18.02A Practice Exam 1

Problem 1 (10) The picture shows a rectangular box, having length 2 and height 1 and width 1. Find the cosine of angle $BOC$.

Problem 2 (10) At what point $(x, y, z)$ does the line given parametrically by

\[ x = 1 + 2t, \quad y = 1 - t, \quad z = 2t \]

intersect the plane through the point $(1, -1, 3)$ and perpendicular to the vector $\mathbf{i} - \mathbf{j} - \mathbf{k}$?

Problem 3 (20) $a = \langle 2, 3, 6 \rangle$ and $b = \langle 6, 2, -3 \rangle$ are two space vectors.

a) Show that $a$ and $b$ are orthogonal.

b) Find the scalar component of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ in the direction of the vector $a$.

c) Find a vector $\mathbf{c}$ having as components three integers with no common factor, such that $a$, $b$, and $c$ form a right-handed system of mutually orthogonal vectors.

d) Using your work in c), find the volume of the parallelepiped spanned by $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{b}$. (This doesn't require the calculation of a determinant!)

Problem 4 (20)

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$. Its matrix of cofactors is (in part) 

\[ C = \begin{pmatrix} -2 & 2 & 4 \\ 3 & -5 & -7 \end{pmatrix} \]

a) (5) Write down the missing middle row of $C$ (be careful about the signs—if in doubt check one of the given entries in $C$):

b) (5) The determinant of $A = -2$. Using this and part (a), find $A^{-1}$.

c) (5) Use the previous parts to solve the system

\[ x + 2y - z = 1, \quad 3x - y + 2z = 2, \quad x + y = 1 \]

d) (5) In the equations in (c), if the coefficient 3 is changed to a certain number $c$, the equations will have no solution. Find $c$, without actually attempting to solve the equations.

Problem 5 (20) Let $\mathbf{r}(t) = \sin 4t \mathbf{i} + \cos 4t \mathbf{j} + 3t \mathbf{k}$.

For the helical motion in space described by the position vector $\mathbf{r}(t)$, where $t =$ time, find

a) the velocity vector $\mathbf{v}$

b) the unit tangent vector $\mathbf{T}$

c) The speed $ds/dt$ and the arclength along the curve between the points where $t = 0$ and $t = \pi$

d) The curvature $k$

e) Show that the curve makes a constant angle with the vertical $k$-direction

Problem 6 (10) A plane motion is given parametrically by $x = \sin t$, $y = \cos 2t$. Write the $x$-$y$ equation giving the path taken by the motion, sketch the path, and describe briefly how the moving point travels along this path.

Problem 7 (10) A roll of Scotch tape having outer radius $a$ is placed so its center is at the origin $O$, and its end $P$ is initially at the point $(a, 0)$ on the $x$-axis. The end of the tape is then pulled upward, so that as it peels from the roll the peeled part is vertical. Using vector methods, give the position vector $OP$ in terms of the parameter $\theta$ (the usual polar angle), for the values $0 \leq \theta \leq \pi$. 