18.02A: IAP 2007. PROBLEM SET 5: SOLUTIONS TO PART II

Problem 5.1. *Let*

$$\mathbf{F} = (4xy + y^2 + 2)\mathbf{i} + (2x^2 + 2xy + y^2)\mathbf{j}.$$

Find a potential function for $\mathbf{F}$ *in two ways: using*

- *a) the algebraic method and*
- *b) the integration method.*

Solution. We calculate the potential using both methods.

- a) We use the algebraic method. We must find a function f where
$$f_x = 4xy + y^2 + 2$$
and
$$f_y = 2x^2 + 2xy + y^2.$$
Integrating, we see that
$$f = 2x^2y + xy^2 + 2x + g(y)$$
and that
$$f = 2x^2y + xy^2 + \frac{1}{3}y^3 + h(x).$$
Solving for $g(y)$ and $h(x)$ we obtain $h(x) = 2x + C$ and $g(y) = \frac{1}{3}y^3 + C$. So
$$f(x, y) = 2x^2y + xy^2 + 2x + \frac{1}{3}y^3 + C.$$
We can easily check that ∇f does indeed equal $\mathbf{F}$.
- b) We use the line integral method. We can arbitrarily set some point of f, so set $f(0, 0) = c$. Then we set
$$f(a, b) = f(0, 0) + \int_C (4xy + y^2 + 2)dx + (2x^2 + 2xy + y^2)dy$$
for each (a, b) where C is a curve from $(0, 0)$ to (a, b). We choose C to be a straight line parameterized as $x = at$ and $y = bt$ for $0 \le t \le 1$. Then the integral becomes
$$\int_0^1 \left[(4abt^2 + b^2t^2 + 2)a + (2a^2t^2 + 2abt^2 + b^2t^2)b\right] dt$$
which is
$$\left[2a^2bt^3 + ab^2t^3 + \frac{1}{3}b^3t^3 + 2at\right]_0^1 = 2a^2b + ab^2 + \frac{1}{3}b^3 + 2a.$$
Thus,
$$f(x, y) = 2x^2y + xy^2 + 2x + \frac{1}{3}y^3 + c.$$

□

Problem 5.2. *Let* $\mathbf{F} = xy\mathbf{i} + xy\mathbf{j}$.

- *a) Show that* **F** *is not a gradient field by taking the appropriate derivatives.*
- *b) Try to find a potential function for* **F** *by the integration method. What goes wrong?*
- *c) Same question for the algebraic method.*

Solution. **F** is not a gradient field.

- a) If **F** is a gradient field, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where $M = xy$ and $N = xy$. But $\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = y$, so **F** is *not* a gradient field.
- b) As in Part II #1, let $f(0,0) = C$. Then

$$f(a,b) = f(0,0) + \int_{(0,0)}^{(a,b)} xydx + xydy.$$

Path 1: Straight line between $(0,0)$ and $(a,b) : x = at$ and $y = bt$.

$$f(a,b) = f(0,0) + \int_0^1 at(bt)adt + at(bt)bdt = \frac{1}{3}(a^2b + ab^2) + C.$$

Path 2: First segment to $(0,a)$, second segment to (a,b).

$$f(a,b) = f(0,0) + \int_0^a xy|_{y=0}dx + \int_0^b xy|_{x=a}dy = ab.$$

But if **F** were a gradient field then $\int_C \mathbf{F} \cdot d\mathbf{r}$ should be path-independent. Since we get different answers for two different paths, **F** cannot be a gradient field.

- c)

$$f_x = xy \Rightarrow f = \frac{1}{2}x^2y + g(y).$$

$$f_y = xy \Rightarrow f = \frac{1}{2}xy^2 + h(x).$$

But it is impossible to find a $g(y)$ that is *only a function of* y and an $h(x)$ that is *only a function of* x that make these two f's equivalent.

□

Problem 5.3. *Let* $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$.

- *a) Calculate the curl of* **F**.
- *b) For each* n *for which* $\operatorname{curl}\mathbf{F} = 0$, *find a potential* g *such that* $\mathbf{F} = \nabla g$. *(Hint: seek a potential* $g = g(r)$. *Watch out for a certain negative* n *value for which the formula is different.)*

Solution. $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$.

