18.02a Practice Midterm Questions, Fall 2006

Problems 1-5 cover material from the first unit. This will not be emphasized on the midterm. It will only appear in the sense that the later material builds on it.

Problems 1-5 take about 1-1.5 hours, problems 6-19 take 2-3 hours.
The actual test will be shorter –designed to take 2 hours, with simpler arithmetic.

Problem 1. Consider the point $P = (20, 0, 0)$, the plane $\mathcal{P} : x + 2y + 3z = 6$, and the point $Q = (1, 1, 1)$ on $\mathcal{P}$.
a) Compute the distance from P to $\mathcal{P}$.

b) Give parametric equations for the line through P and perpendicular to $\mathcal{P}$.

c) Find the point of intersection between $\mathcal{P}$ and the line of part(b). For later reference, call this point R.

d) Find the angle, $\angle PQR$.

e) By computing $|PR|$ directly, verify your answer to part (a).

f) Find the area of the triangle with vertices P, Q and R.

Problem 2. Suppose tape is unwound from a roll in such a way that it is always vertical. Assuming the roll is centered at the origin and has radius 2, and the end of the tape starts at the point $(2, 0)$, give parametric equations for the path traced out by the end of the roll. For what values of your parameter does this make sense?

Problem 3. The motion of a point P is given parametrically by
$\overrightarrow{\mathbf{OP}} = \mathbf{r}(t) = \langle 4\sin t,\, 5\cos t,\, 3\sin t\rangle$.

a) Find $\mathbf{v}$, $\frac{ds}{dt}$, $\mathbf{T}$, κ.

b) Show $\mathbf{r}$ is perpendicular to $3\mathbf{i} - 4\mathbf{k}$. What information about the motion of the point P does this give?

Problem 4. Let $(\sin t, \cos 2t)$ be a parametrized path of a point P in the plane. Give the x-y equation of this path. Sketch and describe how P moves over time.

Problem 5. Let $A_c = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & c \end{pmatrix}$.

a) Let $B = \frac{1}{4}\begin{pmatrix} 3 & 1 & -2 \\ -5 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}$.

Show B is the inverse of A_2 (the subscript indicates $c = 2$). Show your arithmetic carefully.

b) Use part (a) to solve $x + z = 1$, $3x + 2y + z = 0$, $x + y + 2z = 4$.

c) For what c will the system of equations $A_c\mathbf{x} = \mathbf{0}$ have a non-zero solution?

d) For the value of c found in part (c) find a non-zero solution to the system.

e) Compute A_1^{-1}.

(continued)

Problem 6.
a) Find the normal to the level surface $x^3 + y^3 z = 3$ at the point $(1, 1, 2)$.

b) Use level surfaces to show the perpendicular to the graph of $z = f(x, y)$ is given by $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \rangle$. (You can use what you know about gradients and level surfaces.)

Problem 7. Graph the surface and level curves of $z = y^2 - x$.

Problem 8. Let $w = f(x, y)$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Along the path given by $\mathbf{r}$ we have $w = f(x(t), y(t))$.
a) Assuming everything is differentiable, show $\frac{dw}{dt} = \boldsymbol{\nabla} w \cdot \frac{d\mathbf{r}}{dt}$.

b) Suppose the path is along a level curve of f. Show that $\boldsymbol{\nabla} w$ is perpendicular to the level curve.

Problem 9. Suppose $w = f(xy)$ (i.e. $w = f(u)$ with $u = xy$). This implies $x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y} = 0$.

a) Verify this for the function $w = \sin(xy)$.

b) Show this in general.

Problem 10. Suppose $w = x^2 + y^2 + z^2$ and x, y, z are related by $x = f(y, z)$.
a) Suppose $f(y, z) = yz$., find $\left(\frac{\partial w}{\partial z}\right)_x$. Do this twice, once with the chain rule and once with differentials. Do not do it by solving for y in terms of z (except to check your answer).

b) For arbitrary f, Find $\left(\frac{\partial w}{\partial z}\right)_x$ in terms of the formal partials of $f(y, z)$. (Again, do this twice, once with the chain rule and once with differentials.)

Problem 11.
a) Starting from the basic chain rule: $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$, derive the matrix equation relating $\left(\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}\right)$ and $\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)$

b) Write down the matrix equation in the case where (x, y) is as usual and $(u, v) = (r, \theta)$ (polar coordinates).

c) Same question for $(u, v) = (x, \theta)$.

Problem 12. Let $w = x^3 y + x/y$ and $P = (2, 1)$.

a) Compute the gradient $\boldsymbol{\nabla} w|_P$.

b) Compute the directional derivative, $\frac{dw}{ds}$ at P in the direction of $\mathbf{i} + 3\mathbf{j}$.

c) Find a direction at P in which w is not changing.

d) Estimate the value of w at $(2.1, 0.9)$.

Problem 13. Give the equation for the tangent plane to the surface $x^2 + xy^2 + yz^2 = 23$ at the point $(1, 2, 3)$.

(continued)

Problem 14. Suppose you have an open box of volume 4 with dimensions x, y, z. So we all use the same notation, assume the open end is one of the sides with dimensions y and z.

a) By substituting for z write down the unconstrained equation for the surface area of the box.

b) Use part (a) to find the dimensions that minimize the area.

c) Use the second derivative test to verify your answer to part (b) is a minimum

d) In part (a) x and y can be anywhere in a region R. Describe R. What is its boundary?

e) Why can't the minimum area occur on the boundary?

f) Redo the minimization using Lagrange multipliers.

Problem 15. Find the critical points of $x^2 - 2xy^2 + 2y^2$. Classify them as minima, maxima or saddle points.

Problem 16. Evaluate by reversing the limits of integration: $\int_0^1 \int_{x^{\frac{1}{3}}}^1 \mathrm{e}^{y^4}\, dy\, dx$.

Problem 17. Let R be a circular region of radius a and uniform density. Set up (**but do not evaluate**) iterated integrals in polar coordinates for the following moments of inertia. You can make things easier by carefully choosing where to put R for each problem.

a) Moment of inertia of R about its center.

b) Moment of inertia of R about a point on its edge.

c) Moment of inertia of R about a diameter.

d) Moment of inertia of R about a line tangent to the circle.

e) Let S be the region in the first quadrant, bounded below by the x-axis, above by $y = x$ and on the right by the circle of radius 1 with center at $(1, 0)$ (NOTE CENTER). Assume uniform density and write down (but don't evaluate) an integral in polar coordinates for the polar moment of inertia (i.e. moment of inertia about the origin) of S.

Problem 18. Consider the volume with square horizontal cross-section, bounded below by the rectangle in the xy-plane with boundaries, $x = 0$, $y = 0$, $x = 1$, $y = 2$ and bounded above by the graph of $z = x^3 + y^3$. Suppose the density of this volume is given by $\delta(x, y, z) = xyz + 1$. Write down, with limits of integration an iterated triple integral for the mass of this volume. DO NOT EVALUATE.

Problem 19. CHANGE OF VARIABLE IS NOT ON MIDTERM (Fall 2006)
Evaluate $\int\int_R (2x - 3y)^2 (x + y)^2 dx\, dy$, where R is the triangle bounded by the positive x-axis, negative y-axis and the line $y = \frac{2}{3}x - \frac{4}{3}$, by making a change of variable $u = x + y$, $v = 2x - 3y$.