The symplectic geometry of symmetric products and invariants of 3-manifolds with boundary

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builds on work of: R. Lipshitz, P. Ozsváth, D. Thurston; T. Perutz, Y. Lekili
M. Abouzaid, P. Seidel; S. Ma'u, K. Wehrheim, C. Woodward

Low-dimensional topology

- Goal: find invariants to distinguish smooth manifolds
- Dimensions 3 and 4 hardest (Poincaré conjecture, ...)
- Exotic smooth 4-manifolds (homeomorphic, not diffeomorphic)

Smooth 3- and 4-manifold invariants (beyond algebraic topology)

80's Donaldson invariants
90's Seiberg-Witten invariants
00's Ozsváth-Szabó invariants

↓ increasingly computable and versatile

These all associate *numerical* invariants to closed 4-manifolds, and (graded) *abelian groups* to closed 3-manifolds. But the story goes further!

Heegaard-Floer TQFT

Ozsváth-Szabó (2000)

- Y^3 closed $\rightsquigarrow$ $\widehat{HF}(Y)$ abelian group (Heegaard-Floer homology)
- W^4 cobordism ($\partial W = Y_2 - Y_1$) $\rightsquigarrow$ $\widehat{F}_W : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$
- (and more)

Extend to surfaces and 3-manifolds with boundary?

- Σ surface $\rightsquigarrow$ category $\mathcal{C}(\Sigma)$?
- Y^3 with boundary $\partial Y = \Sigma$ $\rightsquigarrow$ object $C(Y) \in \mathcal{C}(\Sigma)$?
- cobordism $\partial Y = \Sigma_2 - \Sigma_1$ $\rightsquigarrow$ functor $\mathcal{C}(\Sigma_1) \to \mathcal{C}(\Sigma_2)$?
- Want: $\widehat{HF}(Y_1 \cup_\Sigma Y_2) = \hom_{\mathcal{C}(\Sigma)}(C(Y_1), C(Y_2))$ (pairing theorem)

This can be done in 2 equivalent ways: bordered Heegaard-Floer homology (Lipshitz-Ozsváth-Thurston 2008, more computable), or geometry of Lagrangian correspondences (Lekili-Perutz 2010, more conceptual).

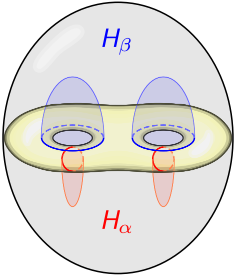
Plan of the talk

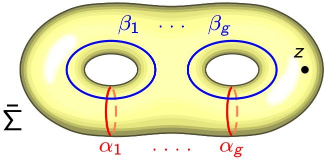
- Heegaard-Floer homology
- Background: Floer homology, Fukaya categories, correspondences
- The Lekili-Perutz approach: correspondences from cobordisms
- The Fukaya category of the symmetric product
- The Lipshitz-Ozsváth-Thurston strands algebra
- Modules and bimodules from bordered 3-manifolds

Heegaard-Floer homology

Y^3 closed 3-manifold admits a *Heegaard splitting* into two handlebodies $Y = H_\alpha \cup_{\bar{\Sigma}} H_\beta$.

This is encoded by a *Heegaard diagram* $(\bar{\Sigma}, \alpha_1 \dots \alpha_g, \beta_1 \dots \beta_g)$. $(g = genus(\bar{\Sigma}))$





unordered g-tuples of points on punctured Σ

Let $T_\alpha = \alpha_1 \times \dots \times \alpha_g$, $T_\beta = \beta_1 \times \dots \times \beta_g \subset \mathrm{Sym}^g(\bar{\Sigma} \setminus z)$

Theorem (Ozsváth-Szabó, $\sim$ 2000)

$\widehat{HF}(Y) := HF(T_\beta, T_\alpha)$ is independent of chosen Heegaard diagram.

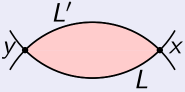
(Floer homology: complex generated by $T_\alpha \cap T_\beta$ = g-tuples of intersections between α and β curves, differential counts holomorphic curves).

