D. Auroux – 28/8/09 – Tropical aspects of mirror symmetry III

We saw yesterday:

toric variety X_Δ $\longleftrightarrow$ $(\mathbb{C}^*)^n$, $W = \sum t^{\alpha_i} z^{\nu_i}$

polytope w/ facets $\langle \nu_i, x\rangle = \alpha_i$ — Laurent polynomial

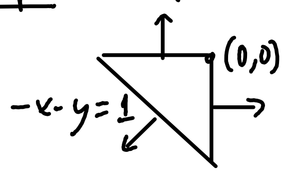
complex geom. of X_Δ $\longleftrightarrow$ sympl. geom. of $((\mathbb{C}^*)^n, W^{-1}(1))$

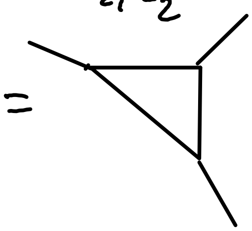
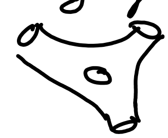
e.g. line bundles on X_Δ (more generally, coherent sheaves) $\longleftrightarrow$ certain Lagrangians in $(\mathbb{C}^*)^n$ with boundary in $W^{-1}(1)$

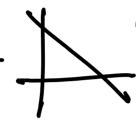
also, symplectic geom. of X_Δ $\longleftrightarrow$ Complex geom. of singularities of W.

- $W^{-1}(1)$ is actually mirror to CY hypersurface = toric divisor in X_Δ

Example: $\mathbb{CP}^2$ $\longleftrightarrow$ $(\mathbb{C}^*)^2$, $W = z_1 + z_2 + \dfrac{t}{z_1 z_2}$



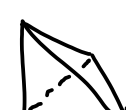
$W^{-1}(1) =$  (actually genus 1) 

This 3-punctured elliptic curve is mirror to the toric divisor $C_0 =$  $\{XYZ=0\} \subset \mathbb{CP}^2$;

- smoothing C_0 to a nonsing. elliptic curve (cx. deform^{n} away from the LCSL point C_0 in moduli space of plane cubics) $\underset{\text{mirror}}{\longleftrightarrow}$ compactifying $\check{C}_0 = W^{-1}(1)$ to a closed elliptic curve (sympl. deform away from large volume limit)

<u>Gross–Siebert programme:</u>

use tropical geometry to construct mirror families near toric degeneration limit. Key idea = encode symplectic / complex geometries into singular integral affine manifolds. The process is too complicated to describe here, but

E.g.: quartic surface $\subset \mathbb{CP}^3$: toric limit $\{XYZW=0\}$ $\longleftrightarrow$ 

(4 coordinate planes $\underset{\text{tropically}}{\leadsto}$ ∂ of simplex).

(2) $\underset{\text{deformation}}{\rightsquigarrow}$ quartic surface $\{XYZW = \varepsilon f_4\} \longleftrightarrow$ affine structure on "tropical K3" $\simeq S^2 - 24$ points

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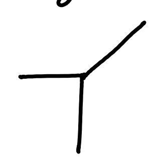
We'll now focus instead on: given an arbitrary hypersurface $H \subset X_\Delta$ in a toric variety (Fano), how can we build a mirror to it?
[joint work in progress w/ Abouzaid & Katzarkov]

* in particular: mirrors to varieties of general type! running examples will be: $H =$ line in $\mathbb{CP}^2$
and $H =$ genus 2 curve of degree $(2,3)$ in $X = \mathbb{P}^1 \times \mathbb{P}^1$.

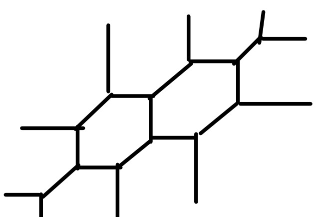
Recipe:

1) consider a tropical degeneration of $H \subset X_\Delta$ (LCSL degeneration of H!)
ie. $H_t : \{f_t = 0\}$ $(t \to 0)$, amoeba converges to some tropical hypersurface Π

Example: line in $\mathbb{P}^2$ $\rightsquigarrow$
e.g. $z_0 + z_1 + z_2 = 0$



Example: $f_t = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow$ e.g. $\Pi =$
f_t degree $(2,3)$ in $((z_0 : z_1), (w_0 : w_1))$

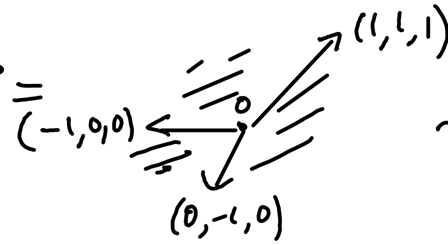


(actual amoeba for $t \neq 0$ small $=$ thickening of Π).

