Homological Mirror Symmetry for Fano Surfaces

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(joint work with L. Katzarkov, D. Orlov)
(after ideas of Kontsevich, Seidel, Hori, Vafa, . . .)

DON'T PANIC !

Mirror Symmetry

Complex manifolds:

(X, J) locally $\simeq (\mathbb{C}^n, i)$

Look at complex analytic subvarieties + holom. vector bundles, or better: coherent sheaves (cokernels of morphisms of holom. bundles with finite resolution)

Intersection theory = Morphisms and extensions of sheaves.

Symplectic manifolds:

(Y, ω) locally $\simeq (\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$
(in $\dim_{\mathbb{R}} 2$, any orientable surface!)

Look at Lagrangian submanifolds:
$L^n \subset Y^{2n}$ with $\omega_{|L} = 0$ (locally $\simeq \mathbb{R}^n \subset \mathbb{R}^{2n}$)
(in $\dim_{\mathbb{R}} 2$, all embedded curves!)

Intersection theory = Floer homology
(discard intersections that cancel by Hamiltonian isotopy)

Mirror symmetry:

Duality between type II A and II B string theories.

D-branes = boundary conditions for open strings.

Homological mirror symmetry (Kontsevich, ...):

A-branes	=	Lagrangian submanifolds,
B-branes	=	coherent sheaves.

only in a weaker sense: derived categories.

Homological Mirror Symmetry Conjecture: Calabi-Yau case

Roughly: X, Y Calabi-Yau ($c_1 = 0$) mirror pair $\Rightarrow$

$$\boxed{\begin{array}{lcl} D^bCoh(X) & \simeq & D\mathcal{F}(Y) \\ D\mathcal{F}(X) & \simeq & D^bCoh(Y) \end{array}}$$

$Coh(X)$ = category of coherent sheaves on X complex mfld.

D^b = bounded derived category

Objects = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0.$

$$\text{Morphisms} = \begin{cases} \text{morphisms of complexes} \\ +\text{formal inverses of quasi-isoms} \end{cases}$$

$\mathcal{F}(Y)$ = Fukaya A_∞-category of (Y, ω). Roughly:

Objects = (some) Lagrangian submanifolds (+flat bundles)

Morphisms: $\mathrm{Hom}(L, L') = CF^*(L, L') = \mathbb{C}^{|L \cap L'|}$ if $L \pitchfork L'$.

(Floer complex, graded by Maslov index)

- Differential $d = m_1 : \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_1)[1]$
- Product $m_2 : \mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L_0, L_2)$ (associative up to homotopy)
- Higher products

$$m_k : \mathrm{Hom}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}(L_{k-1}, L_k) \to \mathrm{Hom}(L_0, L_k)[2-k]$$

(related by A_∞-equations)

Fukaya categories

$\mathcal{F}(Y)$ = Fukaya A_∞-category of (Y, ω).

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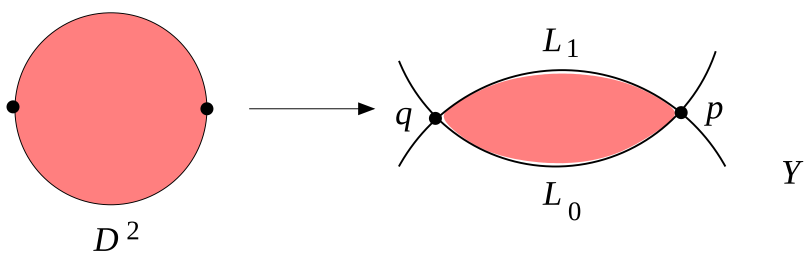
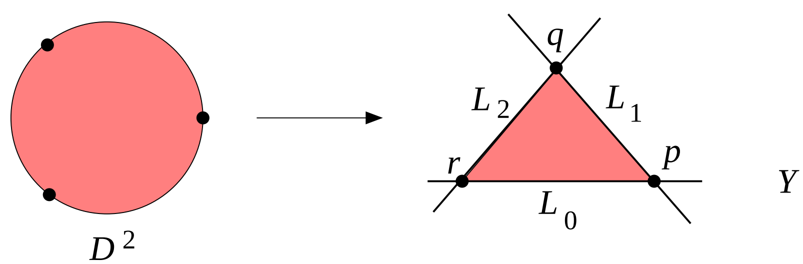
(Floer complex, graded by Maslov index)

