

REGULAR HOMOTOPY OF HURWITZ CURVES

DENIS AUROUX, VIKTOR S. KULIKOV,
AND VSEVOLOD V. SHEVCHISHIN

ABSTRACT. We prove that any two irreducible cuspidal Hurwitz curves C_0 and C_1 (or more generally, curves with A -type singularities) in the Hirzebruch surface \mathbf{F}_N with coinciding homology classes and sets of singularities are regular homotopic; and *symplectically* regular homotopic if C_0 and C_1 are symplectic with respect to a compatible symplectic form.

0. INTRODUCTION

In this paper, we deal with J -holomorphic curves in the projective plane and Hurwitz curves (in particular, algebraic curves) in the Hirzebruch surfaces \mathbf{F}_N which imitate the behavior of plane algebraic curves with respect to pencils of lines (the definition of Hurwitz curves is given in Section 2). We restrict ourselves to the case when Hurwitz curves can have only singularities of the types A_n with $n \geq 0$ (i.e., which are locally given by $y^2 = x^{n+1}$) and also so-called negative nodes (see Section 2).

In [Moi] Moishezon proved the existence of an infinite sequence $\bar{H}_i \subset \mathbf{F}_1$ of generic irreducible cuspidal Hurwitz curves of degree 54 with exactly 378 cusps and 756 nodes which have pairwise distinct braid monodromy type. In particular, they are pairwise non-isotopic, and almost all of them are not isotopic to an algebraic cuspidal curve.

The aim of this article is to prove the following statement.

Theorem 0.1. *Any two irreducible cuspidal Hurwitz curves \bar{H}_0 and \bar{H}_1 in the Hirzebruch surface \mathbf{F}_N having the same homology class and the same numbers of cusps and nodes (or, in presence of negative nodes, differences between numbers of positive and negative nodes) can be connected by a regular homotopy.*

Date: This version: November 2003.

This work was initiated during a stay of the authors at IPAM, UCLA. The first author was supported in part by NSF grant DMS-0244844. The second author was supported in part by RFBR (No. 02-01-00786) and INTAS (No. 00-0269).

Moreover, if \bar{H}_0 and \bar{H}_1 are symplectic with respect to some form ω compatible with the canonical ruling of \mathbf{F}_N , then the regular homotopy between them can be made ω -symplectic.

A regular homotopy is a deformation family $\{\bar{H}_t\}_{t \in [0,1]}$ which is an isotopy except for a finite number of values of t at which the curve undergoes the “standard” transformation of creation or cancellation of a pair of nodes of opposite signs, see Section 2 for the precise definition.

The result remains true if the irreducible Hurwitz curves \bar{H}_0 and \bar{H}_1 are allowed to present arbitrary singularities of type A_n . The necessary and sufficient condition for the existence of a regular homotopy between \bar{H}_0 and \bar{H}_1 then becomes that the numbers of singularities of each type are the same (except in the case of nodes, for which one should compare differences between numbers of positive and negative nodes).

As a corollary of our main theorem, we obtain:

Corollary 0.2. *Let C_0 and C_1 be two ordinary cuspidal irreducible symplectic surfaces in $(\mathbb{C}\mathbb{P}^2, \omega)$, $\deg C_0 = \deg C_1$, pseudoholomorphic with respect to ω -tamed almost-complex structures J_0 and J_1 respectively. If C_0 and C_1 have the same numbers of cusps and nodes, then they can be deformed into each other by a C^1 -smooth symplectic regular homotopy in $\mathbb{C}\mathbb{P}^2$.*

The structure of the rest of this paper is as follows: in Section 1 we give a symplectic isotopy result for curves in Hirzebruch surfaces whose irreducible components are sections. In Section 2 we define Hurwitz curves and regular homotopy, and mention some of their elementary properties. In Section 3 we introduce the main ingredient in the proof of Theorem 0.1, namely braid monodromy factorizations of Hurwitz curves with A -type singularities. The main results are then proved in Section 4, where the outline of a more geometric alternative proof is also given.

Finally, in Section 5, we construct two so-called quasi-positive factorizations of an element in the braid group Br_4 which give a negative answer to the Generalized Garside Problem asking whether the natural homomorphism from the semigroup of quasi-positive braids to the braid group is an embedding.

Acknowledgement. The authors would like to express their gratitude to the Institute for Pure and Applied Mathematics at UCLA for its hospitality during the early stages of the preparation of this paper.

1. SYMPLECTIC ISOTOPY OF SECTIONS

Let X be a Hirzebruch surface \mathbf{F}_k , $k \geq 0$, and E a rational holomorphic curve with self-intersection $E^2 = -k$, which is unique if $k \geq 1$. Furthermore, let ω be a Kähler form compatible with the complex structure $J_{\mathbf{F}_k}$. It is known (see [Li-Liu] or [La-McD]) that (X, ω) is symplectomorphic to \mathbf{F}_1 if k is odd or to \mathbf{F}_0 otherwise, equipped with an appropriate Kähler structure.

Let \mathcal{J} be the set of all C^l -smooth ω -tame almost complex structures on X with a fixed sufficiently large non-integer l . This is a Banach manifold. Let \mathcal{M}_k be the total moduli space of pseudoholomorphic curves C on X in the homology class $[E] \in H^2(X, \mathbb{Z})$. Recall that \mathcal{M}_k is defined as the quotient

$$\mathcal{M}_k := \left\{ (u, J) : \begin{array}{l} J \in \mathcal{J}, \quad u : \mathbb{P}^1 \rightarrow X \text{ is } J\text{-holo-} \\ \text{morphic and represents } [E] \end{array} \right\} / \mathbf{PGL}(2, \mathbb{C}),$$

where the group $\mathbf{PGL}(2, \mathbb{C})$ acts by holomorphic automorphisms on \mathbb{P}^1 . By abuse of notation, we denote the points of \mathcal{M}_k by (u, J) or by (C, J) , emphasizing either the parameterizing map u or the image curve C . The genus formula for pseudoholomorphic curves (see [Mi-Wh]) ensures that all curves in \mathcal{M}_k are embedded. Let $\mathbf{pr}_k : \mathcal{M}_k \rightarrow \mathcal{J}$ be the natural projection given by $\mathbf{pr}_k : (C, J) \mapsto J$. This is a Fredholm operator of \mathbb{R} -index $2(1 - k)$.

Lemma 1.1. *The projection $\mathbf{pr}_k : \mathcal{M}_k \rightarrow \mathcal{J}$ is*

- (1) *an embedding of real codimension $2(k-1)$ if $k \geq 1$; in particular, it is an open embedding for $k = 1$;*
- (2) *a bundle with fiber S^2 over an open subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}$ if $k = 0$.*

Proof. The properties of the projection \mathbf{pr}_k at (C, J) can be described in terms of the *normal sheaf* \mathcal{N}_C of C (see [Iv-Sh-1] or [Iv-Sh-2]). Since C is embedded, \mathcal{N}_C reduces to a line bundle, denoted by N_C , equipped with the Gromov operator $D_{C,J} : L^{1,p}(C, N_C) \rightarrow L^p(C, N_C \otimes \Lambda^{(0,1)})$.

Since in our case C is rational, the operator $D_{C,J}$ cannot have both non-trivial kernel and cokernel, see [H-L-S], [Iv-Sh-1], or [Iv-Sh-2]. This means that \mathbf{pr}_k is of “maximal rank” everywhere on \mathcal{M}_k , i.e., either an immersion, or a submersion, or a local diffeomorphism.

The global injectivity of \mathbf{pr}_k in the case $k \geq 1$ follows easily from the fact that two distinct J -holomorphic curves C and C' must have *positive* intersection number, which would contradict the condition $[C] \cdot [C'] = -k < 0$.

In the case $k = 0$ we first show that every non-empty fiber of pr_0 is compact. Assuming the contrary, we obtain a sequence of J -holomorphic curves C_ν with no limit set in $\text{pr}_0^{-1}(J)$. Applying Gromov's compactness theorem we may assume that the sequence converges to some reducible curve $C^* = \sum m_i C_i^*$. Here the image is understood in the sense of *cycles*. Since the intersection number of every C_i^* with every curve in $\text{pr}_0^{-1}(J)$, say C_1 , is non-negative, and the sum of these indices is zero, we conclude that $[C_i^*] \cdot [C_1] = 0$ for every i .

Therefore each C_i^* represents a homology class which is a positive integer multiple of the primitive class $[C_1]$. Counting the ω -area of C_i^* 's we conclude that C^* consists of a unique non-multiple component homologous to C_1 . Thus (C^*, J) lies in \mathcal{M}_0 .

To show that all non-empty fibers of pr_0 are diffeomorphic, it is sufficient to prove that the complement to the image of pr_0 has real codimension 2. Using Gromov's compactness once more, we can find a reducible J -holomorphic curve $C^* = \sum m_i C_i^*$ for every J lying on the (topological) boundary of the image of pr_0 . By the genus formula, all C_i^* must be embedded rational curves. Moreover, since $c_1(X) \cdot [C^*] = 2$ and the intersection form of $\mathbf{F}_0 = S^2 \times S^2$ is even, for one of them, say C_1^* , we must have $c_1(X) \cdot [C_1^*] \leq 0$. Applying the genus formula we obtain that $[C_1^*]^2 =: k \leq -2$. This implies that J lies in the image of the projection $\text{pr}_k : \mathcal{M}_k \rightarrow \mathcal{J}$, which is locally a submanifold of codimension ≥ 2 . \square

We shall denote the image $\text{pr}_k(\mathcal{M}_k)$ by \mathcal{J}_k . For $k \geq 1$ this is a submanifold of \mathcal{J} of real codimension $2(k-1)$, in general *not closed*.

Lemma 1.2. *Let X be a Hirzebruch surface \mathbf{F}_k , ω a Kähler form on X , $J \in \mathcal{J}_k$ an almost-complex structure, E the corresponding J -holomorphic curve with self-intersection $-k$, and C an irreducible J -holomorphic curve different from E . Let also F be a fiber of the ruling on X . Then*

- (i) C is homologous to $d[E] + f[F]$ with $d \geq 0$ and $f \geq kd$;
- (ii) $c_1(X) \cdot C = d(2-k) + 2f > 0$.

Proof. Fix a point x on X outside E . Observe that there exists a path $h : [0, 1] \rightarrow \mathcal{J}$, $t \in [0, 1]$, which connects $J =: J_0$ with the "standard" structure $J_1 = J_{\text{st}}$ through almost-complex structures $J_t = h(t)$, such that E is J_t -holomorphic for every $t \in [0, 1]$. This means that h takes values in \mathcal{J}_k . Consider the moduli space $\mathcal{M}_h :=$

$$\mathcal{M}_h(X, F, x) := \left\{ (C, t) : \begin{array}{l} t \in [0, 1], C \text{ is a rational } J_t\text{-holomorphic curve} \\ \text{homologous to } F \text{ which passes through } x. \end{array} \right\}$$

together with the natural projection $\text{pr}_h : \mathcal{M}_h \rightarrow [0, 1]$ given by $\text{pr}_h : (C, t) \mapsto t$. The techniques of [Bar-1] and [Bar-2] ensure that $\text{pr}_h : \mathcal{M}_h \rightarrow [0, 1]$ is a diffeomorphism. This implies that there exists a J -holomorphic curve C_0 isotopic to F .

