18.02A Problem Set 7 – IAP 2007 due Thursday Jan.25, 11:45 in 2-106

Part I (20 points)

Lecture 28. Thurs. Jan.18 Triple integrals: rectangular and cylindrical coordinates
Read: Notes I.3, 20.5, 20.6 Work: 5A - 2a,d, 5, 6
Lecture 29. Mon. Jan.22 Spherical coordinates; gravitational attraction
Read: 20.7, Notes G Work: 5B - 1c, 3; 5C-3
Lecture 30. Tues. Jan. 23 Vector fields in 3-space; Surface integrals
Read: Notes V8, V9 Work: 6A - 1, 3, 4; 6B - 1, 2, 3, 4, 5, 8
Lecture 31. Wed. Jan. 24. More surface integrals. Divergence theorem.
Read: Notes V10 Work: 6C - 3, 5, 7a, 8
Lecture 32. Thurs. Jan. 25. Divergence theorem cont'd. Applications, interpretations.
Read: Notes V15 sec.1 for div in ∇ notation - skip Stokes' theorem refs.; Notes v15 sec. 2 to middle p.3

Part II (30 points)

Problem 1. (Friday. 5pts)
A wedge is given by the region above the plane $z = 0$ and below the plane $z = y/2$ where $x^2 + y^2 \leq 1$. The density of the material in the wedge is given by $\rho(x, y, z) = (x^2 + y^2)^{1/2}$. Use cylindrical coordinates to compute the mass of the wedge.

Problem 2. (Mon. 5pts)
Prove by using triple integration in spherical coordinates that the gravitational attraction of a solid sphere of radius a and density 1 on a unit point mass P on its circumference is $\frac{4}{3}\pi Ga$.
(This is the same as if all the mass of the sphere were concentrated at its center. Begin by placing the sphere so that P is at the origin, with the $z-$axis as a diameter.)

Problem 3. (Mon. 5pts)
Let D be a solid circular disk of radius a. If we rotate D around a tangent line, we obtain a donut shape T called a *torus* whose central 'hole' has radius 0. Use spherical coordinates to compute the volume of T. (Can use the fact: $\int \sin^4 x \, dx = \frac{1}{32}(12x - 8 \sin 2x + \sin 4x)$.)

Problem 4. (Tuesday. 4pts: 2+2)
a) What percentage, rounded to the nearest percent, of the Earth's surface is south of Rio de Janeiro? The latitude there is 23°S, that is, 23° south of the Equator.
b) Find the average latitude of all points in the Northern Hemisphere. Our latitude in Boston is 42°N. Are we above or below average? Optional: Identify a city whose latitude is within one degree of the average.

Problem 5. (Wed. 11pts: 2+4+3+2)
Consider the space region bounded below by the right-angled cone $z = (x^2 + y^2)^{1/2}$ and above by the sphere $x^2 + y^2 + z^2 = 2$. These two surfaces intersect in a horizontal circle. Let T be the horizontal disk having this circle as boundary, S the spherical cap forming the upper surface, and U the cone forming the lower surface. Orient S, T, U 'upwards', so the normal vector has a positive $\mathbf{k}$-component.
a) For each of the three surfaces, determine geometrically (without calculation) whether the flux of the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$ is positive or negative.
b) Calculate the flux of $\mathbf{F}$ across each surface (with upwards orientation), directly from the definition of flux.
c) Use the divergence theorem to find the flux of $\mathbf{F}$ outwards from the solid cone bounded by T and U. (In other words, calculate the flux using triple integrals rather than the surface integrals.) Same question with the ice-cream cone bounded by S and U.
d) Show that the answers you found in (c) are consistent with those you found in (b).