Floer homology and Fukaya categories

Σ Riemann surface $\rightsquigarrow M = \mathrm{Sym}^g(\Sigma)$ symplectic manifold (monotone)
Products of disjoint loops/arcs (e.g. $T_\alpha = \alpha_1 \times \cdots \times \alpha_g$) are Lagrangian.

Floer homology = Lagrangian intersection theory,
corrected by holomorphic discs to ensure deformation invariance.

- Floer complex $CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2\, x$ (assuming L, L' transverse)
- differential $\partial : CF(L, L') \to CF(L, L')$
  coeff. of y in ∂x counts holomorphic strips

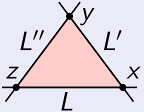


- $HF(L, L') = \mathrm{Ker}\,\partial / \mathrm{Im}\,\partial$.

(For product Lagrangians $T_\alpha, T_\beta \subset \mathrm{Sym}^g(\Sigma)$, intersections = tuples of $\alpha_i \cap \beta_{\sigma(i)}$; holom. curves in $\mathrm{Sym}^g(\Sigma)$ can be seen on Σ. So $\widehat{HF} = HF(T_\beta, T_\alpha)$ fairly easy)

Fukaya category $\mathcal{F}(M)$: objects = Lagrangian submanifolds* (closed)
(monotone, balanced)

- $\hom(L, L') = CF(L, L')$ with differential ∂
- composition $CF(L, L') \otimes CF(L', L'') \to CF(L, L'')$
  coeff. of z in $x \cdot y$ counts holom. triangles



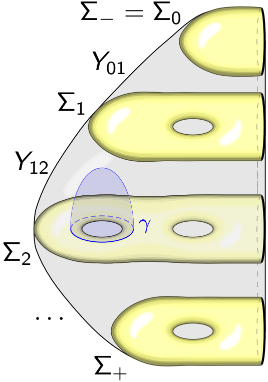
Lagrangian correspondences; the Lekili-Perutz TQFT

- Lagrangian correspondences $M_1 \xrightarrow{L} M_2$ = Lagrangian submanifolds $L \subset (M_1 \times M_2, -\omega_1 \oplus \omega_2)$. These generalize symplectomorphisms (but need not be single-valued); should map Lagrangians to Lagrangians.
- "Generalized Lagrangians" = formal images of Lagrangians under sequences of correspondences; Floer theory extends well.
  $\rightsquigarrow$ extended Fukaya cat. $\mathcal{F}^{\#}(M)$ (Ma'u-Wehrheim-Woodward).
- Correspondences $M_1 \xrightarrow{L} M_2$ induce functors $\mathcal{F}^{\#}(M_1) \to \mathcal{F}^{\#}(M_2)$.

Heegaard-Floer TQFT

- Σ (punctured) surface $\rightsquigarrow$ category $\mathcal{C}(\Sigma) = \mathcal{F}^{\#}(\mathrm{Sym}^g(\Sigma))$
- Y^3 with boundary $\partial Y = \Sigma \rightsquigarrow$ object $\mathbf{T}_Y$: (generalized) Lagrangian submanifold of $\mathrm{Sym}^g(\Sigma)$ (for a handlebody, $\mathbf{T}_Y$ = product torus)
- cobordism $\partial Y = \Sigma_2 - \Sigma_1 \rightsquigarrow$ functor induced by (generalized) Lagr. correspondence $\mathbf{T}_Y : \mathrm{Sym}^{k_1}(\Sigma_1) \longrightarrow \mathrm{Sym}^{k_2}(\Sigma_2)$.
- Pairing theorem: $\widehat{HF}(Y_1 \cup_\Sigma Y_2) \simeq HF(\mathbf{T}_{Y_1}, \mathbf{T}_{-Y_2})$.

