Problem 1.

The solid \( R \) is a cone with vertex \((0,0,10)\) and base on the \(xy\)-plane equal to the disk of radius 10 centered at the origin. We must show that:

\[
\iiint_{R} \text{div}(\vec{F}) \, dV = \iint_{\partial R} \vec{F} \cdot d\vec{S}
\]

where \( \partial R \) is the boundary of \( R \) (in our case, the base and the lateral surface of the cone).

We have \( \text{div}(\vec{F}) = 2 \), so

\[
\iiint_{R} \text{div}(\vec{F}) \, dV = 2 \cdot V \text{ol}(R) = \frac{2000\pi}{3}
\]

On the other hand, the unit normal vector outgoing from the base is \(-\hat{k}\) and \(\vec{F} \cdot -\hat{k} = 0\) so the flux through the base is 0. The lateral surface is given by \( z = f(x,y) = 10 - \sqrt{x^2 + y^2} \), so

\[
d\vec{S} = (-f_x, -f_y, 1) \, dx \, dy = \left(\frac{z}{r}, \frac{y}{r}, 1\right) r \, dr \, d\theta \quad \text{(we switched to polar)}
\]

and

\[
\vec{F} \cdot d\vec{S} = r^2 \, dr \, d\theta.
\]

Hence, the flux is

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \frac{2000\pi}{3} = \frac{2000\pi}{3}
\]

Problem 2.

a) In terms of the parametrization, \( \vec{F} = (as\sin(t) + bc\cos(t)s\sin^2(t))\vec{i} + c\cos^2(t)s\sin(t)\vec{j} + (c\cos^2(t) - t^3)\vec{k} \) and \( dx = -s\sin(t), \, dy = c\cos(t), \, dz = dt \). Integrating from 0 to \(2\pi\) we get:

\[
\int_{0}^{2\pi} [-as\sin^2(t) - c\cos(t)s\sin^3(t) + c\cos^3(t)s\sin(t) + c\cos^2(t) - t^3] \, dt = (1-a)\pi - 4\pi^4
\]

b) \( \nabla \times \vec{F} = (ac\cos(z) - c\cos(z))\vec{j} + (2xy - 2bcxy)\vec{k} \). Therefore, we must have \( a = 1, \, b = 1 \) for the curl to vanish.
c) Using the algebraic method we have that a potential function $f = x\sin(z) - \frac{z^3}{4} + \frac{z^2}{2}$. We have $f(1,0,0) = 0$ and $f(1,0,2\pi) = -4\pi$ as desired.

Problem 3.

a) Orient each of the above four surfaces outward from $R$. Compute $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ for each of these surfaces. Calculate by inspection where you can, to save effort. 

$\nabla \times \vec{F} = -\vec{i} + \vec{j} - 2\vec{k}$. We calculate the next three integrals by inspection:

$S_{xy}$: $\vec{n} = -\vec{k}$ and $\nabla \times \vec{F} \cdot \vec{n} = 2$ thus $\iint_{S_{xy}} \nabla \times \vec{F} \cdot d\vec{S} = 2 \cdot \text{Area}(S_{xy}) = \frac{\pi}{2}$.

$S_{xz}$: $\vec{n} = -\vec{j}$ and $\nabla \times \vec{F} \cdot \vec{n} = -1$ thus $\iint_{S_{xz}} \nabla \times \vec{F} \cdot d\vec{S} = 2 \cdot \text{Area}(S_{xz}) = -\frac{2}{3}$.

$S_{yz}$: $\vec{n} = -\vec{i}$ and $\nabla \times \vec{F} \cdot \vec{n} = 1$ thus $\iint_{S_{yz}} \nabla \times \vec{F} \cdot d\vec{S} = 2 \cdot \text{Area}(S_{yz}) = \frac{2}{3}$.

To calculate $\iint_{S_{top}} \nabla \times \vec{F} \cdot d\vec{S}$ we observe that $S_{xy}$ is the base of $S_{top}$ and we express it as:

$$\iint_{S_{xy}} \nabla \times \vec{F} \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dA = \iint_{S_{xy}} (-\vec{i} + \vec{j} - 2\vec{k}) \cdot (2\vec{x}\vec{i} + 2\vec{y}\vec{j} + \vec{k})\,dA = \iint_{S_{xy}} -2x + 2y - 2\,dA = 2 \int_0^\pi \int_0^2 (r\cos\theta + r\sin\theta - 1)\,r\,d\theta\,d\alpha = -\frac{\pi}{3}$$

b) We calculate the line integral along each of the six segments. We orient $C_x$, $C_y$ and $C_z$ putward from the origin, $C_{xz}$ and $C_{yz}$ downward, and $C_{xy}$ from the $x$-axis to the $y$-axis.

For $C_x$ we have $y, z = 0$ thus $\vec{F}$ has no $\vec{i}$ component, so $\int_{C_x} \vec{F} \cdot d\vec{r} = 0$.

