing packages containing a command for the gsvd (the generalized singular value decomposition). The Jacobi ensemble may be generated in Julia [9] with the command

```
svdvals(randn(m1,n),randn(m2,n))
```

which computes the Jacobi ensemble in cotangent format. One may also obtain the sometimes preferred cosine, sine format by typing

```
_,_,_,C,S,_=svd(randn(m1,n),randn(m2,n)).
```

We believe that if the Jacobi ensemble can be described simply as the generalized singular values of normally distributed matrices, then the underlying mathematical treatment is compelled to follow along.

Traditionally the Jacobi ensemble is presented in cosine squared format as the eigenvalues of a MANOVA matrix [1, 10], or in terms of the eigenvalue density. The MANOVA format was intended for applications to the multivariate analysis of variance, thus the name. The treatment we describe in this paper provides a streamlined approach that gets to the geometrical core of Jacobi by working directly on the Grassmann manifold of random hyperplanes. For example, the Jacobi density reveals itself properly geometrically, as a GSVD Jacobian, rather than as an accident of a random matrix distribution.

We also present the notion that the (cosine, sine) format corresponding to the GSVD is more natural in many settings than the cosine square format of the eigenvalues. This is analogous to the Laguerre ensemble situation where the SVD of a matrix of independent standard normals can be more natural than the squared setting of eigenvalues of Wishart matrices.

2. GRASSMANNIAN, THE TANGENT SPACE AND DIFFERENTIALS

Let π be an n dimensional subspace of $\mathbb{R}^m, n \leqslant m$ and also suppose that we have two reference axis subspaces X and $Y = X^\perp$, of dimensions m_1 and m_2 respectively. (Without loss of generality, we take X and Y to be the spans of the first m_1 and last m_2 columns of $\mathbf{I}_m$ and $m = m_1 + m_2$.) Any unit vector $\boldsymbol{w}$ in π can be projected onto X and Y giving a cosine and sine pair. Generalizing this fact, the entire unit ball in π projects orthogonally onto an ellipsoid in X and also an ellipsoid in Y. Let U $(m_1 \times n)$ and V $(m_2 \times n)$ be unit vectors in the directions of the axes of these respective ellipsoids, the actual axes may be written UC and VS, where C and S are diagonal $n \times n$ matrices of cosine, sine pairs such that $C'C + S'S = \mathbf{I}_n$.

The $m \times n$ matrix W_π in (3) is a good representation for the subspace π capturing the projection onto X and Y. The set of all n-dimensional subspaces π of $\mathbb{R}^m$ is known as the *Grassmann manifold* $G(m, n)$. Every point in the Grassmann manifold where the corresponding cosines are unique may be represented uniquely by (3). If two or more axes have the same length, then the Grassmann points are multiply covered. This is analogous to eigenvectors degenerating into eigenspaces when a matrix has a multiple eigenvalue.

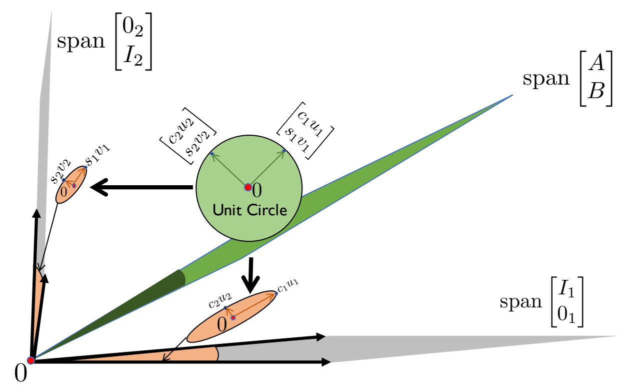

Fig. 1. Ellipses in Jacobi ensemble: the diagonal plane has a green unit circle centered at 0, the random hyperplane, which projects "downward" and leftward into the "horizontal" and "vertical" planes which serve as block coordinate axes. The projection downward leads to the Cosine Ellipse and leftward to the Sine Ellipse. The X and Y multiaxes are the span of $[\mathbf{I}_{m_1}; 0_{m_2}]$ and $[0_{m_1}; \mathbf{I}_{m_2}]$ respectively.