Lekili-Perutz: correspondences from cobordisms



Perutz: Elementary cobordism $Y_{12} : \Sigma_1 \rightsquigarrow \Sigma_2$

$\Longrightarrow$ Lagrangian correspondence

$$\mathbf{T}_{12} \subset \mathrm{Sym}^k(\Sigma_1) \times \mathrm{Sym}^{k+1}(\Sigma_2) \ (k \geq 0)$$

(roughly: k points on $\Sigma_1 \mapsto$ "same" k points on Σ_2 plus one point anywhere on γ)

Lekili-Perutz: decompose Y^3 into sequence of elementary cobordisms $Y_{i,i+1}$, compose all $\mathbf{T}_{i,i+1}$ to get a generalized correspondence $\mathbf{T}_Y$.

$\mathbf{T}_Y : \mathrm{Sym}^{k_-}(\Sigma_-) \to \mathrm{Sym}^{k_+}(\Sigma_+)$ $\quad (\partial Y = \Sigma_+ - \Sigma_-)$

Theorem (Lekili-Perutz)

$\mathbf{T}_Y$ *is independent of decomposition of* Y *into elementary cobordisms.*

- View Y^3 (*sutured:* $\partial Y = \Sigma_+ \cup \Sigma_-$) as cobordism of surfaces w. boundary
- For a handlebody (as cobordism $D^2 \rightsquigarrow \Sigma_g$), $\mathbf{T}_Y \simeq$ product torus
- Y^3 closed, $Y \setminus B^3 : D^2 \rightsquigarrow D^2$, then $\mathbf{T}_Y \simeq \widehat{HF}(Y) \in \mathcal{F}^\#(pt) = \mathrm{Vect}$

Lekili-Perutz vs. bordered Heegaard-Floer

The extended Fukaya category $\mathcal{F}^{\#}(\mathrm{Sym}^g(\Sigma))$ and the generalized Lagrangians $\mathbf{T}_Y$ (for Y^3 with $\partial Y = \Sigma$) constructed by Lekili-Perutz are not very explicit at first glance... unlike

Bordered Heegaard-Floer homology (Lipshitz-Ozsváth-Thurston 2008)

- Σ (decorated) surface $\rightsquigarrow$ (cat. of modules over) dg-algebra $\mathcal{A}(\Sigma, g)$
- Y^3 with $\partial Y = \Sigma \rightsquigarrow \widehat{CFA}(Y)$ (right A_∞) module over $\mathcal{A}(\Sigma, g)$
- pairing: $\widehat{HF}(Y_1 \cup_\Sigma Y_2) \simeq \hom_{\text{mod-}\mathcal{A}}(\widehat{CFA}(-Y_2), \widehat{CFA}(Y_1))$

In fact, by considering specific product Lagrangians in $\mathrm{Sym}^g(\Sigma)$ one gets:

Theorem

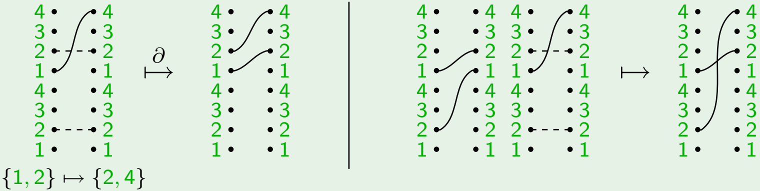
- $\mathcal{F}^{\#}(\mathrm{Sym}^g(\Sigma))$ *embeds fully faithfully into mod-*$\mathcal{A}(\Sigma, g)$
- *Given* Y^3 *with* $\partial Y = \Sigma$, *the embedding maps* $\mathbf{T}_Y$ *to* $\widehat{CFA}(Y)$

The Lipshitz-Ozsváth-Thurston strands algebra $\mathcal{A}(\Sigma, g)$

Describe Σ by a pointed matched circle: segment with $4g$ points carrying labels $1, \dots, 2g, 1, \dots, 2g$ (= how to build $\Sigma = D^2 \cup 2g$ 1-handles)

$\mathcal{A}(\Sigma, g)$ is generated (over $\mathbb{Z}_2$) by g-tuples of {upward strands, pairs of horizontal dotted lines} s.t. the g source labels (resp. target labels) in $\{1, \dots, 2g\}$ are all distinct.