The homology group $H_2(X, \mathbb{Z})$ is a free abelian group with generators E and F , so $[C] = d[E] + f[F]$ with some integers d and f . Since C , E , and C_0 are J -holomorphic, the intersection indices $[C] \cdot [C_0]$ and $[C] \cdot [E]$ are non-negative (see [Mi-Wh]). This gives the inequalities in (i). The equality in part (ii) follows from formulas $c_1(X) \cdot E = 2 - k$ and $c_1(X) \cdot F = 2$. Finally, $d(2 - k) + 2f = 2d + f + (f - kd)$ is non-negative and vanishes only if $d = f = 0$. \square

Theorem 1.3. *Let X be a Hirzebruch surface \mathbf{F}_k , ω_t , $t \in [0, 1]$ a smooth family of symplectic forms on X taming the complex structure $J_{\mathbf{F}_k}$, E a holomorphic curve with self-intersection $-k$, F a fiber of the ruling of X , and C_0 an immersed ω_0 -symplectic surface in X such that every component $C_{0,i}$ of C_0 is homologous to either to $[F]$ or to $[E] + f_i[F]$ with $f_i \geq 1$. Assume also that the surface $C_0 \cup E$ has only positive transversal double points as singularities. Then there exists an ω_t -symplectic isotopy C_t between C_0 and a holomorphic curve C_1 , such that each C_t meets E transversally with positive intersection index.*

The case of main interest is when all $f_i = k$. Then C_0 consists of sections of the line bundle $X \setminus E \cong \mathcal{O}_{\mathbb{P}^1}(k)$. The general case corresponds to a collection of meromorphic sections with various numbers of poles.

Proof. It follows from the assumption of the theorem that both C_0 and E are J_0 -holomorphic curves with respect to a same ω_0 -tame almost complex structure J_0 on X (see e.g. [Iv-Sh-1]). Furthermore, there exists a path $h : [0, 1] \rightarrow \mathcal{J}$, $t \in [0, 1]$, connecting J_0 with the “standard” structure $J_1 = J_{\text{st}}$, such that each $J_t := h(t)$ is ω_t -tame and E is J_t -holomorphic for every $t \in [0, 1]$. This means that h takes values in \mathcal{J}_k . We assume that h is chosen generic enough.

By Lemma 1.2, $f_i \geq k$ and $c_1(X) \cdot [C_{0,i}] = 2 + 2f_i - k$ is strictly positive. On each component $C_{0,i}$ we fix $p_i := c_1(X) \cdot [C_{0,i}] - 1 = 1 + 2f_i - k$ points $x_{i,1}, \dots, x_{i,p_i}$ in generic position. Let $\mathbf{x} = \{x_{i,j}\}$ be the whole collection of these points. Consider the moduli space $\mathcal{M}_h = \mathcal{M}_h(X, C_0, \mathbf{x})$ of deformations of C_0 as J_t -holomorphic curves which have the same *constellation* as C_0 , i.e. such that

- C has the same number of irreducible components as C_0 ;
- each component C_i of C is rational and homologous to the corresponding component $C_{0,i}$ of C_0 ;

- the component C_i of C passes through the same points $x_{i,1}, \dots, x_{i,p_i}$ as $C_{0,i}$ does.

Now, the moduli space is defined as

$$\mathcal{M}_h(X, C_0, \mathbf{x}) := \left\{ (C, t) : \begin{array}{l} t \in [0, 1], C \text{ is a } J_t\text{-holomorphic} \\ \text{curve with the constellation of } C_0 \end{array} \right\}.$$

We denote by $\text{pr}_h : \mathcal{M}_h \rightarrow [0, 1]$ the natural projection given by $\text{pr}_h : (C, t) \mapsto t$. By abuse of notation we write $C \in \mathcal{M}_h$ meaning that (C, t) lies in \mathcal{M}_h for some t .

The expected real dimension of \mathcal{M}_h is 1. The possibility to deform the structures $h(t)$ arbitrarily near the fixed points \mathbf{x} ensures the transversality property of the deformation problem. So \mathcal{M}_h is a manifold of the expected dimension. An important observation of [H-L-S], see also [Bar-1] and [Bar-2], is that in this situation, because the curves C are rational, the projection pr_h has no critical points.

So the statement of the theorem would follow from the properness of $\text{pr}_h : \mathcal{M}_h \rightarrow [0, 1]$. Assuming the contrary, we would find a sequence t_n converging to some $t^* \in [0, 1]$ and a sequence of J_{t_n} -holomorphic curves $C_n \in \mathcal{M}_h$ with no accumulation points in \mathcal{M}_h . By Gromov's compactness theorem, going to a subsequence we may assume that C_n weakly converges to some J^* -holomorphic curve C^* with $J^* = h(t^*)$. Since it is possible to consider the behavior of the components of C_n separately, we may assume that C_0 and every C_n are irreducible. Let $C^* = \sum_{j=1}^l m_j C_j^*$ be the decomposition of C^* into irreducible components, m_j being the corresponding multiplicities. It follows from Lemma 1.2 that there exists exactly one component, say C_1^* , which is homologous to $[E] + f_1^*[F]$, and every remaining component C_j^* is in the homology class $f_j^*[F]$. Moreover, if $[C_0] = [E] + f[F]$, then $\sum_{j=1}^l m_j f_j^* = f$. Applying the genus formula to C_j^* we see that $f_j^* = 1$ for every $j \geq 2$.

Now recall that C^* must pass through the $p = 1 + 2f - k$ marked points x_j used in the definition of \mathcal{M}_h . On the other hand, the genericity of the path $h(t)$ and of the points x_j implies that one can have at most $p_1^* := 1 + 2f_1^* - k$ of the marked points on C_1^* , and at most one such point on C_j^* for $j \geq 2$. Altogether, this allows C^* to pass through $1 + 2f_1^* - k + l - 1$ marked points, which is strictly less than the needed $1 + 2f - k$ unless $l = 1$. But this means that C^* is irreducible and hence lies in \mathcal{M}_h , a contradiction. Thus $\text{pr}_h : \mathcal{M}_h \rightarrow [0, 1]$ is proper. \square

2. HURWITZ CURVES

Definition 2.1. A Hurwitz curve of degree m in the Hirzebruch surface \mathbf{F}_N is the image $\bar{H} = f(\mathcal{S}) \subset \mathbf{F}_N$ of an oriented closed real surface \mathcal{S} by a smooth map $f : \mathcal{S} \rightarrow \mathbf{F}_N \setminus E_N$ such that there exists a finite subset $Z \subset \bar{H}$ with the following properties:

- (i) The restriction of f to $\mathcal{S} \setminus f^{-1}(Z)$ is an embedding, and for any $p \in \bar{H} \setminus Z$, \bar{H} and the fiber $F_{\text{pr}(p)}$ of pr meet at p transversely with intersection index $+1$;
- (ii) for each $p \in Z$ there is a neighborhood $U \subset \mathbf{F}_N$ of p such that $\bar{H} \cap U$ is a complex analytic curve, and the complex orientation of $\bar{H} \cap U \setminus \{p\}$ coincides with the orientation transported from \mathcal{S} by f ;
- (iii) the restriction of pr to \bar{H} is a finite map of degree m .

For any Hurwitz curve \bar{H} there is a unique minimal subset $Z \subset \bar{H}$ satisfying the conditions from Definition 2.1. We denote it by $Z(\bar{H})$. We say that \bar{H} is pr -generic if $\text{pr}|_Z : Z \rightarrow \text{pr}(Z)$ is one-to-one. A fiber of pr is \bar{H} -singular if it meets $Z(\bar{H})$ and \bar{H} -regular otherwise.

A Hurwitz curve \bar{H} has an A_k -singularity at $p \in Z(\bar{H})$ if there is a neighborhood U of p and local analytic coordinates x, y in U such that

- (iv) $\text{pr}|_U$ is given by $(x, y) \mapsto x$;
- (v) $\bar{H} \cap U$ is given by $y^2 = x^{k+1}$.

An “ A_0 -singularity” is in fact a smooth point where \bar{H} becomes tangent to the fiber of pr ; A_1 and A_2 singularities are ordinary nodes and cusps, respectively. Therefore, we will say that \bar{H} is *cuspidal* if all its singularities are of type A_k with $0 \leq k \leq 2$, and *nodal* if it has only A_0 and A_1 singularities. We say that \bar{H} has *A-singularities* if all its singularities are of type A_k with $k \geq 0$.

For our purpose we need to extend the class of admissible singularities of Hurwitz curves described in Definition 2.1 (ii) by allowing the simplest non-holomorphic one.

Definition 2.2. A negative node on a Hurwitz curve \bar{H} is a singular point $p \in Z(\bar{H})$ such that

- (ii⁻) there is a neighborhood $U \subset \mathbf{F}_N$ of p such that $\bar{H} \cap U$ consists of two smooth branches meeting transversely at p with intersection index -1 , and each branch of $\bar{H} \cap U$ meets the fiber $F_{\text{pr}(p)}$ transversely at p with intersection index $+1$.

Definition 2.3. A Hurwitz curve $\bar{H} \subset \mathbf{F}_N$ is called an almost-algebraic curve if \bar{H} coincides with an algebraic curve C over a disc $D(r) \subset \mathbb{P}^1$ and with the union of m pairwise disjoint smooth sections $H_{\infty,1}, \dots, H_{\infty,m}$ of pr over $\mathbb{P}^1 \setminus D(r)$.

Definition 2.4. Two Hurwitz curves \bar{H}_0 and $\bar{H}_1 \subset \mathbf{F}_N$ (possibly with negative nodes) are H -isotopic if there is a continuous isotopy $\phi_t : \mathbf{F}_N \rightarrow \mathbf{F}_N$, $t \in [0, 1]$, fiber-preserving (i.e. $\exists \psi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $\text{pr} \circ \phi_t = \psi_t \circ \text{pr}$), and smooth outside the fibers $F_{\text{pr}(s)}$, $s \in Z(\phi_t(\bar{H}_0))$ such that

- (i) $\phi_0 = \text{Id}$;
- (ii) $\phi_t(\bar{H}_0)$ is a Hurwitz curve for all $t \in [0, 1]$;
- (iii) $\phi_1(\bar{H}_0) = \bar{H}_1$;
- (iv) $\phi_t(E_N) = E_N$ for all $t \in [0, 1]$.

In the specific case of curves with A -singularities, we can in fact assume that ϕ_t is smooth everywhere.

The following theorem was proved in [Kh-Ku].

