Problem 5.1. Let
\[ \mathbf{F} = (4xy + y^2 + 2)i + (2x^2 + 2xy + y^2)j. \]
Find a potential function for \( \mathbf{F} \) in two ways: using

- a) the algebraic method and
- b) the integration method.

**Solution.** We calculate the potential using both methods.

- a) We use the algebraic method. We must find a function \( f \) where
  \[ f_x = 4xy + y^2 + 2 \]
  and
  \[ f_y = 2x^2 + 2xy + y^2. \]
  Integrating, we see that
  \[ f = 2x^2y + xy^2 + 2x + g(y) \]
  and that
  \[ f = 2x^2y + xy^2 + \frac{1}{3}y^3 + h(x). \]
  Solving for \( g(y) \) and \( h(x) \) we obtain \( h(x) = 2x + C \) and \( g(y) = \frac{1}{3}y^3 + C \). So
  \[ f(x, y) = 2x^2y + xy^2 + 2x + \frac{1}{3}y^3 + C. \]
  We can easily check that \( \nabla f \) does indeed equal \( \mathbf{F} \).

- b) We use the line integral method. We can arbitrarily set some point of \( f \), so set \( f(0, 0) = c \). Then we set
  \[ f(a, b) = f(0, 0) + \int_C (4xy + y^2 + 2)dx + (2x^2 + 2xy + y^2)dy \]
  for each \( (a, b) \) where \( C \) is a curve from \( (0, 0) \) to \( (a, b) \). We choose \( C \) to be a straight line parameterized as \( x = at \) and \( y = bt \) for \( 0 \leq t \leq 1 \). Then the integral becomes
  \[ \int_0^1 \left[ (4abt^2 + b^2t^2 + 2)a + (2a^2t^2 + 2abt^2 + b^2t^2)b \right] dt \]
  which is
  \[ \left[ 2a^2bt^3 + ab^2t^3 + \frac{1}{3}b^3t^3 + 2at \right]_0^1 = 2a^2b + ab^2 + \frac{1}{3}b^3 + 2a. \]
  Thus,
  \[ f(x, y) = 2x^2y + xy^2 + 2x + \frac{1}{3}y^3 + c. \]

Problem 5.2. Let \( \mathbf{F} = xyi + xyj \).
• a) Show that $\mathbf{F}$ is not a gradient field by taking the appropriate derivatives.
• b) Try to find a potential function for $\mathbf{F}$ by the integration method. What goes wrong?
• c) Same question for the algebraic method.

**Solution.** $\mathbf{F}$ is not a gradient field.

• a) If $\mathbf{F}$ is a gradient field, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where $M = xy$ and $N = xy$. But $\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = y$, so $\mathbf{F}$ is not a gradient field.

• b) As in Part II #1, let $f(0,0) = C$. Then

$$f(a,b) = f(0,0) + \int_{(0,0)}^{(a,b)} xy \, dx + xy \, dy.$$ 

Path 1: Straight line between $(0,0)$ and $(a,b)$: $x = at$ and $y = bt$.

$$f(a,b) = f(0,0) + \int_{0}^{1} at(bt) \, adt + at(bt) \, bdt = \frac{1}{3} (a^2 b + ab^2) + C.$$ 

Path 2: First segment to $(0,a)$, second segment to $(a,b)$.

$$f(a,b) = f(0,0) + \int_{0}^{a} xy|_{y=0} \, dx + \int_{0}^{b} xy|_{x=a} \, dy = ab.$$ 

But if $\mathbf{F}$ were a gradient field then $\int_{C} \mathbf{F} \cdot d\mathbf{r}$ should be path-independent. Since we get different answers for two different paths, $\mathbf{F}$ cannot be a gradient field.

• c) 

$$f_x = xy \Rightarrow f = \frac{1}{2} x^2 y + g(y).$$

$$f_y = xy \Rightarrow f = \frac{1}{2} xy^2 + h(x).$$

But it is impossible to find a $g(y)$ that is only a function of $y$ and an $h(x)$ that is only a function of $x$ that make these two $f$'s equivalent.

