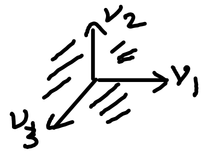
D. Auroux - 27/8/09 - Tropical aspects of mirror symmetry II

Last time: overview of mirror symmetry & its main protagonists
mirror symmetry for $(\mathbb{C}^*)^n$ (complexification of an affine subspace of $\mathbb{R}^n$ vs. conormal construction)

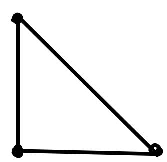
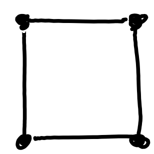
<u>Today</u>: mirror symmetry for toric varieties [Batyrev-Givental, ..., Abouzaid].

<u>Toric variety</u> = (partial) compactification of $(\mathbb{C}^*)^n$ so that torus action extends

Can be described by a complete fan (affine charts = max. cones, gluings given by combinatorics)

e.g.  $= \mathbb{CP}^2$

If we equip with a <u>polarization</u> (= projective embedding), can describe a proj. toric var. as a lattice <u>polytope</u> Δ (moment polytope) dual to the fan: facets have eqns. $\langle v_i, x\rangle = \alpha_i$, $v_i \in \mathbb{Z}^n$ rays of fan, $\alpha_i \in \mathbb{Z}$
vertices $\leftrightarrow$ max. cones of fan

e.g. $\mathbb{CP}^2 =$  $\mathbb{CP}^1 \times \mathbb{CP}^1 =$ 

* in fact, Δ = orbit space for T^n-action by rotation of coords
interior $\leftrightarrow$ open stratum $(\mathbb{C}^*)^n$, and faces $\leftrightarrow$ lower dim. strata (T^n action not free)
(e.g in $\mathbb{CP}^2$, coord axes $\sim$ 3 $\mathbb{CP}^1$)

* the polarization gives a symplectic form (= induced by std one of proj. space into which we embed), such that areas of $\mathbb{CP}^1$'s at edges of Δ $\equiv$ degree as rat! curve in ambient $\mathbb{P}^N$ $\equiv$ length of edge with respect to the integer lattice. (e.g could also embed $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$)

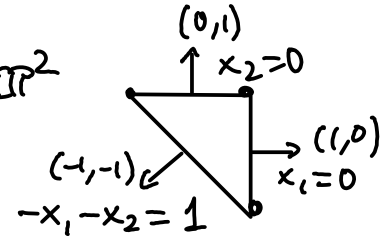
<u>The mirror:</u>

- Toric varieties are not Calabi-Yau, but still there is a version of mirror symmetry in that context. X_Δ with facets $\langle v_i, x\rangle = \alpha_i$

$\Rightarrow$ The mirror of X_Δ is $(\mathbb{C}^*)^n$ equipped with Laurent poly $W(z) = \sum t^{\alpha_i} z^{v_i}$

②

E.g.: $\mathbb{CP}^2$



$\longleftrightarrow \quad (\mathbb{C}^*)^2, \quad W = z_1 + z_2 + \dfrac{t}{z_1 z_2}$

(t = parameter: $t \to 0$ corresponds to large cx. str. limit)

- What does it mean to be mirror to X_Δ?

Various things, e.g. $QH^*(X_\Delta) \simeq Jac(W) := \mathbb{C}[[t]][z_i^{\pm 1}]/\langle \partial_i W \rangle$

quantum cohomology ring — Jacobian ring

and .... coherent sheaves on X_Δ (cx-subvars., vector bundles...) $\longleftrightarrow$ Lagrangians in $(\mathbb{C}^*)^n$ with $\partial L \subseteq W^{-1}(1)$ ← arbitrary constant (+ technical condition)

[actually, derived cat & Fukaya cat.]

ie. complex geometry of $X_\Delta \longleftrightarrow$ sympl geometry of $((\mathbb{C}^*)^n, W^{-1}(1))$

[Also, (compactif.ⁿ of) $W^{-1}(1)$ is mirror to (smoothing of) CY hypersurface H in X_Δ given by $\cup$ toric strata; restricting cx. geometry from X_Δ to H is mirror to restricting Lags. in $((\mathbb{C}^*)^n, W^{-1}(1))$ to their boundary in $W^{-1}(1)$.]

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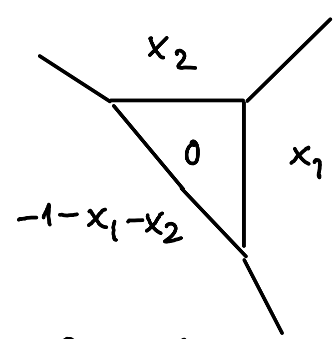
* Viewing $W^{-1}(1)$ via tropical geometry ($t \to 0$)

Look at $\mathrm{Log}_{1/t}(W^{-1}(1))$: amoeba converges to tropical hypersurface Π (= locus where 2 of the terms in $W-1$ are "tied for largest")

ex. for $\mathbb{CP}^2$: $W - c = z_1 + z_2 + \dfrac{t}{z_1 z_2} - 1$

"tropicalization": $\max(x_1, x_2, -1-x_1-x_2, 0) \rightsquigarrow$



General fact: one component of $\mathbb{R}^n - \Pi$ is $\simeq \Delta$.