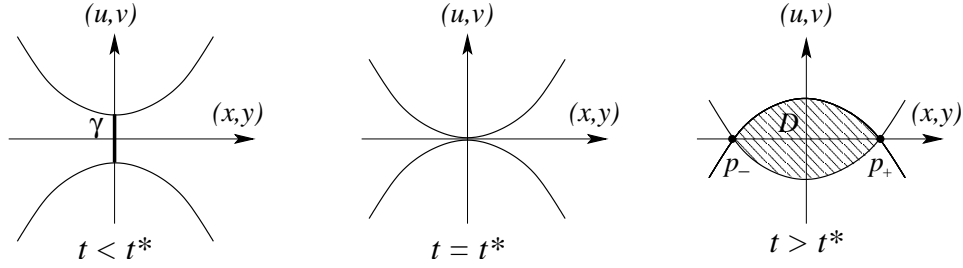
Theorem 2.5. ([Kh-Ku]) Any pr -generic Hurwitz curve $\bar{H} \subset \mathbf{F}_N$ with A -singularities is H -isotopic to an almost-algebraic curve. If moreover \bar{H} is a symplectic surface in \mathbf{F}_N , then this isotopy can be chosen symplectic.

Definition 2.6. A creation of a pair of nodes along a simple curve γ is the transformation of a Hurwitz curve $\bar{H} = \bar{H}_{t^* - \tau}$ given by a homotopy \bar{H}_t , $t \in [t^* - \tau, t^* + \tau]$ and $0 < \tau \ll 1$ with the following properties:

- (1) \bar{H}_t is an isotopy outside a neighborhood U of γ ;
- (2) there exist real coordinates (x, y, u, v) in U such that the projection $\text{pr} : \mathbf{F}_N \rightarrow \mathbb{P}^1$ is given by $\text{pr} : (x, y, u, v) \rightarrow (x, y)$ and the curve γ is given by $\{u \in [-\tau, \tau], x = v = y = 0\}$;
- (3) for every $t \in [t^* - \tau, t^* + \tau]$ the curve $\bar{H}_t \cap U$ consists of two discs which are the graphs of the sections $s_t^\pm : (x, y) \mapsto (u, v) = (\pm(x^2 - (t - t^*)), \pm y)$.

The disc given by $D := \{v = y = 0, x \in [-\sqrt{\tau}, \sqrt{\tau}], x^2 - \tau \leq u \leq \tau - x^2\}$ is called the created Whitney disc.

The inverse transformation is called the cancellation of a pair of nodes along the Whitney disc D .



The curve γ lies in the fiber $\{x = y = 0\}$ and connects two points on $\bar{H} = \bar{H}_{t^*-\tau}$; the created nodes p_- and p_+ have opposite orientations: one is positive, the other negative. The inversion of the time t interchanges the creation and the cancellation operations.

Definition 2.7. *Two Hurwitz curves are regular homotopic if one of them can be obtained from the other by the composition of a finite number of H -isotopies, creations and cancellations of pairs of nodes.*

The definition is motivated by the following claim. The proof is an easy exercise.

Lemma 2.8. *Let $\phi_t : \mathcal{S} \rightarrow \mathbf{F}_N$ be a smooth homotopy of maps of a closed oriented real surface \mathcal{S} to \mathbf{F}_N with the following properties:*

- (1) *for every t , the composition $\text{pr} \circ \phi_t : \mathcal{S} \rightarrow \mathbb{P}^1$ is a ramified covering;*
- (2) *ϕ_t is an H -isotopy in a neighborhood of all its critical points;*
- (3) *ϕ_t is generic with respect to conditions (1) and (2).*

Then ϕ_t is a regular homotopy.

Lemma 2.9. *Let \bar{H} be a Hurwitz curve in \mathbf{F}_N , F an \bar{H} -regular fiber of pr , $\gamma \subset F$ a simple smooth curve in $F \setminus \bar{H}$ with endpoints on \bar{H} , and U any neighborhood of γ in \mathbf{F} . Then there exists a regular homotopy ϕ_t of \bar{H} which creates a pair of nodes along γ and is constant outside U .*

Proof. It follows from the hypotheses of the lemma that γ meets \bar{H} transversally. So, shrinking U if needed, we can find local coordinates (x, y, u, v) in U satisfying the condition (2) of Definition 2.6, such that $U \cap \bar{H}$ consists of two discs which are the graphs of the mutually disjoint sections $s_{t^*-\tau}^\pm : (x, y) \mapsto (u, v) = (\pm(x^2 + \tau), \pm y)$. The result follows. \square

3. FACTORIZATION SEMIGROUPS

In this section we recall the notion of braid monodromy factorization semigroups defined in [Kh-Ku].

3.1. Semigroups over groups. A collection (S, B, α, λ) , where S is a semigroup, B is a group, and $\alpha : S \rightarrow B$, $\lambda : B \rightarrow \text{Aut}(S)$ are homomorphisms, is called a *semigroup S over a group B* if for all $s_1, s_2 \in S$

$$s_1 \cdot s_2 = \lambda(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \rho(\alpha(s_2))(s_1),$$

where $\rho(g) = \lambda(g^{-1})$. If we are given two semigroups $(S_1, B_1, \alpha_1, \lambda_1)$ and $(S_2, B_2, \alpha_2, \lambda_2)$ over, respectively, groups B_1 and B_2 , we call a pair $h = (h_S, h_B)$ of homomorphisms $h_S : S_1 \rightarrow S_2$ and $h_B : B_1 \rightarrow B_2$ a *homomorphism of semigroups over groups* if

- (i) $h_B \circ \alpha_{S_1} = \alpha_{S_2} \circ h_S$,
- (ii) $\lambda_2(h_B(g))(h_S(s)) = h_S(\lambda_1(g)(s))$ for all $s \in S_1$ and all $g \in B_1$.

The *factorization semigroups* defined below constitute, for our purpose, the principal examples of semigroups over groups.

Let $\{g_i\}_{i \in I}$ be a set of elements of a group B . For each $i \in I$ denote by $O_{g_i} \subset B$ the set of all the elements in B conjugated to g_i (the orbit of g_i under the action of B by inner automorphisms). Call their union $X = \cup_{i \in I} O_{g_i} \subset B$ the *full set of conjugates* of $\{g_i\}_{i \in I}$ and the pair (B, X) an *equipped group*.

For any full set of conjugates X there are two natural maps $r = r_X : X \times X \rightarrow X$ and $l = l_X : X \times X \rightarrow X$ defined by $r(a, b) = b^{-1}ab$ and $l(a, b) = aba^{-1}$ respectively. For each pair of letters $a, b \in X$ denote by $R_{a,b;r}$ and $R_{a,b;l}$ the relations defined in the following way:

$R_{a,b;r}$ stands for $a \cdot b = b \cdot r(a, b)$ if $b \neq \mathbf{1}$ and $a \cdot \mathbf{1} = a$ otherwise;

$R_{a,b;l}$ stands for $a \cdot b = l(a, b) \cdot a$ if $a \neq \mathbf{1}$ and $\mathbf{1} \cdot b = b$ otherwise.

Now, put

$$\mathcal{R} = \{R_{a,b;r}, R_{a,b;l} \mid (a, b) \in X \times X, a \neq b \text{ if } a \neq \mathbf{1} \text{ or } b \neq \mathbf{1}\}$$

and introduce a semigroup

$$S(B, X) = \langle x \in X \mid R \in \mathcal{R} \rangle$$

quotient of the free semigroup of words with letters in X by the relation set \mathcal{R} . As will be seen in Section 3.2, the elements of this semigroup represent *factorizations* of elements of the group B with factors in X , up to *Hurwitz equivalence*. Introduce also the *product homomorphism* $\alpha_X : S(B, X) \rightarrow B$ given by $\alpha_X(x) = x$ for each $x \in X$.

Next, we define two actions λ and ρ of the group B on the set X :

$$x \in X \mapsto \lambda(g)(x) = gxg^{-1} \in X.$$

and $\rho(g) = \lambda(g^{-1})$. It is easy to see that the above relation set \mathcal{R} is preserved by the both actions; therefore, ρ and λ define an anti-homomorphism $\rho : B \rightarrow \text{Aut}(S(B, X))$ (*right conjugation action*) and

a homomorphism $\lambda : B \rightarrow \text{Aut}(S(B, X))$ (*left conjugation action*). The action $\lambda(g)$ on $S(B, X)$ is called *simultaneous conjugation* by g .

One can easily show that $(S(B, X), B, \alpha_X, \lambda)$ is a semigroup over B . We call such semigroups the *factorization semigroups over B* . When B is a fixed group, we abbreviate $S(B, X)$ to S_X . By $x_1 \cdot \dots \cdot x_n$ we denote the element in S_X defined by a word $x_1 \dots x_n$.

Notice that $S : (B, X) \mapsto (S(B, X), B, \alpha_X, \lambda)$ is a functor from the category of equipped groups to the category of semigroups over groups. In particular, if $X \subset Y$ are two full sets of conjugates in B , then the identity map $id : B \rightarrow B$ defines an embedding $id_{X,Y} : S(B, X) \rightarrow S(B, Y)$. So for each group B , the semigroup $S_B = S(B, B)$ is a *universal factorization semigroup over B* , which means that each semigroup S_X over B is canonically embedded in S_B by $id_{X,B}$.

Since $\alpha_X = \alpha_B \circ id_{X,B}$, we make no difference between the product homomorphisms α_X and α_B and denote them both simply by α .

Claim 3.1. *For any $s_1, s_2 \in S(B, X)$ we have*

$$s_1 \cdot s_2 = s_2 \cdot \rho(\alpha(s_2))(s_1) = \lambda(\alpha(s_1))(s_2) \cdot s_1.$$

Denote by Σ_m the symmetric group acting on the set $\{1, \dots, m\} = [1, m]$ and by $(i, j) \in \Sigma_m$ the transposition exchanging the elements i and $j \in [1, m]$. The element $h_g = ((1, 2) \cdot (1, 2))^{g+1} \cdot ((2, 3) \cdot (2, 3)) \cdot \dots \cdot ((m-1, m) \cdot (m-1, m)) \in S_{\Sigma_m}$ is called a *Hurwitz element of genus g* .

Lemma 3.2. *The Hurwitz elements h_g are invariant under the conjugation action of Σ_m on S_{Σ_m} .*

Proof. Put

$$G_{h_g} = \{\sigma \in \Sigma_m \mid \lambda(\sigma)(h_g) = h_g\}.$$

To prove Lemma 3.2, it is sufficient to show that $(i, i+1) \in G_{h_g}$ for $1 \leq i \leq m-1$.