- Differential $d = m_1 : \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_1)[1]$
  $\langle m_1(p), q \rangle$ counts pseudo-holomorphic maps
  (in $\dim_{\mathbb{R}} 2$, same as immersed discs with convex corners)



- Product $m_2 : \mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L_0, L_2)$
  $\langle m_2(p, q), r \rangle$ counts pseudo-holomorphic maps



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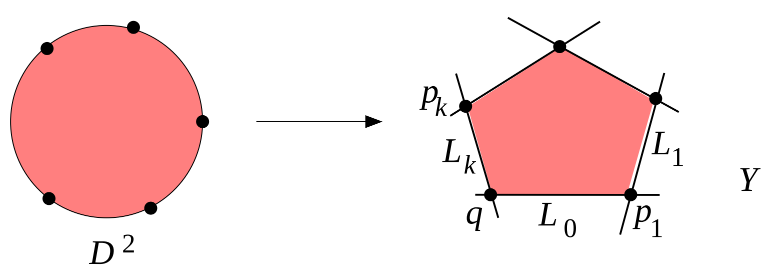


Homological Mirror Symmetry Conjecture: Fano case

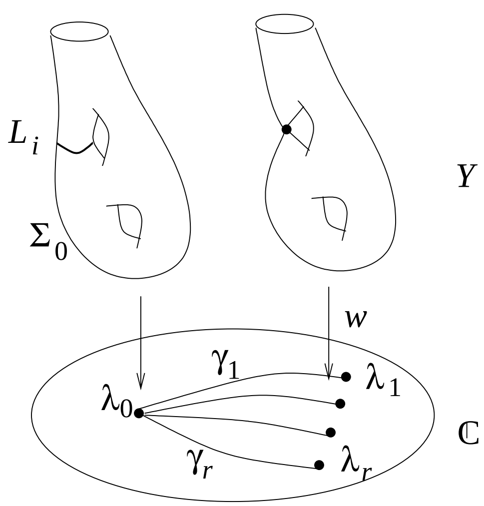
X Fano ($c_1(TX) > 0$) $\overset{M.S.}{\longleftrightarrow}$ "Landau-Ginzburg model"

$$\begin{cases} Y \text{ (non-compact) manifold} \\ w : Y \to \mathbb{C} \text{ "superpotential"} \end{cases}$$

$$\boxed{\begin{array}{lcl} D^bCoh(X) & \simeq & D\mathcal{F}(w) \\ D\mathcal{F}(X) & \simeq & D\,Sing(w) \end{array}}$$

$D\mathcal{F}(w)$ (Lagrangians) and $D\,Sing(w)$ (sheaves) = symplectic and complex geometries of singularities of w.

If $w : Y \to \mathbb{C}$ Lefschetz fibration (isolated non-deg. crit. pts):



$L_i \subset \Sigma_0$ Lagrangian sphere = vanishing cycle associated to γ_i (collapses to crit. pt. by // transport)

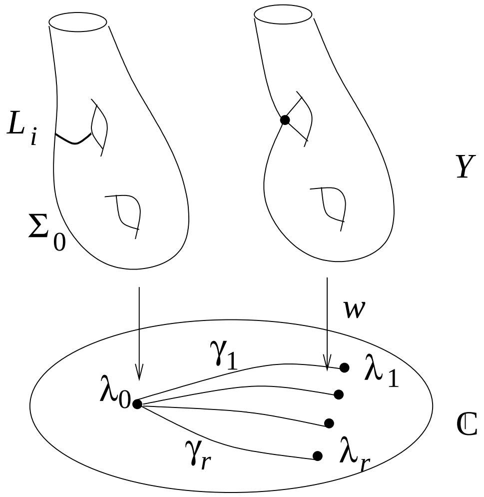
Seidel: $\mathcal{F}(w, \{\gamma_i\})$ finite, directed A_∞-category.

Objects: $L_1, \dots, L_r$.