We'll want to consider Lagr. submflds of $(\mathbb{C}^*)^n$ whose image by $\mathrm{Log}_{1/t}$ is $\subseteq \Delta$, & boundary $\subseteq$ complexifications of the faces of Δ

3 ★ We'll be constructing mirrors to the line bundles $\mathcal{O}(k)$ over X_Δ.

Holom. vector bundle = where defining sections of alg. subvars. actually "live".

Ex: in $\mathbb{CP}^2$, $x^3+y^3+z^3=0$ defines an elliptic curve ...

$x^3+y^3+z^3$ is not a function! ($(x:y:z)$ homogeneous coords. only def^d up to scaling)

though locally over an affine subset we can view it as a function:

e.g. where $z \neq 0$, can fix $z=1$, and think of our section as the function $\left(\frac{x}{z}\right)^3+\left(\frac{y}{z}\right)^3+1$ in the affine chart $\left(\frac{x}{z}:\frac{y}{z}:1\right)$.

but it scales nontrivially when change charts: $\left(\frac{x}{z}\right)^3+\left(\frac{y}{z}\right)^3+1 \rightsquigarrow \left(\frac{x}{y}\right)^3+1+\left(\frac{z}{y}\right)^3$

"change of triv^{n}" by factor $\left(\frac{z}{y}\right)^3$.

This is a section of the line bundle $\mathcal{O}(3)$

(= "regular functions which scale homogeneously with weight 3")

In general, given a projective var. (e.g. polarized toric var.),

$\mathcal{O}(k)$ = bundle which contains homogeneous polynomials of degree k in the coords. of the ambient projective space

($\mathcal{O}=\mathcal{O}(0)$ = regular functions).

★ Want: Lagrangian $L_k \subset (\mathbb{C}^*)^n$ (rel. $W^{-1}(1)$) mirror to $\mathcal{O}(k)\to X_\Delta$?

Claim: L_k should be a graph $p=f(x)$, $f: \Delta(\subset \mathbb{R}^n - T) \to T^n$

(ie. a section of T^n-bundle $(x,p)\mapsto x$).

[Indeed, $\mathcal{O}(k)$ "lives over all of X_Δ" and has rank 1 at each pt

so intersection number [= dim Hom($\mathcal{O}(k), \mathcal{O}_q$) for experts] is 1

$\rightsquigarrow$ recalling points $\overset{\text{MS}}{\Longleftrightarrow}$ T^n's (cotangent fibers = conormals to points)

mirror Lagr. should intersect each T^n in a single point.]

The function f "keeps track of failure of trivialization"; in our case,

$$L_k := \{(x,p)\in \Delta \times (\mathbb{R}/\mathbb{Z})^n \,/\, p_j = -k x_j\}$$

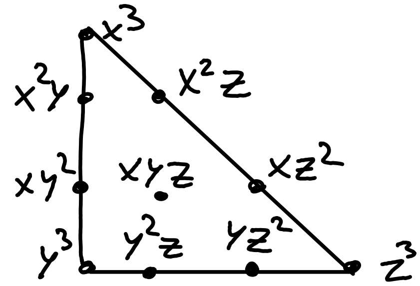
④ Check: • L_k is Lagrangian: $TL_k = \{(v,-kv) \,/\, v\in\mathbb{R}^n\} \subset \underset{x}{\mathbb{R}^n} \oplus \underset{p}{\mathbb{R}^n}$
isotropic wrt $\sum dx_i \wedge dp_i$ ✓

• at ∂L_k: when $x \in F$ facet of Δ, $\langle \nu, x\rangle = \alpha \in \mathbb{Z}$
$\Rightarrow$ complexification $F^{\mathbb{C}} := \{(x,p) \mid \langle \nu,x\rangle = \alpha,\ \langle \nu, p\rangle = 0 \bmod \mathbb{Z}\} \ni (x,-kx)$ ✓

<u>Now</u>: points of $L_0 \cap L_k = \{x\in\Delta \,/\, -kx \equiv 0 \bmod \mathbb{Z}\}$
$= \Delta \cap \left(\frac{1}{k}\mathbb{Z}\right)^n$ ie. <u>integer pts in $k\Delta$</u>.