Applying Claim 3.1, and using the fact that $\lambda(\alpha(s)) = \text{Id}$ whenever $\alpha(s) = \mathbf{1}$, we have

$$\begin{aligned}
h_g &= ((1, 2) \cdot (1, 2))^g \cdot \prod_{j=1}^{m-1} ((j, j+1) \cdot (j, j+1)) = \\
& ((i, i+1) \cdot (i, i+1)) \cdot ((1, 2) \cdot (1, 2))^g \cdot \prod_{j \neq i} ((j, j+1) \cdot (j, j+1)) = \\
& (i, i+1) \cdot \lambda((i, i+1)) \left(((1, 2) \cdot (1, 2))^g \cdot \prod_{j \neq i} (j, j+1)^2 \right) \cdot (i, i+1) = \\
& (i, i+1) \cdot (i, i+1) \cdot \lambda((i, i+1)) \left(((1, 2) \cdot (1, 2))^g \cdot \prod_{j \neq i} (j, j+1)^2 \right) = \\
& \lambda((i, i+1)) \left(((i, i+1) \cdot (i, i+1)) \cdot ((1, 2) \cdot (1, 2))^g \cdot \prod_{j \neq i} (j, j+1)^2 \right) = \\
& \lambda((i, i+1))(h_g),
\end{aligned}$$

since $\alpha((l, s) \cdot (l, s)) = \mathbf{1} \in \Sigma_m$ for any transposition (l, s) . \square

3.2. Hurwitz equivalence. As above, let $X \subset B$ be a union of orbits of the conjugation action λ of B . An ordered set

$$\{x_1, \dots, x_n \mid x_i \in X\}$$

is called a *factorization* (of length $n \in \mathbb{N}$) of $g = x_1 \dots x_n \in B$ in X . Denote by $F_X = \bigcup_{n \in \mathbb{N}} X^n$ the set of all possible factorizations of the elements of B in X . There is a natural map $\varphi : F_X \rightarrow S(B, X)$, given by

$$\varphi(\{x_1, \dots, x_n\}) = x_1 \cdots x_n.$$

The transformations which replace in $\{x_1, \dots, x_n\}$ two neighboring factors (x_i, x_{i+1}) by $(x_{i+1}, \rho(x_{i+1})(x_i))$ or $(\lambda(x_i)(x_{i+1}), x_i)$ and preserve the other factors are called *Hurwitz moves*. Two factorizations are *Hurwitz equivalent* if one can be obtained from the other by a finite sequence of Hurwitz moves.

Claim 3.3. *Two factorizations $x = \{x_1, \dots, x_n\}$ and $x' = \{x'_1, \dots, x'_n\}$ are Hurwitz equivalent if and only if $\varphi(x) = \varphi(x')$.*

3.3. Semigroups over the braid group. In this subsection, $B = Br_m$ is the braid group on m strings. We fix a set $\{a_1, \dots, a_{m-1}\}$ of *standard generators*, i.e., generators being subject to the relations

$$\begin{aligned}
a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} & 1 \leq i < m-1, \\
a_i a_k &= a_k a_i & |i - k| \geq 2.
\end{aligned}$$

For $k \geq 0$ denote by $A_k = A_k(m)$ (resp., by $A_{\bar{k}} = A_{\bar{k}}(m)$) the full set of conjugates of a_1^{k+1} (resp., of a_1^{-k-1}) in Br_m (recall that all the generators a_1, \dots, a_{m-1} are conjugated to each other). Consider the factorization semigroup S_{A_k} as a subsemigroup of the universal

semigroup S_{Br_m} over Br_m . Let $\Delta = \Delta_m$ be the so-called Garside element:

$$\Delta = (a_1 \dots a_{m-1}) \dots (a_1 a_2 a_3) (a_1 a_2) a_1.$$

It is well-known that

$$\Delta^2 = (a_1 \dots a_{m-1})^m$$

is the generator of the center of Br_m . Denote by $\delta^2 = \delta_m^2$ the element in $S_{A_0} \subset S_{Br_m}$ equal to

$$\delta^2 = (a_1 \cdot \dots \cdot a_{m-1})^m.$$

Put also

$$\tilde{\delta}_m^2 = \prod_{l=m}^2 \prod_{k=1}^{l-1} z_{k,l}^2 \in S_{A_1},$$

where $z_{k,l} = (a_{l-1} \dots a_{k+1}) a_k (a_{l-1} \dots a_{k+1})^{-1}$ for $k < l$ (the notation $\prod_{l=m}^2$ means that the product is taken over decreasing values of l from left to right). It is known (see for example [Moi-Te]) that $\alpha(\tilde{\delta}_m^2) = \Delta_m^2$. Moreover,

$$\tilde{\delta}_m^2 = \prod_{k=1}^{m-1} z_{k,m}^2 \cdot \tilde{\delta}_{m-1}^2.$$

Note that the elements $z_{k,l}^2$, $1 \leq k < l \leq m$, generate the group of *pure* braids $P_m = \ker \gamma_m$, where $\gamma_m : Br_m \rightarrow \Sigma_m$ is the natural homomorphism onto the symmetric group Σ_m .

Although we will not be using that fact, it is worth mentioning that the factorizations δ_m^2 and $\tilde{\delta}_m^2$ represent the braid monodromies of a smooth plane curve of degree m and of a configuration of m lines in generic position, respectively.

Lemma 3.4. ([Kh-Ku]) *The element δ^2 is fixed under the conjugation action of Br_m on S_{Br_m} , i.e., $\rho(g)(\delta^2) = \delta^2$ for any $g \in Br_m$.*

Theorem 3.5. ([Kh-Ku]) *The element $\tilde{\delta}_m^2$ is the only element $s \in S_{A_1}$ such that $\alpha(s) = \Delta_m^2$.*

Lemma 3.6. *Let $g \in A_0$ have the same image in Σ_m as z_{k_0, l_0} , i.e., $\gamma_m(g) = \gamma_m(z_{k_0, l_0}) = (k_0, l_0)$. Then there exists $p \in P_m$ such that $g = pz_{k_0, l_0}p^{-1}$. Moreover, p can be chosen so that it can be represented as a positive word in the alphabet $\{z_{i,j}^2\}$.*

Proof. The elements of Br_m can be considered as diffeomorphisms of a punctured disc $D \setminus \{x_1, \dots, x_m\}$ and, in particular, any element $g \in A_0$ can be presented as a half-twist along a simple path l_g connecting some two points, say x_k and x_l , in $(D \setminus \{x_1, \dots, x_m\}) \cup \{x_k, x_l\}$. It follows from the equality $\gamma_m(g) = \gamma_m(z_{k_0, l_0})$ that $\{x_k, x_l\} = \{x_{k_0}, x_{l_0}\}$ and

the paths l_g and $l_{z_{k_0, l_0}}$ are homotopic in the closed disc D as paths with fixed endpoints. Let $h_t : [0, 1] \rightarrow D$, $0 \leq t \leq 1$, be a homotopy connecting l_g and $l_{z_{k_0, l_0}}$, so that $h_0 = l_g$ and $h_1 = l_{z_{k_0, l_0}}$.

Without loss of generality, we can assume that for all t the paths $h_t([0, 1])$ are simple arcs and there is a finite set $T = \{t_1 < \dots < t_q\}$ of values $t \in (0, 1)$ such that the image of h_t remains disjoint from $\{x_1, \dots, x_m\} \setminus \{x_{k_0}, x_{l_0}\}$ for all $t \notin T$, and passes through exactly one of those points, say x_{j_i} , for each $t_i \in T$. Denote by L_i the part of the path $h_{t_i}([0, 1])$ connecting the points x_{k_0} and x_{j_i} . Then it is easy to see that the paths $h_{t^+}([0, 1])$, $t_i < t^+ < t_{i+1}$ are homotopic in $D \setminus \{x_1, \dots, x_m\}$ to the paths obtained from $h_{t^-}([0, 1])$, $t_{i-1} < t^- < t_i$ by either the full-twist along the path L_i or the inverse transformation (depending on whether the map $(t, s) \mapsto h_t(s)$ preserves or reverses orientation near x_{j_i}). Since crossing each value $t_i \in T$ results in conjugation by a pure braid, we conclude that there exists $p \in P_m$ such that $g = pz_{k_0, l_0}p^{-1}$.

To prove that p can be chosen so that it can be represented as a positive word in the alphabet $\{z_{i,j}^2\}$, recall that

$$\Delta_m^2 = \prod_{l=m}^2 \prod_{k=1}^{l-1} z_{k,l}^2 \quad (1)$$

is a generator of the center of the braid group Br_m . Therefore a cyclic permutation of the factors entering in (1) does not change the product. Thus, for any z_{i_0, j_0}^2 , Δ_m^2 can be presented as $\Delta_m^2 = z_{i_0, j_0}^2 \Delta'_{i_0, j_0}$, where Δ'_{i_0, j_0} is a positive word in letters $z_{i,j}^2$. This completes the proof of Lemma 3.6, since

$$z_{i_0, j_0}^{-2} b z_{i_0, j_0}^2 = \Delta'_{i_0, j_0} b (\Delta'_{i_0, j_0})^{-1}$$

for all $b \in Br_m$. □

Consider semigroups $\mathcal{A}_k^0 = S_{\bigcup_{i=0}^k A_i}$ and $\mathcal{A}_k = S_{A_{\bar{1}} \cup (\bigcup_{i=0}^k A_i)}$, $k \geq 0$. We have a natural embedding $\mathcal{A}_k^0 \subset \mathcal{A}_k$. Note that if $g \in A_1$ then $g^{-1} \in A_{\bar{1}}$. For $k \in \mathbb{N} \cup \{\infty\}$, denote by

$$\overline{\mathcal{A}}_k = \langle x \in A_{\bar{1}} \cup \bigcup_{i=0}^k A_i \mid R \in \mathcal{R}_k \cup \overline{\mathcal{R}} \rangle,$$

where \mathcal{R}_k is the set of relations defining \mathcal{A}_k and

$$\overline{\mathcal{R}} = \langle g \cdot (g^{-1}) = (g^{-1}) \cdot g = \mathbf{1} \mid g \in A_1 \rangle$$

is the set of cancellation relations. There is a natural homomorphism of semigroups

$$c : \mathcal{A}_k \rightarrow \overline{\mathcal{A}}_k.$$

We say that two elements $s_1, s_2 \in \mathcal{A}_k$ are *weakly equivalent* if $c(s_1) = c(s_2)$. It is easy to see that $\overline{\mathcal{A}}_k$ can be considered as a semigroup over Br_m with the natural product homomorphism $\overline{\alpha} : \overline{\mathcal{A}}_k \rightarrow Br_m$ such that $\alpha = \overline{\alpha} \circ c$.