Example ($g = 2$)



- Differential: sum all ways of smoothing one crossing.
- Product: concatenation (end points must match).
- Treat  as  +  and set  = 0.

The extended Fukaya category vs. $\mathcal{A}(\Sigma, g)$

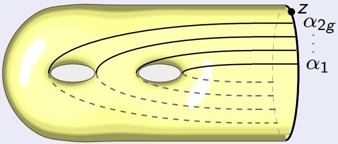
Theorem

$\mathcal{F}^{\#}(\mathrm{Sym}^g(\Sigma))$ *embeds fully faithfully into mod-*$\mathcal{A}(\Sigma, g)$ *(A_∞-modules)*

Main tool: partially wrapped Fukaya cat. $\mathcal{F}^{\#}(\mathrm{Sym}^g(\Sigma), z)$ $(z \in \partial\Sigma)$

Enlarge $\mathcal{F}^{\#}$: add noncompact objects = products of disjoint properly embedded arcs. Roughly, $\hom(L_0, L_1) := CF(\tilde{L}_0, \tilde{L}_1)$, deforming all arcs so that end points of $\tilde{L}_0$ lie *above* those of $\tilde{L}_1$ (without crossing z). Similarly, product is defined by perturbing so that $\tilde{L}_0 > \tilde{L}_1 > \tilde{L}_2$.

(after Abouzaid-Seidel)



Let $D_s = \prod_{i \in s} \alpha_i$ $(s \subseteq \{1 \ldots 2g\}, |s| = g)$. Then:

1. $\bigoplus_{s,t} \hom(D_s, D_t) \simeq \mathcal{A}(\Sigma, g)$
2. the objects D_s generate $\mathcal{F}^{\#}(\mathrm{Sym}^g(\Sigma), z)$

Yoneda embedding and A_∞-modules

Recall: Y^3, $\partial Y = \Sigma \cup D^2 \Rightarrow$ gen. Lagr. $\mathbf{T}_Y \in \mathcal{F}^\#(\mathrm{Sym}^g \Sigma)$ (Lekili-Perutz)

- Yoneda embedding: $\mathbf{T}_Y \mapsto \mathcal{Y}(\mathbf{T}_Y) = \bigoplus_s \hom(\mathbf{T}_Y, D_s)$
  right A_∞-module over $\bigoplus_{s,t} \hom(D_s, D_t) \simeq \mathcal{A}(\Sigma, g)$.
- In fact, $\mathcal{Y}(\mathbf{T}_Y) \simeq \widehat{CFA}(Y)$ (bordered Heegaard-Floer module)
- Pairing theorem: if $Y = Y_1 \cup Y_2$, $\partial Y_1 = -\partial Y_2 = \Sigma \cup D^2$, then
  $$\widehat{CF}(Y) \simeq \hom_{\mathcal{F}^\#}(\mathbf{T}_{Y_1}, \mathbf{T}_{-Y_2}) \simeq \hom_{\text{mod-}\mathcal{A}}(\mathcal{Y}(\mathbf{T}_{-Y_2}), \mathcal{Y}(\mathbf{T}_{Y_1})).$$
- also: (using $\mathcal{A}(-\Sigma, g) \simeq \mathcal{A}(\Sigma, g)^{op}$)
  $$\widehat{CF}(Y) \simeq \mathbf{T}_{Y_1} \circ \mathbf{T}_{Y_2} \simeq \mathcal{Y}(\mathbf{T}_{Y_1}) \otimes_{\mathcal{A}} \mathcal{Y}(\mathbf{T}_{Y_2}).$$

More generally, if $\partial Y = \Sigma_+ \cup -\Sigma_-$ (sutured manifold), the generalized corresp. $\mathbf{T}_Y \in \mathcal{F}^\#(-\mathrm{Sym}^{k_-}\Sigma_- \times \mathrm{Sym}^{k_+}\Sigma_+)$ yields an A_∞-bimodule
$$\mathcal{Y}(\mathbf{T}_Y) = \bigoplus_{s,t} \hom(D_{-,s}, \mathbf{T}_Y, D_{+,t}) \in \mathcal{A}(\Sigma_-, k_-)\text{-mod-}\mathcal{A}(\Sigma_+, k_+)$$
(cf. Ma'u-Wehrheim-Woodward). $\mathcal{Y}(\mathbf{T}_Y) \simeq \widehat{CFDA}(Y)$? (same properties)

