Bush 18.02A pset 4, part II solutions, fall 2006

Problem 1

Method 1, substitution and brute force: This will work, but is more painful and error-prone than method 2.

Method 2, chain rule and product rule (applied all over the place):

$f(x, y) = \mathrm{e}^x \cos y.$

$\Rightarrow \dfrac{df}{dt} = \dfrac{d}{dt}(\mathrm{e}^x \cos y) = \mathrm{e}^x \cos y \dfrac{dx}{dt} - \mathrm{e}^x \sin y \dfrac{dy}{dt}. \Rightarrow$

$$\begin{aligned}
\frac{d^2 f}{dt^2} &= \frac{d}{dt}(\mathrm{e}^x \cos y \frac{dx}{dt} - \mathrm{e}^x \sin y \frac{dy}{dt}) \\
&= \mathrm{e}^x \cos y \left(\frac{dx}{dt}\right)^2 - \mathrm{e}^x \sin y \frac{dy}{dt}\frac{dx}{dt} + \mathrm{e}^x \cos y \frac{d^2x}{dt^2} - \mathrm{e}^x \sin y \frac{dx}{dt}\frac{dy}{dt} - \mathrm{e}^x \cos y \left(\frac{dy}{dt}\right)^2 - \mathrm{e}^x \sin y \frac{d^2y}{dt^2} \\
&= \mathrm{e}^x \cos y\,(4t^2) - \mathrm{e}^x \sin y\,(2t)(-3t^2) + \mathrm{e}^x \cos y(2) - \mathrm{e}^x \sin y\,(2t)(-3t^2) - \mathrm{e}^x \cos y\,(9t^4) - \mathrm{e}^x \sin y\,(-6t) \\
&= \mathrm{e}^x \cos y\,(-9t^4 + 4t^2 + 2) + \mathrm{e}^x \sin y\,(12t^3 + 6t).
\end{aligned}$$

Problem 2

$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} = \dfrac{\partial u}{\partial r} + \dfrac{\partial u}{\partial s}.$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s\,\partial r}\frac{\partial s}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial r\,\partial s}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial x}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s\,\partial r}\right) + \left(\frac{\partial^2 u}{\partial r\,\partial s} + \frac{\partial^2 u}{\partial s^2}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s\,\partial r} + \frac{\partial^2 u}{\partial s^2}.
\end{aligned}$$

$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial t} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial t} = \dfrac{\partial u}{\partial r}(-c) + \dfrac{\partial u}{\partial s}(c).$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 u}{\partial t^2} &= -c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial s}\right) \\
&= -c\left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial s\,\partial r}\frac{\partial s}{\partial t}\right) + c\left(\frac{\partial^2 u}{\partial r\,\partial s}\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial t}\right) \\
&= -c\left(\frac{\partial^2 u}{\partial r^2}(-c) + \frac{\partial^2 u}{\partial s\,\partial r}(c)\right) + c\left(\frac{\partial^2 u}{\partial r\,\partial s}(-c) + \frac{\partial^2 u}{\partial s^2}(c)\right) \\
&= c^2\frac{\partial^2 u}{\partial r^2} - 2c^2\frac{\partial^2 u}{\partial s\,\partial r} + c^2\frac{\partial^2 u}{\partial s^2}.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 u}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s\,\partial r} + \frac{\partial^2 u}{\partial s^2} - \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s\,\partial r} - \frac{\partial^2 u}{\partial s^2} \\
&= 4\frac{\partial^2 u}{\partial s\,\partial r}. \text{ QED}
\end{aligned}$$

(continued)

Problem 3

a) i) $x = r\cos\theta \;\Rightarrow\; w = (r^2 - r^2\cos^2\theta)^{1/2} = (r^2\sin^2\theta)^{1/2} = r\sin\theta \;\Rightarrow\; \boxed{\left(\frac{\partial w}{\partial r}\right)_\theta = \sin\theta.}$

ii) First, $w = (r^2 - x^2)^{1/2} \;\Rightarrow\; \left(\frac{\partial w}{\partial r}\right)_\theta = \frac{1}{2}(r^2 - x^2)^{-1/2}\left(2r - 2x\left(\frac{\partial x}{\partial r}\right)_\theta\right).$

Second, $x = r\cos\theta \;\Rightarrow\; \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta.$

$\Rightarrow \boxed{\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{1}{2}(r^2 - x^2)^{-1/2}(2r - 2x\cos\theta).}$

Writing in terms of r and θ:

$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{1}{2}\frac{1}{r\sin\theta}(2r)(1-\cos^2\theta) = \sin\theta$ (which agrees with part (i)).

b) Intermediate variables: x, r, independent variables x, θ.