The existence of natural embeddings of $S_{A_{\bar{1}}}$ and S_{A_j} ($j = 0, \dots, k$) into \mathcal{A}_k implies that any element $s \in \mathcal{A}_k$ can be written as

$$s = s_{\bar{1}} \cdot s_0 \cdot \dots \cdot s_k,$$

where each $s_j \in S_{A_j}$ is either a product $s_j = g_{j,1} \cdot \dots \cdot g_{j,d_j}$ of elements $g_{j,i} \in Br_m$ conjugated to a_1^{j+1} (resp., to a_1^{-2} for $s_{\bar{1}}$) or the empty word. The number $d(s_j) = d_j$ ($d_j = 0$ if s_j is the empty word) is called the *degree* of s_j in S_{A_j} , $d(s) = (d_{\bar{1}}, d_0, \dots, d_k)$ is called the *multi-degree* of s , and the vector $d(c(s)) = (d_0, d_1 - d_{\bar{1}}, d_2, \dots, d_k)$ is called the *multi-degree* of $c(s) \in \overline{\mathcal{A}}_k$.

3.4. Braid monodromy factorizations of Hurwitz curves with

A-singularities. Let \bar{H} be a **pr**-generic Hurwitz curve with A -type singularities (and possibly negative nodes) of degree m in \mathbf{F}_N . The classical concept of *braid monodromy factorization* of \bar{H} was given a precise definition in [Kh-Ku] using the language of factorization semigroups: a braid monodromy factorization of \bar{H} is an element $\text{bmf}(\bar{H}) \in \mathcal{A}_k \subset \mathcal{A}_\infty$ (for some $k \geq 0$), whose factors describe the monodromy of **pr** around the various points of $Z(\bar{H})$, with the property that $\alpha(\text{bmf}(\bar{H})) = \Delta_m^{2N}$. If two elements $s_1, s_2 \in \mathcal{A}_k$ are braid monodromy factorizations of the same Hurwitz curve \bar{H} , then they are conjugated, i.e., there exists $g \in Br_m$ such that $s_2 = \lambda(g)(s_1)$. A braid monodromy factorization $\text{bmf}(\bar{H})$ of a **pr**-generic Hurwitz curve \bar{H} with A -singularities can be written as $\text{bmf}(\bar{H}) = s_{\bar{1}} \cdot s_0 \cdot s_1 \cdot \dots \cdot s_k$ with

$$s_{\bar{1}} = \prod_{i=1}^{d_{\bar{1}}} (q_{\bar{1},i} a_1^{-2} q_{\bar{1},i}^{-1}) \in S_{\bar{1}}, \quad \text{and} \quad s_j = \prod_{i=1}^{d_j} (q_{j,i} a_1^{j+1} q_{j,i}^{-1}) \in S_{A_j}. \quad (2)$$

The index j (resp., $\bar{1}$) indicates the type of singularity: the factors in $s_{\bar{1}}$ are monodromies around the negative nodes, those in s_0 correspond to the vertical tangency points (i.e., given by $y^2 = x$ in local coordinates), and those in s_j , $j \geq 1$, are the singularities of type A_j .

The converse statement can be also proved in a straightforward way.

Theorem 3.7. ([Moi]) *For any $s \in \mathcal{A}_\infty$ such that $\alpha(s) = \Delta^{2N}$ there exists a Hurwitz curve $\bar{H} \subset \mathbf{F}_N$ with $\text{bmf}(\bar{H}) = s$.*

Recall also the following statements.

Theorem 3.8. ([Ku-Te], [Kh-Ku]) *Two pr-generic curves with A-type singularities \bar{H}_1, \bar{H}_2 in \mathbf{F}_N are H-isotopic if and only if $\text{bmf}(\bar{H}_1) = \text{bmf}(\bar{H}_2)$. If \bar{H}_1, \bar{H}_2 are symplectic surfaces then this isotopy can be chosen symplectic.*

Proposition 3.9. ([Kh-Ku]) *For any element $s \in \mathcal{A}_\infty^0$ such that $\alpha(s) = \Delta_m^{2N}$ there is an almost algebraic curve $\bar{C} \subset \mathbf{F}_N$ with $\text{bmf}(\bar{C}) = s$.*

The following theorem is a generalization of Theorem 3.5:

Theorem 3.10. *The element $(\tilde{\delta}_m^2)^N$ is the only element $s \in S_{A_1}$ such that $\alpha(s) = \Delta_m^{2N}$.*

Proof. As in the proof of Theorem 3.5, one can show that the orbit of $(\tilde{\delta}_m^2)^N$ under the conjugation action of Br_m on S_{A_1} consists of the single element $(\tilde{\delta}_m^2)^N$ (see [Kh-Ku], Proposition 1.2).

Consider an element $s \in S_{A_1}$ such that $\alpha(s) = \Delta_m^{2N}$. By Proposition 3.9, there is an almost algebraic curve $\bar{H} \subset \mathbf{F}_N$ with $\text{bmf}(\bar{H}) = s$. After rescaling, we can assume that \bar{H} is a symplectic curve which coincides with an algebraic curve over a disc $D \subset \mathbb{P}^1$. We can change the complex structure on \mathbf{F}_N over the complement $\mathbb{P}^1 \setminus D$ so that it coincides with the standard complex structure near the exceptional section E_N and \bar{H} becomes a pseudoholomorphic curve. By Theorem 1.3, the curve \bar{H} is symplectically isotopic to an algebraic curve \bar{C} , with $\text{bmf}(\bar{C}) = s$, since the isotopy preserves braid monodromy factorization (recall that each irreducible component is a section of \mathbf{F}_N).

This isotopy result implies the uniqueness of s up to conjugation by an element of Br_m ; hence s is conjugated to $(\tilde{\delta}_m^2)^N$, and therefore, by the observation made at the beginning of the proof, $s = (\tilde{\delta}_m^2)^N$. Alternatively, one can also check directly that $\text{bmf}(\bar{C}) = (\tilde{\delta}_m^2)^N$. Indeed, \bar{C} is the union of m holomorphic sections of \mathbf{F}_N in generic position; the calculation is the same as for a generic configuration of m lines in $\mathbb{C}\mathbb{P}^2$ (or can be reduced to that case by viewing \mathbf{F}_N as a fiber sum of N copies of \mathbf{F}_1 which is $\mathbb{C}\mathbb{P}^2$ with one point blown up). \square

Lemma 3.11. *Let \bar{H} be a Hurwitz curve in \mathbf{F}_N and $p_+, p_- \in Z(\bar{H})$ two nodal singular points, such that the fibers of \mathbf{F}_N through p_\pm contain no other points from $Z(\bar{H})$. Assume that there exists a simple smooth arc $\gamma \subset \mathbb{P}^1$ with the following properties:*

- (1) *The endpoints of γ are the projections $q_\pm := \text{pr}(p_\pm)$ of the nodal points p_\pm ;*
- (2) *The interior of γ does not meet the branch locus $\text{pr}(Z(\bar{H}))$ of \bar{H} ;*

- (3) Let $q_0 \in \mathbb{P}^1 \setminus Z(\bar{H})$ be a base point, F_0 the fiber of \mathbf{F}_N over q_0 , $\gamma_- \subset \mathbb{P}^1 \setminus Z(\bar{H})$ a smooth immersed path from q_0 to q_- , $\gamma_+ \subset \mathbb{P}^1 \setminus Z(\bar{H})$ the path from q_0 to q_+ obtained as the composition of γ_- and γ , and μ_{\pm} the braid monodromies of \bar{H} at p_{\pm} along γ_{\pm} in the braid group $Br_m(F_0)$; then $\mu_+ \cdot \mu_- = 1 \in Br_m(F_0)$.

Then one can cancel the pair of nodal points (p_+, p_-) by a regular homotopy ϕ_t constant outside any given neighborhood $U \subset F_N$ of $\text{pr}^{-1}(\gamma)$.

Proof. Set $X_{\gamma} := \text{pr}^{-1}(\gamma)$, $H_{\gamma} := \bar{H} \cap X_{\gamma}$, $F_q := \text{pr}^{-1}(q)$, $H_q := \bar{H} \cap F_q$, and $\gamma^{\circ} := \gamma \setminus \{q_-, q_+\}$. Then X_{γ} is diffeomorphic to $\gamma \times F_0$ and $H_{\gamma} \subset X_{\gamma}$ consists of m smooth arcs, which we denote by $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m$. The assumptions imply that two of these arcs, say $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have common endpoints p_-, p_+ , at which they meet transversally, and the remaining $m - 2$ are embedded and disjoint from each other and from $\tilde{\gamma}_1, \tilde{\gamma}_2$.

Fixing a metric on \mathbf{F}_N , and considering the induced metrics on each fiber F_q , we can construct families $\delta_{\pm}(q)$ of arcs which depend smoothly on $q \in \gamma^{\circ}$ and have the following properties:

- (1) each $\delta_{\pm}(q)$ is a smooth embedded arc in $F_q \setminus H_q$ with endpoints $\tilde{\gamma}_1(q) := \tilde{\gamma}_1 \cap F_q$ and $\tilde{\gamma}_2(q) := \tilde{\gamma}_2 \cap F_q$;
- (2) for q close to q_{\pm} , the arc $\delta_{\pm}(q)$ is the geodesic in F_q connecting $\tilde{\gamma}_1(q)$ with $\tilde{\gamma}_2(q)$.

The monodromy hypothesis (3) implies that for every $q \in \gamma^{\circ}$ the arcs $\delta_+(q)$ and $\delta_-(q)$ are isotopic in F_q relative to their endpoints. Consequently, we may assume that $\delta_+(q) = \delta_-(q) =: \delta(q)$ for all $q \in \gamma^{\circ}$. Now we see that the union $\sqcup_{q \in \gamma^{\circ}} \delta(q) \sqcup \{p_-, p_+\}$ forms an embedded Whitney disk D with the following properties:

- (1) D is the image of a smooth embedding $f : \Delta \rightarrow \mathbf{F}_n$ of the disc $\Delta := \{(x, u) \in \mathbb{R}^2 : x \in [-1, 1]; -(1 - x^2) \leq u \leq 1 - x^2\}$;
- (2) the corner points p_{\pm} of D are the images of the corner points $(\pm 1, 0)$ of Δ ;
- (3) the images of the boundary arcs $I_{\pm} := \{(x, u) : x \in]-1, 1[, u = \pm(1 - x^2)\}$ of Δ are the arcs $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

This makes it possible to cancel the nodes p_+ and p_- along the Whitney disc D . \square

Definition 3.12. *The construction described in the proof of the lemma is called the cancellation of the pair of nodes p_-, p_+ along the path γ .*

We leave to the reader the proof of the fact that the cancellation of a pair of nodes along a path γ —if it is possible—depends only on

the isotopy class of γ relative to its endpoints. Observe that the monodromy condition is necessary and sufficient for the cancellability of a pair of nodes.

It follows from Lemma 3.11 that if a Hurwitz surface \bar{H}_2 is obtained from a Hurwitz surface \bar{H}_1 by the creation of a pair of nodes, then

$$\text{bmf}(\bar{H}_2) = \text{bmf}(\bar{H}_1) \cdot g \cdot (g^{-1}),$$

where g is some element belonging to A_1 . Therefore, we obtain the following statement.

