1. We use cylindrical coordinates: \( x = r \cos \theta, \ y = r \sin \theta, \) and \( z, \) where in our case \( 0 \leq z \leq y/2 = (r \sin \theta)/2, \) from which we get \( y \geq 0, \) i.e., \( 0 \leq \theta \leq \pi. \) And from \( x^2 + y^2 \leq 1 \) we get \( r \leq 1. \) Then the mass is

\[
M = \iiint_R \delta \, dV = \iiint_R r^2 \, dr \, d\theta \, dz = \int_0^\pi \int_0^1 \int_0^{(r \sin \theta)/2} r^2 \, dz \, dr \, d\theta.
\]

Computing this, we have

- **Inner:**
  \[
  \int_0^{(r \sin \theta)/2} r^2 \, dz = \left( r^3 \sin \theta \right)/2.
  \]

- **Middle:**
  \[
  \int_0^1 \left( r^3 \sin \theta \right)/2 \, dr = (\sin \theta)/8.
  \]

- **Outer:**
  \[
  \int_0^\pi \left( \sin \theta \right)/8 \, d\theta = (-\cos \theta)/8|_0^\pi = 1/4.
  \]

2. Place the sphere so the point mass \( P \) is at the origin. Let \( Q \) be diametrically opposite to \( P \) (so it has coordinates \((0, 2a)\)), and form the right triangle with vertices \( P, \) \( Q, \) and \( R, \) where \( R \) is an arbitrary point on the surface of the sphere. By trigonometry, we get that \( \rho, \) the distance from \( P \) to \( R, \) is \( 2a \cos \phi, \) where \( \phi \) is the third coordinate of \( R \) in spherical coordinates. The gravitational attraction is in the \( \hat{k} \) direction by symmetry and has magnitude

\[
G \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2a \cos \phi} \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta.
\]

Evaluating, we have

- **Inner (drop \( G \)):** \( \cos \phi \sin \phi \cdot 2a \cos \phi = 2a \cos^2 \phi \sin \phi. \)

- **Middle:** \(-2a \cos^3 \phi |_0^{\pi/2} = 2a/3. \)

- **Outer (include \( G \)):** \( G \cdot 2\pi \cdot \frac{2a}{3} = \frac{4\pi a}{3} G. \)

3. We use spherical coordinates \( x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ z = \rho \cos \phi. \) Place the central “hole” of the torus at the origin of the coordinate system. Then \( 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi, \) and if we take a section of the torus with a plane passing through the \( z \)-axis, we see, by an argument similar to the one in the previous problem, that \( 0 \leq \rho \leq 2a \sin \phi. \) For the volume of the torus \( T \) we obtain

\[
\iiint_T \, dV = \iiint_T \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{2a \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.
\]

Evaluating, we obtain

- **Inner:** \( \int_0^{2a \sin \phi} \rho^2 \sin \phi \, d\rho = \frac{(2a \sin \phi)^3 \sin \phi}{3}. \)

- **Middle:** \( \int_0^{2a \sin \phi} \rho^3 \sin^4 \phi \, d\phi \cdot 12\pi = \frac{8a^3}{3} \cdot \frac{1}{32} \cdot 12\pi = a^3 \pi. \)
4. (a) Assume the Earth is spherical with radius $R$ so its surface has total area $4\pi R^2$. The region south of Rio has latitude ranging from 113 to 180 degrees, so it has area
\[
\int_0^{2\pi} \int_{113/180}^{\pi} R^2 \sin \phi \, d\phi \, d\theta = 2\pi R^2 \left[ 1 + \cos \left( \frac{113}{180} \pi \right) \right].
\]
Hence the percentage is approximately 30%.

(b) The surface area of the northern hemisphere is $2\pi R^2$. The average value of $\phi$ is
\[
\frac{1}{2\pi R^2} \int_0^{2\pi} \int_0^{\pi/2} \phi R^2 \sin \phi \, d\phi \, d\theta = \int_0^{\pi/2} \phi \sin \phi \, d\phi = \left[-\phi \cos \phi\right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos \phi) \, d\phi
= [\sin \phi]_0^{\pi/2} = 1.
\]
Thus the average value of $\phi$ is 1 radian, or approximately 57 degrees, so the average latitude is approximately $90 - 57 = 33$ degrees.

5. It is easy to see that the intersection of the sphere and the cone is the circle of radius 1 in the plane $z = 1$.

(a) The field $(-x, -y)$ is horizontal and points radially inwards (towards the $z$-axis), so the flux is negative across $S$ (the normal points outward), zero across $T$ (the normal is perpendicular to the field), and positive across $U$ (the normal points inward).

(b) Across $T$, the flux is zero (since the normal vectors to $T$ are perpendicular to the field).
Across $S$ (the spherical cap, where $\rho = \sqrt{2}$ and $\phi < \pi/4$), $dS = \rho^2 \sin \phi \, d\phi \, d\theta$ and the unit normal is $1/\rho(x, y, z)$, so $\mathbf{F} \cdot \mathbf{n} = -1/\rho(x^2 + y^2) = -\rho \sin^2 \phi = -\sqrt{2} \sin^2 \phi$, so the flux is
\[
\int_0^{2\pi} \int_0^{\pi/4} \left(-\sqrt{2} \sin^2 \phi\right)(2 \sin \phi) \, d\phi \, d\theta
= -2\sqrt{2}(2\pi) \int_0^{\pi/4} \sin^3 \phi \, d\phi
= -4\pi \sqrt{2} \int_0^{\pi/4} \sin \phi (1 - \cos^2 \phi) \, d\phi
= -4\pi \sqrt{2} \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^{\pi/4}
= -\frac{(8\sqrt{2} - 10)\pi}{3}.
\]
Now across the cone \( U \), \( \mathbf{dS} = (-f_x, -f_y, 1) \, dA = (-x/r, -y/r, 1) \, dA \), so \( \mathbf{F} \cdot \mathbf{dS} = r^2 \, dr \, d\theta \). Thus the flux is

\[
\int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta = (2\pi) \left( \frac{1}{3} \right) = \frac{2\pi}{3}.
\]

(c) The divergence of \((-x, -y)\) is just \(-2\), so by the divergence theorem, the flux out of the cone is \(-2\) times the volume of the cone, which is just \(\pi/3\). Thus the flux out of the cone is \(-2\pi/3\). This differs by the value computed in part (b) because the divergence theorem assumes that the surface is positively oriented (normals point outward) whereas we oriented the cone so that the normals pointed inward in (b). Note that the flux through the top \( T \) is 0.

For the flux through the entire “ice cream cone,” we need the volume of the solid, which we compute in spherical coordinates:

\[
\iiint dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
= 2\pi \int_0^{\pi/4} \frac{1}{3} \rho^3 \sin \phi \, d\rho
= 2\pi \left( \frac{2\sqrt{2}}{3} \right) \int_0^{\pi/4} \sin \phi \, d\phi
= \frac{4(\sqrt{2} - 1)\pi}{3}.
\]

Since the divergence is \(-2\), the total flux out of the “ice cream cone” is \(-\frac{8(\sqrt{2} - 1)\pi}{3}\).

(d) As we mentioned in (c), the computation in (b) agrees with that of (c) for the cone, once we take into account the orientation of the cone (we used the negative orientation in (b), so we have to flip the sign for it to be consistent with (c)). A similar accounting of signs yields agreement for the ice cream cone.