We want to show $\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{w_x}{r_x} + w_r.$

$$\left(\frac{\partial w}{\partial r}\right)_\theta = w_x\left(\frac{\partial x}{\partial r}\right)_\theta + w_r\left(\frac{\partial r}{\partial r}\right)_\theta = w_x\left(\frac{\partial x}{\partial r}\right)_\theta + w_r. \qquad (\star)$$

To find $\left(\frac{\partial x}{\partial r}\right)_\theta$ we differentiate $r = r(x,\theta)$ implicitly:

$\left(\frac{\partial r}{\partial r}\right)_\theta = r_x\left(\frac{\partial x}{\partial r}\right)_\theta + r_\theta\left(\frac{\partial \theta}{\partial r}\right)_\theta.$

We know $\left(\frac{\partial r}{\partial r}\right)_\theta = 1$ and $\left(\frac{\partial \theta}{\partial r}\right)_\theta = 0. \;\Rightarrow\; 1 = r_x\left(\frac{\partial x}{\partial r}\right)_\theta \;\Rightarrow\; \left(\frac{\partial x}{\partial r}\right)_\theta = \frac{1}{r_x}.$

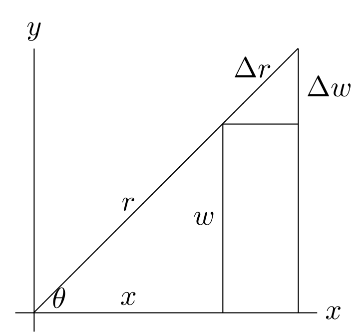
Substitute this in $(\star)$ $\Rightarrow \left(\frac{\partial w}{\partial r}\right)_\theta = w_x\frac{1}{r_x} + w_r.$ QED

Applying this to the problem: $w_x = -\frac{x}{(r^2-x^2)^{1/2}},\; w_r = \frac{r}{(r^2-x^2)^{1/2}},\; r = \frac{x}{cos\theta} \;\Rightarrow\; r_x = \frac{1}{cos\theta}.$

$\Rightarrow \left(\frac{\partial w}{\partial r}\right)_\theta = -\frac{x\cos\theta}{(r^2-x^2)^{1/2}} + \frac{r}{(r^2-x^2)^{1/2}} = -\frac{r\cos^2\theta}{r\sin\theta} + \frac{r}{r\sin\theta} = \frac{r\sin^2\theta}{r\sin\theta} = \sin\theta.$ (Same as above.)

c) We know $w = y$.

The picture shows that (holding θ constant) $\Delta w/\Delta r = \sin\theta$.

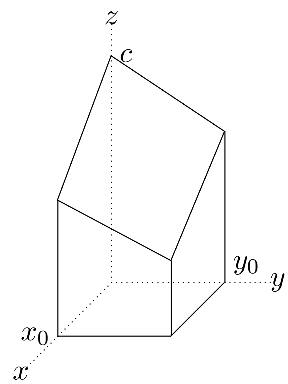


(continued)

Problem 4

Equation of top plane: $z = ax + by + c$.

Intercepts x_0, y_0, c.



Vol $= \iint_R z\,dA = \int_0^{x_0}\int_0^{y_0} ax + by + c\;dy\,dx$.

$$\text{Inner: } axy + b\frac{y^2}{2} + cy\Big|_0^{y_0} = ay_0x + \frac{by_0^2}{2} + cy_0.$$

$$\text{Outer: } \int_0^{x_0} ay_0x + \frac{by_0^2}{2} + cy_0\,dx = \frac{ax_0^2y_0}{2} + \frac{bx_0y_0^2}{2} + cx_0y_0 = x_0y_0\left(\frac{ax_0}{2} + \frac{by_0}{2} + c\right).$$

Length of four legs: c, $ax_0 + c$, $by_0 + c$, $ax_0 + by_0 + c$

$\Rightarrow$ average $= c + ax_0/2 + by_0/2$.

$\Rightarrow$ volume = (area base)$\cdot$ (average of legs). QED

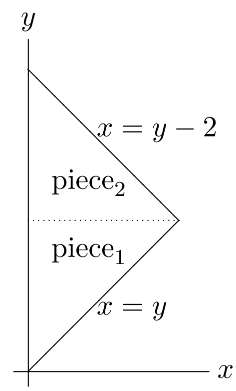
Problem 5

The original integral is over the triangular region shown.

Changing the order of integration we need to break it into two pieces.

Piece 1: $y : 0$ to 1; For fixed y, $x : 0$ to y.

Piece 2: $y : 1$ to 2; For fixed y, $x : 0$ to $2 - y$.



$\Rightarrow$ integral $= \int_0^1\int_0^y \frac{x}{y}\,dx\,dy + \int_1^2\int_0^{2-y}\frac{x}{y}\,dx\,dy$.

$$\text{Inner}_1 = \frac{x^2}{2y}\Big|_0^y = \frac{y}{2}.$$

$$\text{Inner}_2 = \frac{x^2}{2y}\Big|_0^{2-y} = \frac{(2-y)^2}{2y} = \frac{2}{y} - 2 + \frac{y}{2}.$$

$$\text{Outer}_1 = \frac{y^2}{4}\Big|_0^1 = \frac{1}{4}.$$

$$\text{Outer}_2 = 2\ln y - 2y + \frac{y^2}{4}\Big|_1^2 = 2\ln 2 - 4 + 1 + 2 - \frac{1}{4}.$$

Integral = outer$_1$ + outer$_2$ = $\boxed{2\ln 2 - 1.}$

You could do this double integral in the order given, but it involves more difficult integrals.