Claim 3.13. *Two pr-generic curves $\bar{H}_1, \bar{H}_2 \subset \mathbf{F}_N$ with A -singularities are regular homotopic in the class of Hurwitz curves with A -singularities and negative nodes if and only if*

$$c(\text{bmf}(\bar{H}_1)) = c(\text{bmf}(\bar{H}_2)).$$

4. REGULAR HOMOTOPY

4.1. Weak equivalence. Let $s \in \mathcal{A}_k$. Consider the subgroup $(Br_m)_s$ of Br_m generated by the factors of s , and denote by $\gamma_s : (Br_m)_s \rightarrow \Sigma_m$ the restriction of $\gamma_m : Br_m \rightarrow \Sigma_m$ to $(Br_m)_s$. For a given $d = (d_{\bar{1}}, d_0, \dots, d_k)$ and $b \in Br_m$ set

$$\mathcal{A}_k(d, b) = \langle s \in \mathcal{A}_k \mid d(s) = (d_{\bar{1}}, d_0, \dots, d_k), \alpha(s) = b \rangle.$$

Theorem 4.1.

- (i) *For any multi-degree $d = (d_{\bar{1}}, d_0, \dots, d_k)$ and for any $b \in Br_m$, the set $c(\mathcal{A}_k(d, b))$ consists of a finite number of elements.*
- (ii) *Let s', s'' be two elements in $\mathcal{A}_k(d, \Delta^{2N})$, $N \in \mathbb{N}$, such that $\gamma_{s'}((Br_m)_{s'}) = \gamma_{s''}((Br_m)_{s''}) = \Sigma_m$. Then s' and s'' are weakly equivalent.*

Proof. Consider an element $s \in \mathcal{A}_k(d, b)$. It can be represented as a product $s = s_{\bar{1}} \cdot s_0 \cdot \dots \cdot s_k$, where $s_i = g_{i,1} \cdot \dots \cdot g_{i,d_i} \in S_{A_i}$ and all $g_{i,j} \in A_i$, i.e., all $g_{i,j}$ are conjugated to a_1^{i+1} (or a_1^{-2} in the case of $g_{\bar{1},j}$; to simplify the notation, we assume that $d_i \geq 1$ for all i , the adaptation to the case $d_i = 0$ is trivial and left to the reader). By Lemma 3.6, there exists $p_{i,j} \in P_m$ such that $g_{i,j} = p_{i,j}^{-1} z_{k_{i,j}, l_{i,j}}^{i+1} p_{i,j}$ for some $z_{k_{i,j}, l_{i,j}}$. For a fixed presentation of s as the product $s = s_{\bar{1}} \cdot s_0 \cdot \dots \cdot s_k$ with $s_i = g_{i,1} \cdot \dots \cdot g_{i,d_i} \in S_{A_i}$, put $\bar{s}_i = (k_{i,1}, l_{i,1}) \cdot \dots \cdot (k_{i,d_i}, l_{i,d_i}) \in S_{\Sigma_m}$.

Claim 4.2. *Let s', s'' be two elements in $\mathcal{A}_k(d, b)$ such that for some factorizations $s' = s'_{\bar{1}} \cdot s'_0 \cdot \dots \cdot s'_k$ and $s'' = s''_{\bar{1}} \cdot s''_0 \cdot \dots \cdot s''_k$ the ordered collections $\bar{s}' = (\bar{s}'_{\bar{1}}, \bar{s}'_0, \dots, \bar{s}'_k)$ and $\bar{s}'' = (\bar{s}''_{\bar{1}}, \bar{s}''_0, \dots, \bar{s}''_k)$ of elements from S_{Σ_m} coincide. Then s' and s'' are weakly equivalent.*

Proof. As above, let $s'_i = g'_{i,1} \cdots g'_{i,d_i} \in S_{A_i}$, where all $g'_{i,j} \in A_i$, and $s''_i = g''_{i,1} \cdots g''_{i,d_i} \in S_{A_i}$ where $g''_{i,j} \in A_i$. Since $\bar{s}' = \bar{s}''$, up to Hurwitz equivalence we can assume that $g'_{i,j}$ and $g''_{i,j}$ are conjugated by pure braids to powers of a same element $z_{k_{i,j},l_{i,j}}$. So, by Lemma 3.6, for all i, j there exists $p_{i,j} \in P_m$ such that $g''_{i,j} = p_{i,j} g'_{i,j} p_{i,j}^{-1}$. Moreover, we can assume that each $p_{i,j} = \prod_{q=1}^{M_{i,j}} z_{k_{q,i,j},l_{q,i,j}}^2$. Put $M = \sum_i \sum_j M_{i,j}$.

Consider

$$\tilde{\delta}_m^{-2} = \prod_{l=2}^m \prod_{k=l-1}^1 z_{k,l}^{-2} \in S_{A_1}.$$

Note that $c(\tilde{\delta}_m^2 \cdot \tilde{\delta}_m^{-2}) = \mathbf{1}$ in $\overline{\mathcal{A}}_k$ for any $k \geq 1$. Therefore for any $s \in \mathcal{A}_k$ the elements s and $s \cdot (\tilde{\delta}_m^2 \cdot \tilde{\delta}_m^{-2})^M$ are weakly equivalent.

It follows from Claim 3.1 that $\tilde{\delta}_m^2$ commutes with all elements of S_{Br_m} . Therefore, after a suitable sequence of Hurwitz moves $(\tilde{\delta}_m^2)^{M_{i,j}}$ can be rewritten as

$$(\tilde{\delta}_m^2)^{M_{i,j}} = \prod_{q=1}^{M_{i,j}} z_{k_{q,i,j},l_{q,i,j}}^2 \cdot \delta_{i,j}$$

for some $\delta_{i,j} \in S_{A_1}$. Now

$$\begin{aligned} s'' \cdot (\tilde{\delta}_m^2 \cdot \tilde{\delta}_m^{-2})^M &= \left(\prod_i \prod_{j=1}^{d_i} g''_{i,j} \right) \cdot (\tilde{\delta}_m^2 \cdot \tilde{\delta}_m^{-2})^M \\ &= \left(\prod_i \prod_{j=1}^{d_i} g''_{i,j} \cdot (\tilde{\delta}_m^2)^{M_{i,j}} \right) \cdot (\tilde{\delta}_m^{-2})^M \\ &= \left(\prod_i \prod_{j=1}^{d_i} \left(p_{i,j} g'_{i,j} p_{i,j}^{-1} \cdot \prod_{q=1}^{M_{i,j}} z_{k_{q,i,j},l_{q,i,j}}^2 \cdot \delta_{i,j} \right) \right) \cdot (\tilde{\delta}_m^{-2})^M \\ &= \left(\prod_i \prod_{j=1}^{d_i} \left(\prod_{q=1}^{M_{i,j}} z_{k_{q,i,j},l_{q,i,j}}^2 \cdot g'_{i,j} \cdot \delta_{i,j} \right) \right) \cdot (\tilde{\delta}_m^{-2})^M. \end{aligned}$$

In fact, the specific contribution of positive nodes ($i = 1$) can be treated more directly by observing that for any j the element $\tilde{\delta}_m^2$ can be written as $\tilde{\delta}_m^2 = g'_{1,j} \cdot \delta_{1,j}$ (this follows from the conjugation invariance of $\tilde{\delta}_m^2$, see Theorem 3.5), so that $s''_1 \cdot (\tilde{\delta}_m^2)^{d_1} = s''_1 \cdot \prod_{j=1}^{d_1} (g'_{1,j} \cdot \delta_{1,j})$; this allows us to decrease M to $\sum_{i \neq 1} \sum_j M_{i,j} + d_1$.

Applying the relations $a \cdot b = b \cdot r(a, b)$, we can move $g'_{i,j}$ to the left and obtain that

$$s'' \cdot (\tilde{\delta}_m^2 \cdot \tilde{\delta}_m^{-2})^M = s' \cdot \delta' \cdot (\tilde{\delta}_m^{-2})^M$$

for some $\delta' \in S_{A_1}$. We have $\alpha(\delta') = \Delta^{2M}$, since $\alpha(s') = \alpha(s'')$. It follows from Theorem 3.10 that $\delta' = (\tilde{\delta}_m^2)^M$. Therefore, s' and s'' are weakly equivalent. \square

Since the group Σ_m is finite, the set of collections $\bar{s} = (\bar{s}_1, \bar{s}_0, \dots, \bar{s}_k)$ where each \bar{s}_i is a product of d_i transpositions in S_{Σ_m} is also finite. This completes the proof of Theorem 4.1 (i).

To prove Theorem 4.1 (ii), as above, put

$$s' = s'_1 \cdot s'_0 \cdot s'_1 \cdot \dots \cdot s'_k, \quad (3)$$

where $s'_i = g'_{i,1} \cdot \dots \cdot g'_{i,d_i} \in S_{A_i}$ and all $g'_{i,j} \in A_i$, and similarly

$$s'' = s''_1 \cdot s''_0 \cdot s''_1 \cdot \dots \cdot s''_k, \quad (4)$$

where $s''_i = g''_{i,1} \cdot \dots \cdot g''_{i,d_i} \in S_{A_i}$ and each $g''_{i,j} \in A_i$. To prove (ii), it is sufficient to show that s' and s'' have factorizations of the form (3) and (4) such that the collections $\bar{s}' = (\bar{s}'_1, \bar{s}'_0, \dots, \bar{s}'_k)$ and $\bar{s}'' = (\bar{s}''_1, \bar{s}''_0, \dots, \bar{s}''_k)$ of elements of S_{Σ_m} coincide with each other.

Note that the image $\gamma_m(g'_{i,1}) \in \Sigma_m$ (respectively, $\gamma_m(g''_{i,1}) \in \Sigma_m$) is either a transposition if i is even or $\mathbf{1} \in \Sigma_m$ if i is odd. By assumption, the elements $\gamma_m(g'_{i,j})$ (respectively, $\gamma_m(g''_{i,j})$) with even i generate the group Σ_m .

For any $a \in A_i$ and $b \in Br_m$, we have

$$\gamma_m(\rho(a)(b)) = \gamma_m(\lambda(a)(b)) = \gamma_m(a)\gamma_m(b)\gamma_m(a) \quad (5)$$

if i is even and

$$\gamma_m(\rho(a)(b)) = \gamma_m(\lambda(a)(b)) = \gamma_m(b) \quad (6)$$

if i is odd.