- Actually tropicalization of equation f_t defines a piecewise linear convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ whose domains of linearity are exactly the components of $\mathbb{R}^n - \Pi$.
(φ is def.^{d} up to adding a global affine function ...).

e.g.: for line in $\mathbb{P}^2$, $\varphi = \max(x_1, x_2, 0)$

2) consider the (noncompact) convex polytope
$$P = \{x_{n+1} \geq \varphi(x_1, \dots, x_n)\} \subseteq \mathbb{R}^{n+1} \quad \text{(overgraph of } \varphi\text{)}.$$
(facets of $P \equiv$ components of $\mathbb{R}^n - \Pi$)
This defines a toric variety Y of $\dim_{\mathbb{C}} n+1$

<u>Ex:</u> for line $\subset \mathbb{P}^2$, $P =$ 

$\rightsquigarrow Y \cong \mathbb{C}^3$

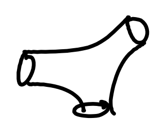
<u>NB:</u>
- The fan for Y (dual to P) has rays generated by all $(-\vec{\alpha}, 1) \in \mathbb{Z}^{n+1}$ where $\vec{\alpha} \in \mathbb{Z}^n$ = weight of a monomial in f_t
- The toric divisors in $Y \leftrightarrow$ the components of $\mathbb{R}^n \setminus \Pi$ ($\leftrightarrow$ facets of P)

  E.g. for line $\subset \mathbb{P}^2$, $3 \times \mathbb{C}^2$

  for genus 2 curve $\subset \mathbb{P}^1 \times \mathbb{P}^1$, 12 components:
  - 2 compact toric surfaces (e.g. $\widehat{\mathbb{P}}^2_{3 \text{ points}}$).
  - 10 non compact ($\mathbb{C}^2$, $\mathbb{P}^1 \times \mathbb{C}$, and blowups)

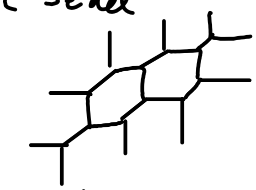
3) structure of $Y \rightsquigarrow \exists$ (monomial) function $W : Y \to \mathbb{C}$ s.t. the divisor of zeros of W is exactly $\coprod$ toric hypersurfaces (ie. W vanishes with order 1 on each toric hypersurface assoc. to a facet of P).

<u>Claim:</u> (Y, W) is mirror to the open hypersurface $\mathring{H} = H \cap (\mathbb{C}^*)^n$ (intersect with open stratum in X).

<u>Ex:</u> H = line in $\mathbb{CP}^2 \rightsquigarrow \mathring{H}$ = pair of pants $\mathbb{P}^1 \setminus 3$ pts 

mirror: $(\mathbb{C}^3, W = xyz)$. $\mathrm{crit}(W)$ = coord. axes 

Cone side of HMS verified by { Abouzaid-Seidel) Seidel

Mirror of open genus 2 curve in $(\mathbb{C}^*)^2$  Π

= (toric 3-fold Y, W vanishing on the 12 toric surfaces depicted by $(\mathbb{R}^2, \Pi)$. $\mathrm{crit}(W) = \bigcup \mathbb{P}^1$'s and $\mathbb{C}$'s as depicted by Π)

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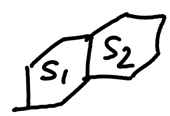
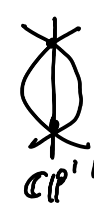
⚠ We've actually considered a degenerating family H_t ...
symplectic structure on H_t indep^t of t
Complex structure $\to$ LCSL as $t \to 0$
on mirror: complex structure, which we've specified, indep^t of t
symplectic structure actually scales by $\log t$
$\to$ large vol. limit as $t \to 0$

4) Compactifying: ie. mirror of H_t rather than $H_t \cap (\mathbb{C}^*)^n$?
This is more complicated, specially when H_t is of general type (the interesting case)
In general: the mirror remains Y, but should modify W by adding other terms to it. Main terms = monomials corresponding to rays in fan for X.

Ex: • line $\subset \mathbb{CP}^2 \longleftrightarrow \mathbb{C}^3$, $\tilde{W} = xyz + x + y + z$
(NB: "equivalent" to $W = z + \frac{1}{z}$ by dim^l reduction procedure).
e.g. both have 2 nondegenerate critical points.

• genus 2 curve $\subset \mathbb{CP}^1 \times \mathbb{CP}^1 \longleftrightarrow Y$, $\tilde{W} = \ldots$

NB: $Y \supset$ two compact toric surfaces 
$\to W$ must remain constant $(=0)$ on them
$W^{-1}(0) = S_1 \cup S_2 \cup S_0$
S_0: noncompact, smoothing of the 10 noncompact components in $W^{-1}(0)$

crit W = 
3 $\mathbb{CP}^1$'s

$S_i \cap S_j \simeq \mathbb{CP}^1$
$S_0 \cap S_1 \cap S_2 = 2$ pts.

(matches prediction of Katzarkov; HMS proof by Seidel)

NB: what we build is in fact a mirror to: $X \times \mathbb{C}$ blown up along $H \times 0$. By various results (Bondal-Orlov, ...) this is pretty much the same as H, but it has effective $(-K)$ which makes it suitable for geom construction of mirror.