□

**Problem 5.3.** Let $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$.

• a) Calculate the curl of $\mathbf{F}$.
• b) For each $n$ for which $\text{curl} \mathbf{F} = 0$, find a potential $g$ such that $\mathbf{F} = \nabla g$. (Hint: seek a potential $g = g(r)$. Watch out for a certain negative $n$ value for which the formula is different.)

**Solution.** $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$.

• a) 

$$\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j});$$

$$\text{curl} \mathbf{F} = \frac{\partial (y r^n)}{\partial x} - \frac{\partial (x r^n)}{\partial y} = nr^{n-1} \frac{x}{r} - n xr^{n-1} \frac{y}{r} = 0.$$
\begin{itemize}
  \item b) If \( g = g(r) \), then \( g_x = g'(r) \frac{2}{r} \) and \( g_y = g'(r) \frac{2}{r} \) (by the chain rule), so

  \[ \nabla g = \frac{g'(r)}{r} (x \mathbf{i} + y \mathbf{j}) \]

  We must find \( g \) such that \( \frac{g'(r)}{r} = r^n \), i.e. \( g'(r) = r^{n+1} \). Two cases: \( n \neq -2 \):

  \[ g(r) = \frac{1}{n+2} r^{n+2} \] \( \text{or} \)

  \[ g(r) = \ln(r) \]

  \( \square \)

\end{itemize}

\textbf{Problem 5.4.} Consider \( \mathbf{F} = \nabla (x^2y + xy^2) \). Let \( C \) be the semicircle having its midpoint at the origin and running from \((-1,1)\) to \((1,1)\).

\begin{itemize}
  \item a) Write the integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) in the \( \int_C M \, dx + N \, dy \) form.
  \item b) Evaluate the integral in two different but easy ways: using i) the Fundamental Theorem, ii) path-independence. Show the calculations in each case.
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{curve_C.png}
\caption{The curve \( C \)}
\end{figure}

\begin{itemize}
  \item a) \( \mathbf{F} = \nabla (x^2y + xy^2) = (2xy + y^2, x^2 + 2xy) \)

  \[ L: x = t, \, dx = dt, \, y = 1, \, dy = 0. \]

  \[ \int_C (2xy + y^2) \, dx + (x^2 + 2xy) \, dy \]

  \item b) By Fundamental Theorem of Calculus for Line Integrals:

  \[ x^2y + xy^2 \bigg|_{(-1,1)}^{(1,1)} = 2 - (1 - 1) = 2. \]

  Using path independence:

  \[ \int_L \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} (2x + 1) \, dx = x^2 + x \bigg|_{-1}^{1} = 2 - (1 - 1) = 2. \]

  \( \square \)
Problem 5.5.  • a) For what simple closed (positively oriented) curve $C$ in the plane does the line integral
\[
\oint_C (x^2 y + y^3 - y)dx + (3x + 2y^2 x + e^y)dy
\]
have the largest positive value? (Hint: use Green’s Theorem).
  • b) What is this maximum value?

Solution. We use Green’s Theorem.
  • a)
\[
\oint_C (x^2 y + y^3 - y)dx + (3x + 2y^2 x + e^y)dy = \iint_R (3 + 2y^2) - (x^2 + 3y^2 - 1) dA = \iint_R (4 - x^2 - y^2) dA,
\]
where $R$ is the region enclosed by $C$. The integrand is positive inside the circle $x^2 + y^2 = 4$ and negative outside, so the integral is biggest when $C$ is the circle $x^2 + y^2 = 4$.
  • b)
\[
\iint_{x^2 + y^2 < 4} (4 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^2 (r - 4^2) r dr d\theta = 2\pi \left[2r^2 - \frac{1}{4} r^4\right]_0^2 = 8\pi.
\]
\[
\square
\]