Bush 18.02A pset 2, part II solutions, fall 2006

Problem 1

a) The paremeter t is in units of time.

Usual unit circle is $(x, y) = (\cos t, \sin t)$ (counterclockwise at unit speed, starting at $(1, 0)$).

Reverse direction: $(x, y) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t)$.

Shift start to $(-1, 0)$: $(x, y) = (\cos(t + \pi), -\sin(t + \pi) = (-\cos t, \sin t)$.

$\Rightarrow$ position vector = $\boxed{\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}.}$

b) Circular CCW at constant speed $\Rightarrow$ $(x, y) = 10(\cos\omega t, \sin\omega t)$.

Speed $= \sqrt{(x')^2 + (y')^2} = 10\omega = 60 \Rightarrow \omega = 6. \Rightarrow$ $\boxed{\mathbf{r}(t) = 10\cos(6t)\,\mathbf{i} + 10\cos(6t)\,\mathbf{j}.}$

c) RPM is revolutions (or cycles) per minute.

60 rpm $\Leftrightarrow$ 120π radians/minute $\Rightarrow$ $\boxed{\mathbf{r}(t) = 10\cos(120\pi\,t)\,\mathbf{i} + 10\sin(120\pi\,t)\,\mathbf{j}).}$

d) Because they are so easy, we don't show all the algebra for the integrals. The important thing is to remember the lower limit of 0.

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \mathbf{a}(t)\,dt = -\mathbf{i} + \sin(t)\,\mathbf{i} + (\cos t - 1)\,\mathbf{j} + t\,\mathbf{k} = (\sin t - 1)\,\mathbf{i} + (\cos t - 1)\,\mathbf{j} + t\,\mathbf{k}.$$

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t)\,dt = \mathbf{j} + (-\cos t + 1 - t)\,\mathbf{i} + (\sin t - t)\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}.$$

$$\Rightarrow \boxed{\mathbf{r}(t) = (1 - \cos t - t)\,\mathbf{i} + (1 + \sin t - t)\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}.}$$

Problem 2

a) Position of jet $= \mathbf{r}(t) = (1, 1, 0) + t\,(-5, 0, 1) = (1 - 5t, 1, t)$.

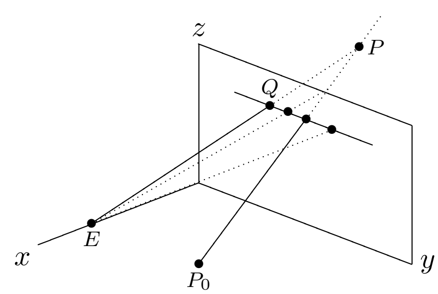
When the jet is at P the eye at point E will see it at Q on the yz-plane.

$\Rightarrow$ Q = intersection of the line $\overrightarrow{\mathbf{EP}}$ with the yz-plane.

Line $\overrightarrow{\mathbf{EP}}$ is parameterized by

$$(x, y, z) = E + u(P - E) = (1, 0, 0) + u(-5t, 1, t).$$

Q = point on $\overrightarrow{\mathbf{EP}}$ with $x = 0 \Rightarrow 1 - 5u\,t = 0 \Rightarrow u = 1/5t$.



$\Rightarrow$ $\boxed{Q = (0, 1/5t, 1/5) \quad \text{or} \quad y = 1/5t,\ z = 1/5.}$

b) Point on screen is $\mathbf{r}(t) = \dfrac{1}{5t}\,\mathbf{j} + \dfrac{1}{5}\,\mathbf{k}. \Rightarrow \mathbf{v}(t) = \dfrac{1}{5t^2}\,\mathbf{j}.$

The velocity always points in same direction (along $\mathbf{j}$) implies the trajectory is along a line. QED

From part (a): $\boxed{\text{As } t \to \infty \quad \text{we have} \quad (y, z) \to (0, 1/5).}$

(continued)

c) We do this problem two ways, geometrically and algebraically.
Let $\mathbf{v}$ be the common direction vector for parallel trajectories.
Any line with direction vector $\mathbf{v}$ is of the form $P_0 + t\,\mathbf{v}$.

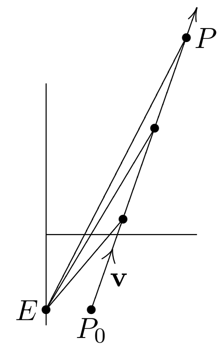


Figure 1

Geometrically: The (two dimensional) figure 1 shows that as $t \to \infty$ the line $\overrightarrow{\mathbf{EP}}$ becomes parallel to $\mathbf{v}$, i.e. it heads towards the line $E + u\,\mathbf{v}$.
$\Rightarrow$ the image on the screen heads towards the point Q_0 where the line $E + u\,\mathbf{v}$ intersects the yz-plane. The figure in part (a) shows that the image on the screen is a straight line. Therefore the screen image of three parallel trajectories looks like figure 2.

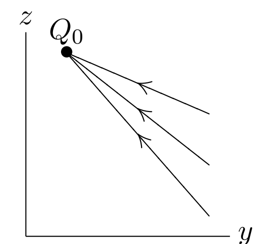


Figure 2

Algebraically: The geometric version has a certain hand-waving quality to it. We can confirm the geometric picture convincingly with symbols.
Take an arbitrary parametrized line $P = (a + \alpha t, b + \beta t, c + \gamma t)$.
As in part (a), to the eye at $E = (1, 0, 0)$ the point P will appear on the screen at the point Q where $\overrightarrow{\mathbf{EP}}$ intersects the yz-plane.