- a)

$$\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j});$$

$$\operatorname{curl}\mathbf{F} = \frac{\partial(yr^n)}{\partial x} - \frac{\partial(xr^n)}{\partial y} = nyr^{n-1}\frac{x}{r} - nxr^{n-1}\frac{y}{r} = 0.$$

- b) If $g = g(r)$, then $g_x = g'(r)\frac{x}{r}$ and $g_y = g'(r)\frac{y}{r}$ (by the chain rule), so

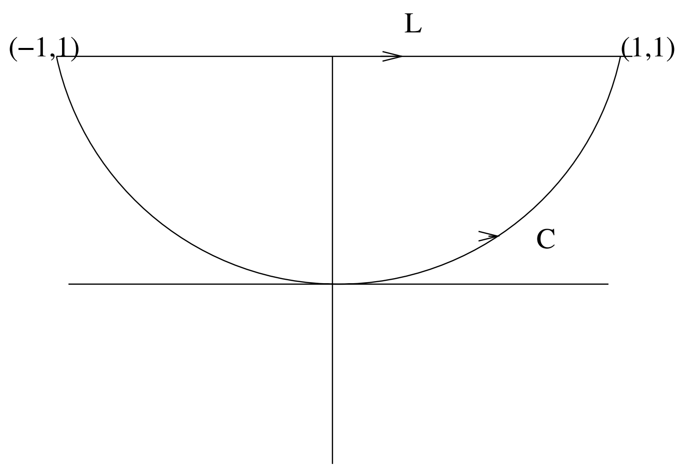
$$\nabla g = \frac{g'(r)}{r}(x\mathbf{i} + y\mathbf{j}).$$

We must find g such that $\frac{g'(r)}{r} = r^n$, i.e. $g'(r) = r^{n+1}$. Two cases: $n \neq -2 :$ $g(r) = \frac{1}{n+2}r^{n+2}$. $n = -2 : g(r) = \ln(r)$.

□

Problem 5.4. *Consider* $\mathbf{F} = \nabla(x^2y + xy^2)$. *Let* C *be the semicircle having its midpoint at the origin and running from* $(-1, 1)$ *to* $(1, 1)$.

- *a) Write the integral* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *in the* $\int_C M dx + N dy$ *form.*
- *b) Evaluate the integral in two different but easy ways: using i) the Fundamental Theorem, ii) path-independence. Show the calculations in each case.*



Solution.

FIGURE 1. The curve C

- a) $\mathbf{F} = \nabla(x^2y + xy^2) =< 2xy + y^2, x^2 + 2xy >$
  L: $x = t$, $dx = dt$, $y = 1$, $dy = 0$.

$$\int_C (2xy + y^2)dx + (x^2 + 2xy)dy$$

- b) By Fundamental Theorem of Calculus for Line Integrals:

$$x^2y + xy^2\Big|_{(-1,1)}^{(1,1)} = 2 - (1 - 1) = 2.$$

Using path independence:

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} (2x + 1)dx = x^2 + x\Big|_{-1}^{1} = 2 - (1 - 1) = 2.$$

□

Problem 5.5. • *a) For what simple closed (positively oriented) curve C in the plane does the line integral*

$$\oint_C (x^2y + y^3 - y)dx + (3x + 2y^2x + e^y)dy$$

have the largest positive value? (Hint: use Green's Theorem).

- *b) What is this maximum value?*

Solution. We use Green's Theorem.

- a)

$$\oint_C (x^2y+y^3-y)dx+(3x+2y^2x+e^y)dy = \iint_R (3+2y^2)-(x^2+3y^2-1)dA = \iint_R (4-x^2-y^2)dA,$$

where R is the region enclosed by C. The integrand is positive inside the circle $x^2 + y^2 = 4$ and negative outside, so the integral is biggest when C is the circle $x^2 + y^2 = 4$.

- b)

$$\iint_{x^2+y^2<4} (4 - x^2 - y^2)dA = \int_0^{2\pi}\int_0^{2} (r - 4^2)rdrd\theta = 2\pi \left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 8\pi.$$

□