If i is odd and r is even, it follows from (5) and (6) that after the following composition of Hurwitz moves

$$\begin{aligned} g_{r,s} \cdot \tilde{s} \cdot g_{i,j} &= \lambda(g_{r,s})(\tilde{s}) \cdot \lambda(g_{r,s})(g_{i,j}) \cdot g_{r,s} = \\ &= \lambda(g_{r,s})(\tilde{s}) \cdot \lambda(\lambda(g_{r,s})(g_{i,j}))(g_{r,s}) \cdot \lambda(g_{r,s})(g_{i,j}) = \\ &= \lambda(\lambda(g_{r,s})(g_{i,j}))(g_{r,s}) \cdot \rho(\lambda(\lambda(g_{r,s})(g_{i,j}))(g_{r,s}))(\lambda(g_{r,s})(\tilde{s})) \cdot \lambda(g_{r,s})(g_{i,j}), \end{aligned}$$

we will have

- (i) $\gamma_m(\lambda(\lambda(g_{r,s})(g_{i,j}))(g_{r,s})) = \gamma_m(g_{r,s})$;
- (ii) $\gamma_m(\rho(\lambda(\lambda(g_{r,s})(g_{i,j}))(g_{r,s}))(\lambda(g_{r,s})(\tilde{s}_t))) = \gamma_m(\tilde{s}_t)$ for each factor \tilde{s}_t of \tilde{s} ;
- (iii) $\gamma_m(\lambda(g_{r,s})(g_{i,j})) = \gamma_m(g_{r,s})\gamma_m(g_{i,j})\gamma_m(g_{r,s})$.

Therefore, since the elements $\gamma_m(g'_{r,s})$ with even r generate the group Σ_m , after a finite sequence of such Hurwitz moves we can obtain factorizations of s' and s'' such that $\bar{s}'_i = \bar{s}''_i$ for odd i . After that, we move all factors s'_i entering in s' for odd i (and, similarly, all factors s''_i entering in s'' for odd i) to the left by Hurwitz moves.

Now, to prove that there exist factorizations of s' and s'' with $\bar{s}'_i = \bar{s}''_i$ for even i , note that s' and s'' are braid monodromy factorizations of two irreducible Hurwitz curves with A -singularities $f_j : \mathcal{S}_j \rightarrow \bar{H}_j \subset \mathbf{F}_N \setminus E_N$, $j = 1, 2$, of degree m in \mathbf{F}_N (irreducibility follows from the assumption that $\gamma_{s'}((Br_m)_{s'}) = \gamma_{s''}((Br_m)_{s''}) = \Sigma_m$). Since s' and s'' have the same multi-degree, \mathcal{S}_1 and \mathcal{S}_2 have the same geometric genus g . Applying the Hurwitz theorem to the morphisms $\mathbf{pr} \circ f_j$, we have

$$\tilde{\gamma}_m(s') = \tilde{\gamma}_m(s'') = h_g \in S_{\Sigma_m},$$

where h_g is the Hurwitz element of genus g and $\tilde{\gamma}_m : \mathcal{A}_\infty \rightarrow S_{\Sigma_m}$ is the homomorphism induced by the natural epimorphism $\gamma_m : Br_m \rightarrow \Sigma_m$. To go further, we need to assign markings by integers to the various factors in $\tilde{\gamma}_m(s')$ and $\tilde{\gamma}_m(s'')$. Each factor $\gamma_m(g'_{i,j})$ of h_g corresponding to $g'_{i,j}$ with even i is marked by the integer i , and similarly for $\gamma_m(g''_{i,j})$. Then, to prove (ii) of Theorem 4.1, we must show that $\tilde{\gamma}_m(s') = \tilde{\gamma}_m(s'')$ as products with marked factors. Thus, Theorem 4.1 (ii) follows from

Claim 4.3. *Two marked factorizations of h_g with coinciding sets of marks are Hurwitz equivalent.*

Proof. It is sufficient to show that

$$(i, i+1)_{j_1} \cdot (i+1, i+2)_{j_2} = (i, i+1)_{j_2} \cdot (i+1, i+2)_{j_1}$$

For this let us perform the Hurwitz move $a \cdot b \mapsto (aba^{-1}) \cdot a$ three times. Then we obtain

$$\begin{aligned} (i, i+1)_{j_1} \cdot (i+1, i+2)_{j_2} &= (i, i+2)_{j_2} \cdot (i, i+1)_{j_1} = \\ (i+1, i+2)_{j_1} \cdot (i, i+2)_{j_2} &= (i, i+1)_{j_2} \cdot (i+1, i+2)_{j_1} \end{aligned}$$

□

4.2. Proof of the main results. Theorem 0.1 follows from Theorems 3.7, 3.8, Claim 3.13, and Theorem 4.1. The assumption that the curves \bar{H}_i are irreducible implies that the images by γ_m of the factors in $\text{bmf}(\bar{H}_i)$ are transpositions acting transitively on $[1, m]$, and hence generate Σ_m as required in order to apply Theorem 4.1 (ii).

To prove Corollary 0.2, observe that, for any cuspidal symplectic surface C in (\mathbb{P}^2, ω) , of degree m , and pseudoholomorphic with respect to some ω -tamed almost-complex structure J , it is possible to define a braid monodromy factorization with respect to a generic pencil of J -lines (see Section 4 of [Kh-Ku]); by deforming the almost-complex structure to the standard one, the surface C is symplectically C^1 -smoothly isotopic in \mathbb{P}^2 to a symplectic surface C' which becomes a Hurwitz curve in \mathbf{F}_1 after blowing up a point in \mathbb{P}^2 . Therefore we can apply Theorem 0.1.

4.3. An alternative proof. We now describe a more geometric approach to the proof of Theorem 0.1.

Step 1. As above, we first reduce to the case where the combinatorics of the branched coverings $\mathbf{pr}_0 = \mathbf{pr}|_{\bar{H}_0} : \bar{H}_0 \rightarrow \mathbb{P}^1$ and $\mathbf{pr}_1 = \mathbf{pr}|_{\bar{H}_1} : \bar{H}_1 \rightarrow \mathbb{P}^1$ are the same. More precisely, let $Z^0(\bar{H}_j)$ be the set of all points of $Z(\bar{H}_j)$ other than nodes of any orientation, and let $\Delta_j = \mathbf{pr}(Z^0(\bar{H}_j))$. Because the numbers of singular points of each type coincide, there exists a diffeomorphism of pairs $\beta : (\mathbb{P}^1, \Delta_0) \rightarrow (\mathbb{P}^1, \Delta_1)$ which maps the projections of the A_i -singularities of \bar{H}_0 to the projections of the A_i -singularities of \bar{H}_1 for all i . Also choose an identification between the fibers of \mathbf{pr}_0 and \mathbf{pr}_1 above the base point in \mathbb{P}^1 .

Then the irreducibility of \bar{H}_0 and \bar{H}_1 makes it possible to modify the diffeomorphism β in such a way that the monodromy of \bar{H}_0 along any closed loop $\gamma \in \pi_1(\mathbb{P}^1 - \Delta_0)$ involves the same sheets of the m -fold covering \mathbf{pr}_0 as the monodromy of \bar{H}_1 along the closed loop $\beta(\gamma) \in \pi_1(\mathbb{P}^1 - \Delta_1)$. In particular, if we consider only the A_i -singularities with even i , this means that the symmetric group-valued monodromies $\psi_j : \pi_1(\mathbb{P}^1 - \Delta_j) \rightarrow \Sigma_m$ of the m -fold coverings \mathbf{pr}_0 and \mathbf{pr}_1 satisfy the identity $\psi_1 \circ \beta_* = \psi_0$. The argument, which involves composing β with a suitable sequence of elementary braids, is essentially identical to that given in Section 4.1 to prove that the ordered collections \bar{s}' and \bar{s}'' can be assumed to coincide, and so we will not repeat it; once again the crucial ingredient is a Hurwitz theorem for marked factorizations in the symmetric group (Claim 4.3).

By performing a suitable H -isotopy on \bar{H}_0 (obtained by lifting a homotopy between Id and β among diffeomorphisms of \mathbb{P}^1), we can restrict ourselves to the case where $\beta = \text{Id}$.

Step 2. Because the branching data for \mathbf{pr}_0 and \mathbf{pr}_1 agree, there exists a Riemann surface S and a covering map $f : S \rightarrow \mathbb{P}^1$, with simple branching above the points of $\Delta_0 = \Delta_1$ corresponding to A_i -singularities with even i , such that \bar{H}_0 and \bar{H}_1 are the pushforwards via f of the graphs of smooth sections σ_0 and σ_1 of the line bundle $L = f^*(\mathcal{O}_{\mathbb{P}^1}(N))$ over S . Choose local holomorphic coordinates on \mathbb{P}^1 and local holomorphic trivializations of $\mathcal{O}_{\mathbb{P}^1}(N)$ over small neighborhoods of every point of Δ_0 , and lift them to local holomorphic coordinates on S and trivializations of L near the points of $f^{-1}(\Delta_0)$.

For each $p \in \Delta_0$ corresponding to an A_i -singularity with i even, there exists a unique branch point of f , say $\bar{p} \in S$, such that $f(\bar{p}) = p$. By construction the sections of L obtained by lifting Hurwitz curves are holomorphic over a neighborhood of \bar{p} . A Hurwitz curve $\bar{H} = f_*(\text{graph}(\sigma))$ has an A_i -singularity above p if and only if the jet of

σ at \bar{p} (its power series expansion in the local coordinates) has all odd degree terms vanishing up to degree $i - 1$, and has a non-zero coefficient in its degree $i + 1$ term. Define $\mathcal{C}_p \subset C^\infty(L)$ to be the space of smooth sections of L which are holomorphic over a fixed given neighborhood of \bar{p} and satisfy $\sigma^{(k)}(\bar{p}) = 0$ for all odd $k = 1, \dots, i - 1$, and let $\mathcal{B}_p = \{\sigma \in \mathcal{C}_p, \sigma^{(i+1)}(\bar{p}) \neq 0\}$.

For each $p \in \Delta_0$ corresponding to an A_i -singularity with i odd, there exist two points $p', p'' \in f^{-1}(p)$ such that $\sigma(p') = \sigma(p'')$. To make sense of this equality, recall that $L = f^*O_{\mathbb{P}^1}(N)$, so that the fibers of L at p' and p'' are canonically isomorphic (the same is true over neighborhoods of p' and p''). The sections of L obtained by lifting Hurwitz curves are holomorphic over neighborhoods of p', p'' , and the presence of an A_i -singularity with $i = 2r + 1 \geq 3$ means that the jets of σ at p' and p'' coincide up to order r and differ at order $r + 1$. Let $\mathcal{C}_p \subset C^\infty(L)$ be the space of smooth sections of L which are holomorphic over fixed given neighborhoods of p' and p'' and satisfy $\sigma^{(k)}(p') = \sigma^{(k)}(p'')$ for all $0 \leq k \leq r$, and let $\mathcal{B}_p = \{\sigma \in \mathcal{C}_p, \sigma^{(r+1)}(p') \neq \sigma^{(r+1)}(p'')\}$.

Finally, let $\mathcal{C} = \bigcap_{p \in \Delta_0} \mathcal{C}_p$ and $\mathcal{B} = \bigcap_{p \in \Delta_0} \mathcal{B}_p$. Note that \mathcal{C} is an infinite-dimensional complex vector space, and \mathcal{B} is the complement of a finite union of hyperplanes in \mathcal{C} . By construction the sections σ_0 and σ_1 describing \bar{H}_0 and \bar{H}_1 belong to \mathcal{B} .