$\overrightarrow{\mathbf{EP}}$ is parametrized by $(x, y, z) = E + u\,(P - E) = (1, 0, 0) + u\,(a - 1 + \alpha t, b + \beta t, c + \gamma t)$.
The point Q is the point on $\overrightarrow{\mathbf{EP}}$ with $x = 0 \;\Rightarrow\; 1 + u(a - 1 + \alpha t) = 0 \;\Rightarrow\; u = \dfrac{1}{1 - a - \alpha t}$

$$\Rightarrow \boxed{y = \frac{b + \beta t}{1 - a - \alpha t},\ z = \frac{c + \gamma t}{1 - a - \alpha t}.}$$

We have to show (y, z) (1) lies along a line and (2) tends to a limit as $t \to \infty$.
(1) We compute velocity. After the quotient rule and some algebra we have

$$(y', z') = \left(\frac{\beta(1 - a) + b\alpha}{(1 - a - \alpha t)^2}, \frac{\gamma(1 - a) + c\alpha}{(1 - a - \alpha t)^2} \right) = \frac{1}{(1 - a - \alpha t)^2} \left(\beta(1 - a) + b\alpha,\ \gamma(1 - a) + c\alpha \right).$$

$\Rightarrow$ the velocity always points in same direction (along $(\beta(1 - a) + b\alpha,\ \gamma(1 - a) + c\alpha)$) $\Rightarrow$ the trajectory on screen is along a line.
(2) From the boxed formula for (x, y) we have $\boxed{\lim_{t \to \infty} (y, z) = (-\beta/\alpha, -\gamma/\alpha).}$

(It's easy to check this is the intersection of $E + t\,\mathbf{v}$ and the yz-plane.)
So the algebra confirms our geometric intuition. What a relief!

(continued)

Problem 3

a) The key is to break $\mathbf{r} = \overrightarrow{\mathbf{OP}}$ into simpler vectors: $\overrightarrow{\mathbf{OP}} = \overrightarrow{\mathbf{OT}} + \overrightarrow{\mathbf{TP}}$.

Easily, $\overrightarrow{\mathbf{OT}} = a(\cos\theta, \sin\theta)$.

$\overrightarrow{\mathbf{TP}}$ has direction $(\sin\theta, -\cos\theta)$ and length $a\theta$ (i.e. perpendicular to $\overrightarrow{\mathbf{OT}}$ with the same length as the unwound thread) $\Rightarrow$ $\overrightarrow{\mathbf{TP}} = a\theta\,(\sin\theta, -\cos\theta)$

$\Rightarrow$ $\boxed{\mathbf{r} = (x, y) = (a\cos\theta + a\theta\,\sin\theta,\ a\sin\theta - a\theta\,\cos\theta).}$

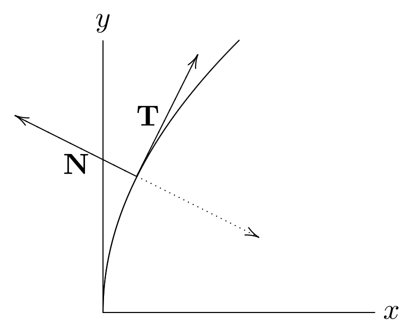
b) $\dfrac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\,\cos\theta = a\theta\,\cos\theta$ and $\dfrac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\,\sin\theta = a\theta\,\sin\theta$.

$\Rightarrow$ $\boxed{\dfrac{ds}{d\theta} = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} = a\theta.}$ For the true speed we need $\boxed{\dfrac{ds}{dt} = \dfrac{ds}{d\theta}\dfrac{d\theta}{dt} = a\theta\,\dfrac{d\theta}{dt}.}$

Problem 4

$\mathbf{r}(t) = (at^2, 2at) \Rightarrow \mathbf{v}(t) = (2at, 2a) \Rightarrow \dfrac{ds}{dt} = |\mathbf{v}| = 2a\sqrt{1+t^2}$.

$\Rightarrow$ $\boxed{\text{Unit tangent} = \mathbf{T} = \left(\dfrac{t}{\sqrt{1+t^2}}, \dfrac{1}{\sqrt{1+t^2}}\right).}$



For the book, $\mathbf{N}$ is found by rotating $\mathbf{T}$ $\pi/2$ radians counterclockwise.

$\Rightarrow$ $\boxed{\mathbf{N} = \left(-\dfrac{1}{\sqrt{1+t^2}}, \dfrac{t}{\sqrt{1+t^2}}\right).}$

(This is shown as the solid normal vector in the picture.)

It is more standard to define κ as positive and have $\mathbf{N}$ point to the concave side of the curve. With these definitions: $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, where $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

This leads to several nice formulas using velocity and acceleration.

1) $\kappa = \dfrac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3}$.

2) $\mathbf{v}\times(\mathbf{a}\times\mathbf{v}) = \kappa\,|v|^4\,\mathbf{N}$. ($\Rightarrow$ $\mathbf{N}$ is the direction vector of $\mathbf{v}\times(\mathbf{a}\times\mathbf{v})$.)

Using formula (2) we get

$\mathbf{v}\times(\mathbf{a}\times\mathbf{v}) = (2at\,\mathbf{i} + 2a\,\mathbf{j})\times(2a\,\mathbf{i}\times(2at\,\mathbf{i} + 2a\,\mathbf{j})) = (2at\,\mathbf{i} + 2a\,\mathbf{j})\times 4a^2\,\mathbf{k} = 8a^3(\mathbf{i} - t\,\mathbf{j})$.

$\Rightarrow$ $\mathbf{N} = \frac{1}{1+t^2}(\mathbf{i} - t\,\mathbf{j})$. (This is shown as the dotted normal vector in the picture.)