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) = \mathbb{C}^{|L_i \cap L_j|} & \text{if } i < j \\ \mathbb{C} \cdot \mathrm{Id} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

Products: $(m_k)_{k \geq 1}$ = Floer theory for Lagrangians $\subset \Sigma_0$.

Fukaya-Seidel categories



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Products: $(m_k)_{k \geq 1}$ = Floer theory for Lagrangians $\subset \Sigma_0$.

$$m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$$

– trivial unless $i_0 < \cdots < i_k$
– count discs in Σ_0 w/ boundary in $\bigcup L_i$ (Floer theory)

Remarks:

- $\langle L_1, \ldots, L_r \rangle$ = exceptional collection generating $D\mathcal{F}$.
- objects are also Lefschetz thimbles (discs bounded by L_i)
- in our case, no technical issues such as bubbling etc.
- coefficient ring: $R = \mathbb{C}$, count w/ coef. $\pm \exp(-\int_{D^2} u^*\omega)$

Theorem. (Seidel) *Changing $\{\gamma_i\}$ affects $\mathcal{F}(w, \{\gamma_i\})$ by mutations; $D\mathcal{F}(w)$ depends only on $w : (Y, \omega) \to \mathbb{C}$.*

Example: weighted projective planes

(cf. work of Seidel on $\mathbb{CP}^2$)

$X = \mathbb{CP}^2(a,b,c) = (\mathbb{C}^3 - \{0\})/(x,y,z) \sim (t^a x, t^b y, t^c z)$ (Fano orbifold).

$D^bCoh(X)$ generated by exceptional collection
$O_X, O_X(1), \dots, O_X(N-1)$ $(N = a+b+c)$
(Homogeneous coords. x, y, z are sections of $O(a), O(b), O(c)$)

$\mathrm{Hom}(O(i), O(j)) \simeq$ degree $(j-i)$ part of symmetric algebra $\mathbb{C}[x,y,z]$ (degs. a, b, c)
All in degree 0 (no Ext's); composition = obvious.

Mirror: $Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, $w = x + y + z$.
$(Y \simeq (\mathbb{C}^*)^2$ if $gcd(a,b,c) = 1)$
$\mathbb{Z}/N$ $(N = a+b+c)$ acts by diagonal mult.; complex conjugation.

We choose ω invariant under $\mathbb{Z}/N$ and complex conj.
($\Rightarrow [\omega] = 0$ exact)

Theorem. $D\mathcal{F}(w) \simeq D^bCoh(X)$

(should also work in higher dimensions...)

Non-commutative deformations

$X = \mathbb{CP}^2(a, b, c)$;
$Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, $w = x + y + z$,
ω invariant under $\mathbb{Z}/N$ and complex conj. ($\Rightarrow$ exact):

Theorem. $D\mathcal{F}(w) \simeq D^b Coh(X)$

Can deform $FS(w)$ by changing $[\omega]$ (& introducing a B-field).

Choose $\tau \in \mathbb{C}$, and take $\boxed{\int_{S^1 \times S^1} [\omega + iB] = \tau}$
(keeping $\mathbb{Z}/N$-invariance) ($S^1 \times S^1$ generates $H_2(Y, \mathbb{Z}) \simeq \mathbb{Z}$)
$\longrightarrow$ deformed category $D\mathcal{F}(w)_\tau$.

$\Longleftrightarrow$ non-commutative deformation X_τ of X:
deform polynomial algebra $\mathbb{C}[x, y, z]$ to

$$yz = \mu_1 \, zy, \quad zx = \mu_2 \, xz, \quad xy = \mu_3 \, yx,$$

with $\boxed{\mu_1^a \mu_2^b \mu_3^c = e^{-\tau}}$

Theorem. $\forall \tau \in \mathbb{C}$, $D\mathcal{F}(w)_\tau \simeq D^b Coh(X)_\tau$.