We identify n-dimensional subspaces of $\mathbb{R}^m$ (the Grassmann manifold) with matrices of the form (3), noting there is an irrelevant coordinate breakdown corresponding to cosines with multiplicity greater than 1. We also note that there are n phases to the columns of U and V which will be divided out later. The tangent space to the matrices Y_π consists of matrices

$$\mathrm{d}W_\pi = \begin{bmatrix} \mathrm{d}UC + U\mathrm{d}C \\ \mathrm{d}VS + V\mathrm{d}S \end{bmatrix} = \begin{bmatrix} \mathrm{d}UC - US\mathrm{d}\Theta \\ \mathrm{d}VS + VC\mathrm{d}\Theta \end{bmatrix}.$$

It is useful to complete U, V, and W_π to square orthogonal matrices, $\widetilde{U} = [U\ U^\perp]$ and $\widetilde{V} = [V\ V^\perp]$, and

$$\widetilde{W_\pi} = [W_\pi\ W_\pi^\perp], \quad W_\pi^\perp = \begin{bmatrix} -US & U^\perp & \\ VC & & V^\perp \end{bmatrix},$$

with $W_\pi^\perp$ broken into three block columns each having $n,\ m_1-n,\ m_2-n$ columns. We then may calculate

$$(W_\pi^\perp)'\mathrm{d}W_\pi = \begin{bmatrix} -SU'\mathrm{d}UC + CV'\mathrm{d}VS + d\Theta \\ (U^\perp)'\mathrm{d}UC \\ (V^\perp)'\mathrm{d}VS \end{bmatrix}, \quad (4)$$

where the block multiplication is viewed as $(2\times 3)^T(2\times 1)$. Notice $U'\mathrm{d}U$ and $V'\mathrm{d}V$ are anti-symmetric differential matrices.

We recall that the Grassmann manifold is $(m-n)n$ dimensional. The $(m-n)\times n$ matrix in (4) captures the dimensionality of the manifold. We ignore the $n\times n$ antisymmetric matrix of differentials $W_\pi'\mathrm{d}W_\pi$ as it corresponds to a direction where $W_\pi \to W_\pi Q$ for some orthogonal matrix which does not change the span.

We refer readers to [11] where, for example, in Table 2.2 we have the equation $W'\Delta = 0$, with Δ being the tangent directions, (or $Y'\Delta = 0$ in the notation of that paper) to indicate that we do not care about $W'\mathrm{d}W$. We also explain the quotient space language of horizontal and vertical spaces of differential geometry to understand the significance of the horizontal direction $(W_\pi^\perp)'\mathrm{d}W_\pi$. For readers uninterested in the jargon of differential geometry, suffice it to say that directions that spin the basis of a subspace, but do not move the subspace are treated as a zero tangent on the Grassmann manifold.

3. JACOBIAN COMPUTATION

We simply wedge together the differentials that are the matrix entries on both sides of (4) to directly get the answer:

$$\begin{aligned}((W_\pi^\perp)'\mathrm{d}W_\pi)^\wedge = &\prod_{i<j}(c_i^2-c_j^2)\prod_{i=1}^n c_i^{m_1-n}\prod_{i=1}^n s_i^{m_2-n} \\ &\times (\mathrm{d}\Theta)^\wedge(\widetilde{U}'\mathrm{d}U)^\wedge(\widetilde{V}'\mathrm{d}V)^\wedge.\end{aligned} \quad (5)$$

We have introduced the notation $()^\wedge$ to denote the wedge product of the enclosed differentials. This is nothing but a shorthand for a determinant. We sometimes omit the wedge between multiplicands understanding its presence. In the case of (5), we have the determinant of an $(m-n)n$ by $(m-n)n$ matrix that is entirely diagonal or block diagonal 2×2 matrices. For example, the wedge product of the (j,i) and (i,j) entries of $(-SU'\mathrm{d}UC + CV'\mathrm{d}VS + \mathrm{d}\Theta)$ is

$$\begin{aligned}&(s_jc_i(U'\mathrm{d}U)_{ij} - c_js_i(V'\mathrm{d}V)_{ij}) \\ &\qquad \wedge(-s_ic_j(U'\mathrm{d}U)_{ij} + c_is_j(V'\mathrm{d}V)_{ij}) \\ &= (s_j^2c_i^2 - s_i^2c_j^2)(U'\mathrm{d}U)_{ij}(V'\mathrm{d}V)_{ij}.\end{aligned}$$

We then simplify $(s_j^2c_i^2 - s_i^2c_j^2) = (c_i^2-c_j^2)$. Notice that $U'\mathrm{d}U$ and $V'\mathrm{d}V$ are anti-symmetric matrices, thus wedging the diagonal entries simply yields $(\mathrm{d}\Theta)^\wedge$.