For $C_y$ we have $x, z = 0$ thus $\vec{F}$ has no $\vec{j}$ component, so $\int_{C_y} \vec{F} \cdot d\vec{r} = 0$.

For $C_z$ we have $x, y = 0$ so $\int_{C_z} \vec{F} \cdot d\vec{r} = \int_{C_z} zdz = \int_0^2 dz = \frac{1}{2}$.

For the other three:

On $C_{xz}$ we have $y = 0$ thus $\int_{C_{xz}} \vec{F} \cdot d\vec{r} = \int_{C_{xz}} (z-x)\,dz$. Parametrizing by $0 \leq t \leq 1$, $x = t$, $z = 1 - t^2$ we get $\int_{C_{xz}} = \int_0^1 (1 - t^2 - t)(-2t)\,dt = \frac{1}{6}$.

On $C_{yz}$ we have $x = 0$ thus $\int_{C_{yz}} \vec{F} \cdot d\vec{r} = \int_{C_{yz}} (z-y)\,dz$. Parametrizing by $0 \leq t \leq 1$, $y = t$, $z = 1 - t^2$ we get $\int_{C_{yz}} = \int_0^1 (1 - t^2 - t)(-2t)\,dt = \frac{1}{6}$.

On $C_{xy}$ we have $z = 0$ thus $\int_{C_{xy}} \vec{F} \cdot d\vec{r} = \int_{C_{xy}} ydx - xdy$. Parametrizing by $0 \leq t \leq \frac{\pi}{2}$, $x = \cos t$, $y = \sin t$, we get $\int_{C_{xy}} = \int_0^{\frac{\pi}{2}} (-\sin^2 t - \cos^2 t)\,dt = -\frac{\pi}{2}$.

For each of the four surfaces we orient the curves according to the right hand rule and apply Stokes’ Theorem:
\begin{align*}
\iint_{S_{xy}} &= \int_{C_y} - \int_{C_{xy}} - \int_{C_x} = 0 - \left(-\frac{\pi}{2}\right) - 0 = \frac{\pi}{2} \\
\iint_{S_{xz}} &= \int_{C_x} - \int_{C_{xz}} - \int_{C_z} = -\frac{2}{3} \\
\iint_{S_{yz}} &= \int_{C_z} + \int_{C_{yz}} - \int_{C_y} = 0 - \left(-\frac{\pi}{2}\right) - 0 = \frac{2}{3} \\
\iint_{S_{top}} &= \int_{C_{xz}} + \int_{C_{zy}} - \int_{C_{yz}} = \frac{\pi}{2}
\end{align*}

These values agree with part (a). Summing the four surface integrals sums the corresponding line integrals. But that gives a positive and negative contribution for each of the six segments. Hence the sum must be 0.

Problem 4.

a) \text{div}(\nabla \times \vec{G}) = 0 \text{ for any } \vec{G} \text{ with continuous 2nd partials. We have } \text{div}(\vec{F}) = 0. \text{ By the divergence theorem:}

\[ \iiint_S \vec{F} \, dS = \iiint_D \text{div}(\vec{F}) \, dV = 0 \]

b) Using Stokes' Theorem: Draw a simple closed \( C \) on \( S \). It divides \( S \) into regions \( S_1 \) and \( S_2 \) and applying Stokes’ theorem (with appropriate orientations on \( C \)) we get:

\[ \oint_C \vec{G} \cdot d\vec{r} = \iint_{S_1} \vec{F} \cdot d\vec{S} \]

and

\[ \oint_{-C} \vec{G} \cdot d\vec{r} = \iint_{S_2} \vec{F} \cdot d\vec{S} \]

thus

\[ \iint_S \vec{F} \cdot d\vec{S} = \oint_C \vec{G} \cdot d\vec{r} + \oint_{-C} \vec{G} \cdot d\vec{r} = 0 \]

Problem 5.

a) Assume \( \vec{F} = \nabla \times \vec{G} \). We have:

\[ \oint_C \vec{G} \cdot d\vec{r} = \iint_R \nabla \times \vec{G} \cdot d\vec{S} = \iint_R adS = a \cdot \text{Area}(R) \]

Since \( \vec{F} \cdot \vec{n} = \langle x, y, z \rangle \cdot \frac{(x, y, z)}{a} \). Similarly,
\[
\oint_C \vec{G} \cdot d\vec{r} = \iint_{R'} \nabla \times \vec{G} \cdot d\vec{S} = \iint_{R'} adS = a \cdot \text{Area}(R')
\]

We conclude that \( a \cdot \text{Area}(R) = -a \cdot \text{Area}(R') \) which is a contradiction since both quantities are positive.

b) Using problem 3 or 6H-1 \( \nabla \cdot \nabla \times \vec{G} = 0 \) while for \( \vec{F} = \langle x, y, z \rangle \) we have \( \nabla \cdot \vec{F} = 3 \). This is a contradiction.