Problem 1
a) First we work in the $xy$-plane and then transfer our answer to 3 dimensions. The direction in the $xy$-plane of maximum increase of $z$ is in the direction of $\nabla z = (-4x, -6y) \Rightarrow \nabla z|_{(1,1)} = (-4, -6)$.

$(\text{in the } xy\text{-plane, the fastest increase is in direction of } (-4, -6)).$

If the projection of her path in the $xy$-plane has tangent vector $a i + b j$ then the tangent vector of her path on the mountain is $a i + b j + (\nabla z \cdot (a, b)) k$.

$(\text{along the mountain the fastest increase is in direction of } -4i - 6j + 52k.$

b) We need to show that the tangent at each point along the graph of $y = x^{3/2}$ is parallel to $\nabla z$.

Point on graph: $P = (x, x^{3/2})$.

Tangent vector to $y = x^{3/2}$ at $P$ is $v = (1, \frac{3}{2}x^{1/2})$.

$\nabla z|_P = (-4x, -6x^{3/2}) = -4xv$. So they are parallel. QED

(The 3d picture at right shows the path along the mountain.

Problem 2
a) First note that geometrically $F$ is always tangent to circles centered at the origin. That is, it is always perpendicular to lines from the origin.

We are going to use an indirect argument: we will assume $F$ is a gradient and show this leads to a contradiction.

Suppose $F = \nabla f$ for some function $f$ \Rightarrow $f_x = -y$, $f_y = x$.

$\nabla f$ is perpendicular to level curves \Rightarrow the lines $y = ax$ are level curves (follows from our geometric note).

\Rightarrow $f(x, y) = g(y/x)$ for some function $g(u)$.

\Rightarrow $f_x = -\frac{y}{x^2} g'(\frac{y}{x}) = -y$

\Rightarrow $g'(y/x) = x^2$.

This is impossible (e.g. $g'(2/1) = g'(4/2)$ but $1^2 \neq 2^2$).

So our supposition $F = \nabla f$ must be false. QED

b) If $F = \nabla f$ then $f_x = -y$ and $f_y = x$. \Rightarrow $f_{xy} = -1$ and $f_{yx} = 1$. This is impossible since $f_{xy} = f_{yx}$. \Rightarrow $F$ is not the gradient of any function.

(continued)
Problem 3

a) Line 1: \((1,0,1) + t(0,1,-1) \Rightarrow x = 1, y = t, z = 1 - t.\)
Line 2: \((0,1,0) + u(1,0,1) \Rightarrow x = u, y = 1, z = u.\)

b) \(A = (1, t, 1 - t), \quad B = (u, 1, u)\)
\(\Rightarrow w(t, u) = |AB|^2 = (u - 1)^2 + (1 - t)^2 + (u + t - 1)^2.\)
\(\Rightarrow \frac{\partial w}{\partial t} = -2(1 - t) + 2(u + t - 1) = 4t + 2u - 4,\)
\(\frac{\partial w}{\partial u} = 2(u - 1) + 2(u + t - 1) = 4u + 2t - 4.\)

Critical point when \(\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u} = 0 \Rightarrow \begin{cases} 4t + 2u - 4 = 0 \\ 2t + 4u - 4 = 0 \end{cases}.\)
Solving by any method you like \(\Rightarrow u = 2/3, t = 2/3.\)
\(\Rightarrow A_0 = (1,2/3,1/3), \quad B_0 = (2/3,1,2/3), \quad |A_0B_0| = 1/\sqrt{3}.\)

c) \(w_{tt} = 4, \quad w_{uu} = 4, \quad w_{ut} = 2 \Rightarrow D = w_{tt}w_{uu} - w_{ut}^2 = 12 > 0 \Rightarrow \text{max or min.}\)
Finally \(w_{tt} > 0 \Rightarrow \text{minimum.}\)

Problem 4

\(f(x,y) = x^2 - 2xy + 7y^2\) (function).
\(g(x,t) = x^2 + 4y^2 = 1\) (constraint).

Lagrange: \(\nabla f = \lambda \nabla g\)
\(\Rightarrow \begin{cases} 2x - 2y = \lambda 2x \\ -2x + 14y = \lambda 8y \\ x^2 + 4y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x - y = \lambda x \\ -x + 7y = 4\lambda y \\ x^2 + 4y^2 = 1 \end{cases}.\)

There are several methods of solving these equations. (Two are given below)
They all lead to:
\((x,y) = (1/\sqrt{5},-1/\sqrt{5})\) or \((-1/\sqrt{5},1/\sqrt{5})\), \(f(x,y) = 2, \text{ maximum.}\)
\((x,y) = (2/\sqrt{5},1/2\sqrt{5})\) or \((-2/\sqrt{5},-1/2\sqrt{5})\), \(f(x,y) = 3/4, \text{ minimum.}\)

Method 1: Solve symmetrically: Take the two equations with \(\lambda\) and multiply to make the left hand sides the same
\(\Rightarrow 4xy - 4y^2 = 4\lambda xy\)
\(-x^2 + 7xy = 4\lambda xy\)
\(\Rightarrow 4xy - 4y^2 = -x^2 + 7xy \Rightarrow 4y^2 = x^2 - 3xy.\)
The constraint equation can be written as \(4y^2 = 1 - x^2,\) combining this with the equation just above gives \(x^2 - 3xy = 1 - x^2 \Rightarrow y = \frac{2x^2 - 1}{3x^2}.\)

Substitute in the constraint equation \(\Rightarrow x^2 + 4\left(\frac{2x^2 - 1}{3x^2}\right)^2 = 1\)
\(\Rightarrow 9x^4 + 16x^4 - 16x^2 + 4 = 9x^2 \Rightarrow 25x^4 - 25x^2 + 4 = 0 \Rightarrow (5x^2 - 4)(5x^2 - 1) = 0\)
\(\Rightarrow x = \pm 2/\sqrt{5}, \quad \pm 1/\sqrt{5} \ldots\)

(continued)
Method 2: This uses the matrix methods we learned earlier.

The equations with $\lambda$ can be rewritten as

\[
\begin{align*}
(x - \lambda) - y &= 0 \\
-x + (7 - \lambda)y &= 0
\end{align*}
\]

This has non-zero solutions only when

\[
\begin{vmatrix}
x - \lambda & -1 \\
-1 & 7 - 4\lambda
\end{vmatrix} = 0
\]

\[4\lambda^2 - 11\lambda + 6 = 0 \Rightarrow (4\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2 \text{ or } 3/4.
\]

$\lambda = 2 \Rightarrow x = -y$. So, constraint $\Rightarrow x = \pm 1/\sqrt{5}$…

$\lambda = 3/4 \Rightarrow x = 4y$. So, constraint $\Rightarrow x = \pm 2/\sqrt{5}$…

We provide two graphical views of the problem. In the first we show the constraint as a dotted line which touches the level curves tangentially at the critical points. The second shows the graph of $f$ and the corresponding graph of the constraint curve.