Geometric Interpretation of the Jacobian: A small perturbation to subspace π can shift weight between subspaces X and Y, as well as rotate the projected ellipsoids. Correspondingly $(\mathrm{d}\Theta)^\wedge$ measures the weight shift, $(\widetilde{U}'\mathrm{d}U)^\wedge$ measures the rotation of the X ellipsoid, and $(\widetilde{V}'\mathrm{d}V)^\wedge$ measures the rotation of the Y ellipsoid. The term $\prod_{i<j}(c_i^2-c_j^2)\prod_{i=1}^n c_i^{m_1-n}\prod_{i=1}^n s_i^{m_2-n}$ indicates a repulsion away from multiple cosines, and if the exponents are positive, repulsion from X and Y as well.

4. GENERALIZATION TO $\beta = 2, 4$ AND BEYOND

There is nothing particularly special about having worked in real geometrical space. We could work in complex, and quaternion space, or even general β-dimensional space (formally) [2].

A β-dimensional volume takes linear measurements to the power β. Furthermore, unlike the real case, the general β skew-symmetric matrix has $\beta-1$ parameters on the diagonal. For example, a real anti-symmetric matrix has zeros on the diagonal, while complex anti-Hermitian matrices have pure imaginary diagonals (one dimension!). The quaternion case, of anti-self dual matrices consists of the "purely" imaginary quaternions which is a dimension 3 (i,j,k) subspace of the quaternions. We infer that a β-dimensional object with zero real part, the "purely" imaginary "ghosts" [2], ought to have dimension $\beta-1$, which is consistent with the general formulas. The generalization is thus simply a Jacobian of the form

$$\begin{aligned}&\prod_{i<j}(c_i^2-c_j^2)^\beta\prod_i c_i^{\beta(m_1-n)+(\beta-1)}\prod_i s_i^{\beta(m_1-n)+(\beta-1)} \\ &\qquad \times(\mathrm{d}\Theta)^\wedge(\widetilde{U}'\mathrm{d}U)^\wedge(\widetilde{V}'\mathrm{d}V)^\wedge.\end{aligned} \quad (6)$$

We believe that this "cosine squared" form, while traditional, obfuscates more than it reveals.

5. VOLUMES OF β GEOMETRIC OBJECTS

The sequence $2, 2\pi, 4\pi, 2\pi^2, 8\pi^2/3, \ldots$ may be familiar to many readers. This is the volume (surface area) of the unit hypersphere in n real dimensions:

$$V_{\text{sphere}}(n;\beta=1) = 2\pi^{n/2}/\Gamma(n/2).$$

Our "ghost" view of the world [2] is that this is also the volume of the unit hypersphere in 1 β dimension:

$$V_{\text{sphere}}(n=1;\beta) = 2\pi^{\beta/2}/\Gamma(\beta/2).$$

The obvious generalization suggests the definition

$$V_{\text{sphere}}(n;\beta) = 2\pi^{\beta n/2}/\Gamma(\beta n/2).$$

We introduce the generalized idea of a ghost "phase." Readers may think of a complex number on the unit circle, or a unit quaternion, or even the "signs" ± 1. The content of a collection of n ghost phases may be defined as

$$V_{\text{phases}}(n,\beta) = V_{\text{sphere}}(1;\beta)^n.$$

The real Stiefel manifold is the collection of $m \times n$ orthogonal matrices Y with $Y^T Y = \mathbf{I}$. Numerical linear algebra has long represented such matrices in compact form as products of so called "'Householder" reflectors, each one of which may be thought of as having one direction on a unit sphere of the right size. Generalizing to ghosts, we have

$$V_{\text{Stiefel}}(m, n; \beta) = \prod_{i=1}^{n} V_{\text{sphere}}(m - i; \beta), \ m \geq n.$$

Finally we may interpret the quotient manifold volume with ghost geometry as

$$V_{\text{Grassmann}}(m, n; \beta) = V_{\text{Stiefel}}(m, n; \beta)/V_{\text{Stiefel}}(n, n; \beta).$$

Then integrating out the U and the V in (6) results in $V_{\text{Stiefel}}(m_1, n; \beta)V_{\text{Stiefel}}(m_2, n; \beta)$. The integral of this quantity gives the volume of the Grassmann mainfold multiply covered by the n phases. Dividing by thie volume of the phases we get a formula for the volume of the Grassmann manifold. Dividing further by the Grassmann volume yields a probability density (i.e., integrates to 1)

$$\frac{V_{\text{Stiefel}}(m_1, n; \beta)V_{\text{Stiefel}}(m_2, n; \beta)}{V_{\text{phases}}(n; \beta)V_{\text{Grassmann}}(m, n; \beta)} \times \prod_{i<j}(c_i^2 - c_j^2)^\beta \prod_i c_i^{\beta(m_1-n)+(\beta-1)} \prod_i s_i^{\beta(m_1-n)+(\beta-1)} \mathrm{d}\theta_i,$$

with $\theta_1 \leq \ldots \leq \theta_n$. While we used our intuition to obtain a "ghost" formula, the Selberg integral confirms that density integrates to 1, hence is a true probability density.