Future directions

- $HF^{\pm}$ for bordered 3-manifolds? (in computable form) (algebraic model for filtered $\mathcal{F}^{\#}$ of closed symmetric product?)
- 4-manifold invariants: use this technology to relate Perutz invariants of broken Lefschetz fibrations to Ozsváth-Szabó?
- similar constructions in Khovanov homology (after Seidel-Smith)?

References

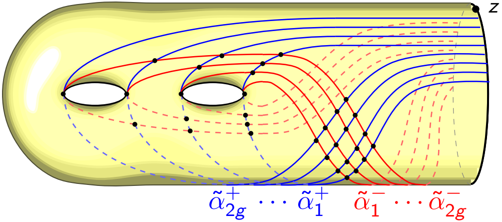
- D. Auroux, *Fukaya categories and bordered Heegaard-Floer homology.* Proceedings of ICM 2010, pp. 917–941 (arXiv:1003.2962).
- R. Lipshitz, P. Ozsváth, D. Thurston, *Bordered Heegaard Floer homology: invariance and pairing*, preprint (arXiv:0810.0687).
- Y. Lekili, T. Perutz, in preparation.

See also work of:

- Rumen Zarev, on "bordered sutured Floer homology".
- Tova Brown, on cobordism maps for 4-manifolds with corners.

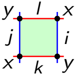
$\bigoplus \hom(D_s, D_t) \simeq \mathcal{A}(\Sigma, k)$

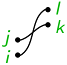
By def. of $\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma), z)$, $\hom(D_s, D_t) = CF(\tilde{D}_s^+, \tilde{D}_t^-)$ $\left(\tilde{D}_s^{\pm} = \prod_{i\in s} \tilde{\alpha}_i^{\pm}\right)$

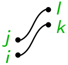


Dictionary: points of $\tilde{\alpha}_i^+ \cap \tilde{\alpha}_j^-$ $\longleftrightarrow$ strands 

(intersections on central axis $\longleftrightarrow$ ⋅- - ⋅ ⋅- - ⋅) } generators = k-tuples

- Differential: y appears in ∂x iff 

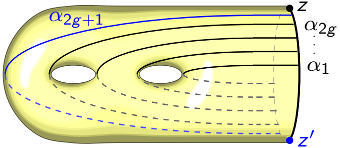
$\longleftrightarrow$ $x =$ 

and $y =$ 

- Similarly for product (triple diagram); all diagrams are "nice"

$\{D_s = \prod_{i\in s} \alpha_i\}_{s\subseteq\{1\dots 2g\}}$ generate $\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma), z)$

- $\pi : \Sigma \xrightarrow{2:1} \mathbb{C}$ induces a Lefschetz fibration $f_k : \mathrm{Sym}^k(\Sigma) \to \mathbb{C}$ with $\binom{2g+1}{k}$ critical points. Its thimbles = products of α_i $(1 \leq i \leq 2g+1)$ generate $\mathcal{F}(f_k) \simeq \mathcal{F}(\mathrm{Sym}^k\Sigma, \{z, z'\})$ (Seidel)



- These $\binom{2g+1}{k}$ objects also generate $\mathcal{F}^{\#}(\mathrm{Sym}^k\Sigma, z)$.

Uses: acceleration functor $\mathcal{F}(\mathrm{Sym}^k\Sigma, \{z, z'\}) \to \mathcal{F}(\mathrm{Sym}^k\Sigma, z)$ (Abouzaid-Seidel)

- $\alpha_{i_1} \times \cdots \times \alpha_{2g+1} \simeq$ twisted complex built from $\{\alpha_{i_1} \times \cdots \times \alpha_j\}_{j=1}^{2g}$

Uses: arc slides are mapping cones