Outline of argument

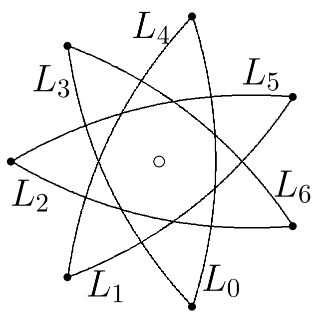
$Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, $w = x + y + z$:

crit $w = \{\lambda \in \mathbb{C},\ \lambda^{a+b+c} = \frac{(a+b+c)^{a+b+c}}{a^a b^b c^c}\} = \{\lambda_j,\ 0 \le j < N\}$

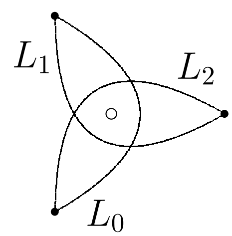
$\lambda_0 \in \mathbb{R}_+$, $\lambda_j = \lambda_0 \exp(\frac{-2\pi i\, j}{a+b+c})$

Reference fiber: $\Sigma_0 = w^{-1}(0)$; arcs γ_j = straight lines.
$\Rightarrow$ vanishing cycles $L_j \subset \Sigma_0$.
If ω is $\mathbb{Z}_N$-invariant, then $L_j = \exp(\frac{-2\pi i\, j}{a+b+c}) \cdot L_0$.

Visualize L_j and intersections via projection $\pi_x : \Sigma_0 \to \mathbb{C}^*$.
($b + c$-fold branched covering, with $a + b + c$ branch points)



$(a, b, c) = (4, 2, 1)$



$(a, b, c) = (1, 1, 1)$

$\Rightarrow$ Description of $\mathcal{F}(w, \{\gamma_j\})$:

- Objects: L_j, $0 \le j < N$.
- $\bigoplus_{i<j} CF^*(L_i, L_j)$ = free module of rank $3N$, generators

$$
\begin{aligned}
x_i &\in CF^*(L_i, L_{i+a}), & \bar{x}_i &\in CF^*(L_i, L_{i+b+c}),\\
y_i &\in CF^*(L_i, L_{i+b}), & \bar{y}_i &\in CF^*(L_i, L_{i+a+c}),\\
z_i &\in CF^*(L_i, L_{i+c}), & \bar{z}_i &\in CF^*(L_i, L_{i+a+b}).
\end{aligned}
$$

Outline of argument

Description of $\mathcal{F}(w, \{\gamma_j\})$:

- Objects: L_j, $0 \le j < N$.
- $\bigoplus_{i<j} CF^*(L_i, L_j)$ = free module of rank $3N$, generators

$$\begin{array}{ll} x_i \in CF^*(L_i, L_{i+a}), & \bar{x}_i \in CF^*(L_i, L_{i+b+c}), \\ y_i \in CF^*(L_i, L_{i+b}), & \bar{y}_i \in CF^*(L_i, L_{i+a+c}), \\ z_i \in CF^*(L_i, L_{i+c}), & \bar{z}_i \in CF^*(L_i, L_{i+a+b}). \end{array}$$

- for suitable graded Lagrangian lifts of L_j,

$$\deg(x_i, y_i, z_i) = 1, \quad \deg(\bar{x}_i, \bar{y}_i, \bar{z}_i) = 2.$$

- $m_k = 0$ for $k \neq 2$.
- only non-zero compositions:

$$\begin{array}{ll} m_2(x_i, y_{i+a}) = \alpha\, \bar{z}_i, & m_2(x_i, z_{i+a}) = \alpha'\, \bar{y}_i, \\ m_2(y_i, z_{i+b}) = \alpha\, \bar{x}_i, & m_2(y_i, x_{i+b}) = \alpha'\, \bar{z}_i, \\ m_2(z_i, x_{i+c}) = \alpha\, \bar{y}_i, & m_2(z_i, y_{i+c}) = \alpha'\, \bar{x}_i. \end{array}$$

If $[\omega] = 0$ then $\alpha = \alpha'$ ($\Rightarrow$ exterior algebra), in general

$$\frac{\alpha}{\alpha'} = \exp\Big(-\frac{1}{a+b+c}\int_{S^1\times S^1} \omega + iB\Big).$$

Then pass to dual exceptional collection by "full mutation" (change $\{\gamma_j\}$ to $\{\gamma'_j\}$ with base point at infinity)

$\Rightarrow$ exterior algebra becomes truncated symmetric algebra.