6. RELATED WORK

It is useful to compare our W_π format (3) to some closely related formats. Suppose W_ρ is an $m \times m_1$ matrix whose columns form an orthonormal basis for a reference hyperplane ρ of dimension m_1. Also let W_π be any $m \times n$ matrix whose columns form an orthonormal basis for the random hyperplane π of dimension n. The cosines of the canonical angles are the svd of $W'_\rho W_\pi$. If either π or ρ is random with uniform measure then the cosines are the Jacobi ensemble in cosine format. Furthermore, the singular values of $W'_\rho W_\pi$ are the same as those of $W'_\rho W_\pi W'_\pi$. These singular values are the square roots of the eigenvalues of $W_\pi W'_\pi W_\rho W'_\rho W_\pi W'_\pi = P_\pi P_\rho P_\pi$, where P denotes the symmetric projector onto the subspace. This is the format favored by Collins [12].

Starting with Collins format, we can recover our favored format by letting $W_\rho = W_X$ be the first m_1 columns of $\mathbf{I}_m$. We then can take U and C to be the left singular vectors and the singular values of $W'_X W_\pi$. Analogously, V and S are the left singular vectors and singular values of $W'_Y W_\pi$, when W_Y are the last m_2 columns of $\mathbf{I}_m$.

The exact distribution of the largest angle between random subspaces may be found in [13]. Statistical applications of the largest roots of the Jacobi ensemble with a computation of the Tracy-Widom asymptotics may be found in [10]. A beta-Jacobi model may be found in [14]. Concentration of the linear statistics of the Jacobi ensemble through combinatorial computations may be found in [15] . The Jacobi corner process for general β is introduced in [16]. Non-central wishart based MANOVA ensemble distributions with zonal polynomials may be found in [17] A general beta with general covariance sampling method is presented in [18]. Other notable works include a series of seminal papers by Forrester [19, 20, 21, 22] and of course his encyclopedic monograph [23].

7. WHAT IS MY β?

We conjecture that the β parameter ought to be valuable for machine learning problems. The first beta estimator application was proposed in 2005 by the first author and Chan for estimating β for the sample eigenvalue spacings and set up as an interactive web application [24]. In this section, we propose a novel learning problem for the Jacobi ensemble.

Suppose we are given a sample $\boldsymbol{\lambda}_i \in [0,1]^n$, $i = 1, \cdots, N$ from the eigenvalues of (1) where the underlying (unobserved) N independently and identically distributed (iid) data matrix pairs (A_i, B_i) with $A_i \in \mathbb{R}^{m_1 \times n}$ and $B_i \in \mathbb{R}^{m_2 \times n}$ are thought of as β-Gaussian models, Our goal is to estimate the β that best describes the underlying data generating process (i.e., the generative model of A, B).

To this end, we resort to the standard approach of maximum likelihood estimation and find the optimal β that minimizes the negative loglikelihood (NLL). Figure 2 shows the NLL vs. β for $m_1 = m_2 = 20, n = 10$ with $N = 50$ and the true generating β's can be recovered at the minimum of the corresponding curves. Standard optimization routines such as Gradient descent could be used to find an optimal β.

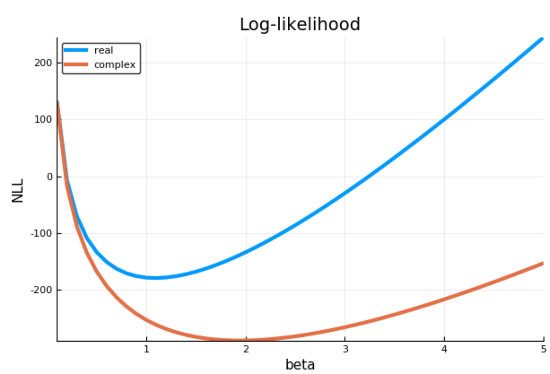

Fig. 2. Loglikehood of the β-Jacobi ensemble. The blue line corresponds to the real case ($\beta = 1$) and the orange one is the complex case ($\beta = 2$.)

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