Lemma 4.4. *\mathcal{B} is path-connected.*

Proof. Given $\sigma \in \mathcal{B}$ and a point $p \in \Delta_0$, define the phase $\varphi_\sigma(p) \in \mathbb{R}/2\pi\mathbb{Z}$ by the formulas $\varphi_\sigma(p) = \arg(\sigma^{(i+1)}(\bar{p}))$ if p is an A_i -singularity with i even, and $\varphi_\sigma(p) = \arg(\sigma^{(r+1)}(p'') - \sigma^{(r+1)}(p'))$ if p is an A_i -singularity with $i = 2r + 1$ odd.

Given any $\sigma_0, \sigma_1 \in \mathcal{B}$, choose θ_0 distinct from $\pi + \varphi_{\sigma_0}(p) - \varphi_{\sigma_1}(p)$ for all $p \in \Delta_0$, and let $\bar{\sigma}_0 = e^{i\theta_0}\sigma_0$. Considering the arc $\{e^{i\theta}\sigma_0, \theta \in [0, \theta_0]\}$, it is easy to see that σ_0 and $\bar{\sigma}_0$ belong to the same path-connected component of \mathcal{B} . Finally, since the phases of $\bar{\sigma}_0$ and σ_1 are never the opposite of each other, the line segment $\{t\bar{\sigma}_0 + (1 - t)\sigma_1, t \in [0, 1]\}$ is entirely contained in \mathcal{B} . \square

Step 3. To conclude the argument, consider a homotopy σ_t between σ_0 and σ_1 in \mathcal{B} . For generic t , the pushforward of the graph of σ_t is a Hurwitz curve \bar{H}_t presenting, in addition to the non-nodal singularities of \bar{H}_0, \bar{H}_1 , an unspecified number of nodes of either orientation. These correspond to pairs of points q, q' belonging to the same fiber of $f : S \rightarrow \mathbb{P}^1$, distinct from any of the previously considered points in $f^{-1}(\Delta_0)$, and such that $\sigma_t(q) = \sigma_t(q')$. Self-intersections of higher order may also occur sporadically in the homotopy $\{\bar{H}_t\}_{t \in [0, 1]}$, when

the coincidence between $\sigma_t(q)$ and $\sigma_t(q')$ extends to higher-order terms in the jets; however, by Lemma 2.8 they can always be removed by an arbitrarily small perturbation of \bar{H}_t , except in the case of creations and cancellations of pairs of nodes. Hence we may assume that $\{\bar{H}_t\}$ is a regular homotopy.

5. AN EXAMPLE OF A BRAID HAVING TWO INEQUIVALENT FACTORIZATIONS IN S_{A_0}

In this section we show that the Generalized Garside Problem has a negative solution for $m \geq 4$, i.e., the product homomorphism $\alpha : S_{A_0(m)} \rightarrow Br_m$ is not an embedding for $m \geq 4$.

Proposition 5.1. *There are two elements $s_1, s_2 \in S_{A_0(4)}$ such that $s_1 \neq s_2$, but $\alpha(s_1) = \alpha(s_2) \in Br_4$.*

Proof. Put $a = a_1, b = a_2, c = a_3$, where $\{a_1, a_2, a_3\}$ is a set of standard generators of Br_4 . Consider the elements

$$\begin{aligned} s_1 &= (c^{-2}b)a(c^{-2}b)^{-1} \cdot b \cdot (ac^3)b(ac^3)^{-1} \cdot (ac^5b^{-1})a(ac^5b^{-1})^{-1} \cdot c \cdot c, \\ s_2 &= b \cdot (ac)b(ac)^{-1} \cdot (ac)b(ac)^{-1} \cdot (ac)^2b(ac)^{-2} \cdot (ac)^2b(ac)^{-2} \cdot (ac)^3b(ac)^{-3}. \end{aligned}$$

We have $s_1, s_2 \in S_{A_0(4)}$. Let us show that $\alpha(s_1) = \alpha(s_2)$. Indeed,

$$\begin{aligned} \alpha(s_1) &= (c^{-2}b)a(c^{-2}b)^{-1}b(ac^3)b(ac^3)^{-1}(ac^5b^{-1})a(ac^5b^{-1})^{-1}c^2 \\ &= c^{-2}bab^{-1}c(cbc)ac(cbc)cb^{-1}aba^{-1}c^{-3} \\ &= c^{-2}ba(b^{-1}cbcb)ac(bc bcb^{-1})aba^{-1}c^{-3} \\ &= c^{-2}ba(cbb)ac(bbc)aba^{-1}c^{-3} \\ &= c^{-2}(bacb)^3a^{-1}c^{-3}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \alpha(s_2) &= b(ac)b(ac)^{-1}(ac)b(ac)^{-1}(ac)^2b(ac)^{-2}(ac)^2b(ac)^{-2}(ac)^3b(ac)^{-3} = \\ &= bacbbacbacbbac b(ac)^{-3} = (bacb)^3a^{-3}c^{-3}, \end{aligned}$$

and to prove that $\alpha(s_1) = \alpha(s_2) \in Br_4$, it is sufficient to check that $c(bacb) = (bacb)a$ and $a(bacb) = (bacb)c$. We have

$$\begin{aligned} (bacb)a &= bcaba = bcbab = cbcab = c(bacb); \\ a(bacb) &= babcb = (bacb)c. \end{aligned}$$

To prove that $s_1 \neq s_2$, consider the homomorphism $\tilde{\gamma}_4 : S_{Br_4} \rightarrow S_{\Sigma_4}$ determined by the natural epimorphism $\gamma_4 : Br_4 \rightarrow \Sigma_4$. We have

$$\gamma_4(a) = (1, 2), \quad \gamma_4(b) = (2, 3), \quad \gamma_4(c) = (3, 4).$$

Therefore,

$$\begin{aligned} \tilde{\gamma}_4(s_1) &= \\ \tilde{\gamma}_4((c^{-2}b)a(c^{-2}b)^{-1} \cdot b \cdot (ac^3)b(ac^3)^{-1} \cdot (ac^5b^{-1})a(ac^5b^{-1})^{-1} \cdot c \cdot c) &= \\ \tilde{\gamma}_4(bab^{-1} \cdot b \cdot (ac)b(ac)^{-1} \cdot (acb)a(acb)^{-1} \cdot c \cdot c) &= \\ (1, 3) \cdot (2, 3) \cdot (1, 4) \cdot (2, 4) \cdot (3, 4) \cdot (3, 4) & \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}_4(s_2) &= \\ \tilde{\gamma}_4(b \cdot (ac)b(ac)^{-1} \cdot (ac)b(ac)^{-1} \cdot (ac)^2b(ac)^{-2} \cdot (ac)^2b(ac)^{-2} \cdot (ac)^3b(ac)^{-3}) &= \\ \tilde{\gamma}_4(b \cdot (ac)b(ac)^{-1} \cdot (ac)b(ac)^{-1} \cdot b \cdot b \cdot (ac)b(ac)^{-1}) &= \\ (2, 3) \cdot (1, 4) \cdot (1, 4) \cdot (2, 3) \cdot (2, 3) \cdot (1, 4). & \end{aligned}$$

Now it is easy to see that $s_1 \neq s_2$, since the groups $(\Sigma_4)_{\tilde{\gamma}_4(s_1)}$ and $(\Sigma_4)_{\tilde{\gamma}_4(s_2)}$ are not isomorphic. Indeed, $(\Sigma_4)_{\tilde{\gamma}_4(s_1)} = \Sigma_4$ and

$$(\Sigma_4)_{\tilde{\gamma}_4(s_2)} = \{(1, 4), (2, 3)\} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

□

REFERENCES

- [Bar-1] BARRAUD, J.-F.: *Curves pseudo-holomorphes équisingulières en dimension 4*. PhD Thesis, Toulouse, 1998; Bull. Soc. Math. France, **128** (2000), 179–206.
- [Bar-2] BARRAUD, J.-F.: *Nodal symplectic spheres in $\mathbb{C}\mathbb{P}^2$ with positive self-intersection*. Internat. Math. Res. Notices, 1999, No. 9, 495–508.
- [H-L-S] HOFER, H., LIZAN, V., SIKORAV, J.-C.: *On genericity for holomorphic curves in four-dimensional almost-complex manifolds*. J. of Geom. Anal., **7**, 149–159, (1998).
- [Iv-Sh-1] IVASHKOVICH, S., SHEVCHISHIN, V.: *Deformations of noncompact complex curves, and envelopes of meromorphy of spheres*. (Russian) Mat. Sb. **189** (1998), No.9, 23–60; translation in Sb.Math. **189** (1998), No.9–10, 1335–1359.
- [Iv-Sh-2] IVASHKOVICH, S., SHEVCHISHIN, V.: *Structure of the moduli space in a neighborhood of a cusp-curve and meromorphic hulls*. Invent. Math., **136**(1999), 571–602.
- [Kh-Ku] KHARLAMOV V., KULIKOV VIK.S.: *On Braid Monodromy Factorizations*. Izv. Math. **67:3** (2003); see also [ArXiv:mathAG/0302113](https://arxiv.org/abs/mathAG/0302113)
- [Ku-Te] KULIKOV VIK. S., TEICHER M.: *Braid monodromy factorization and diffeomorphism types*. Izv. Math. **64:2** (2000), 311–341.
- [La-McD] LALONDE, F.; MCDUFF, D.: *The classification of ruled symplectic manifolds*. Math. Res. Lett., **3**, 769–778, (1996).
- [Li-Liu] LI, T.-J.; LIU, A.-K.: *Symplectic structure on ruled surfaces and a generalized adjunction formula*. Math. Res. Lett., **2**, 453–471, (1995).
- [Mi-Wh] MICALLEF, M., WHITE, B.: *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*. Ann. Math., **139**, 35–85 (1994).

- [Moi] MOISHEZON B.: *The arithmetic of braids and a statement of Chisini*. Contemporary Math. **164** (1994), 151-175.
- [Moi-Te] MOISHEZON B., TEICHER M. *Braid group technique in complex geometry, I: line arrangements in $\mathbb{C}P^2$* . Contemporary Math. **78** (1988), 425-555.

DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE MA 02139-4307, USA,
and CENTRE DE MATHÉMATIQUES, ECOLE POLYTECHNIQUE, 91128 PALAISEAU,
FRANCE

E-mail address: `auroux@math.mit.edu`, `auroux@math.polytechnique.fr`

STEKLOV MATHEMATICAL INSTITUTE, GUBKINA STR., 8, 119991 MOSCOW,
RUSSIA

E-mail address: `kulikov@mi.ras.ru`

MATHEMATISCHES INSTITUT, ABTEILUNG FÜR REINE MATHEMATIK, ALBERT-
LUDWIGS-UNIVERSITÄT, ECKERSTRASSE 1, D-79104 FREIBURG, GERMANY

E-mail address: `sewa@email.mathematik.uni-freiburg.de`