<u>Matches with</u>: classical fact about toric varieties:
$\exists$ basis of global sections of $\mathcal{O}(k)$ given by integer pts in $k\Delta$.
basis element corresponding to a given integer point $\equiv$ homogeneous monomial whose order of vanishing along facet $F \leftrightarrow$ lattice distance from point to facet

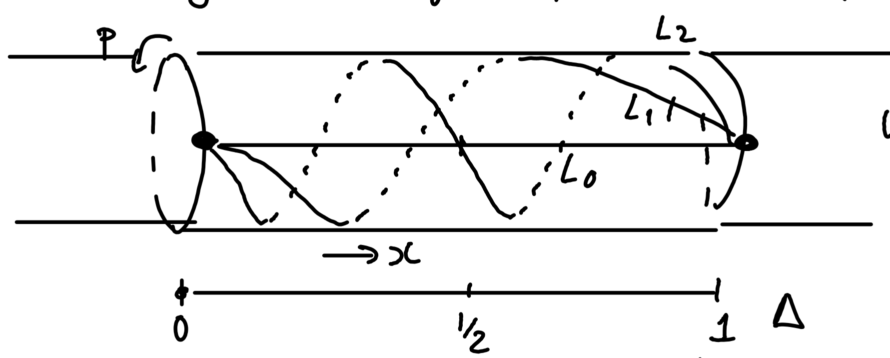
<u>Ex</u>: $\mathbb{CP}^2$, $\mathcal{O}(3)$ = homogeneous deg. 3 polynomials:



<u>Hence</u>: $\underset{\text{cx-side}}{\mathrm{Hom}(\mathcal{O},\mathcal{O}(k))} \simeq \underset{\text{sympl. side}}{HF(L_0,L_k)} \simeq \mathbb{C}^{|k\Delta\cap\mathbb{Z}^n|}$: mirror symmetry ok.
(for $k\geq 0$).

<u>Ex</u>: • $\mathbb{CP}^1$: sections of $\mathcal{O}(k) = a_0x^k + a_1x^{k-1}y + \dots + a_ky^k$ dim. $k+1$.
$(x:y)$

$\Longleftrightarrow$



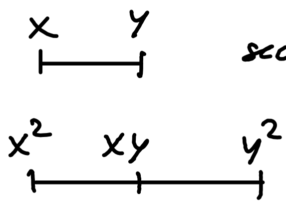
$w^{-1}(1) = 2$ pts

• Also: $\mathrm{Hom}(\mathcal{O}(j),\mathcal{O}(k)) \simeq HF(L_j,L_k) = \mathbb{C}^{|(k-j)\Delta\cap\mathbb{Z}^n|}$ also ok ✓
$(j\leq k)$ ↑ given by mult$^{\underline{n}}$ by deg. $(k-j)$ homogeneous poly.

- More work required: <u>product maps</u> $HF(L_0, L_j) \otimes HF(L_j, L_k) \to HF(L_0, L_k)$
  correspond to: given $q \in j\Delta \cap \mathbb{Z}^n$, $q' \in (k-j)\Delta \cap \mathbb{Z}^n$ $(\rightsquigarrow q+q' \in k\Delta \cap \mathbb{Z}^n)$
  mult^{n} on basis elts is given by $\langle q \rangle \cdot \langle q' \rangle = \langle q+q' \rangle$
  matches with product law on homogeneous polynomials [Abouzaid]

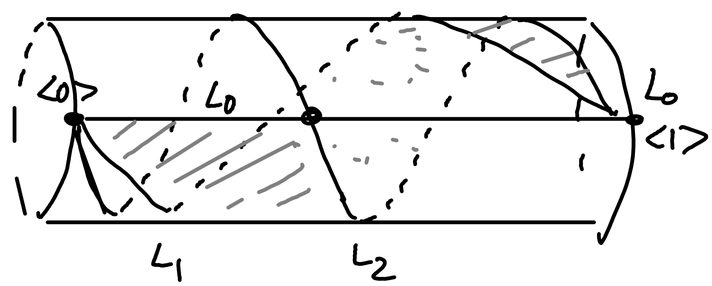
★ This is important because of a classical thm: the der. cat. $D^b\mathrm{Coh}(X_\Delta)$ is generated by the $\mathcal{O}(k)$'s $\Rightarrow$ with enough homological algebra, can use verification for $\mathcal{O}(k) \longleftrightarrow L_k$ to essentially prove HMS conjecture.

<u>Example:</u> $\mathbb{CP}^1$:
$(x:y)$



sections of $\mathcal{O}(1)$

sections of $\mathcal{O}(2)$.



$x \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$
multiply $\langle 0 \rangle \in HF(L_0, L_1)$
with $\langle 1 \rangle \in HF(L_1, L_2)$
$y \in \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2))$
$\rightarrow$ get $\langle 1 \rangle \in HF(L_0, L_2)$?
$xy \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(2))$.

Floer product counts (holomorphic) triangles bounded by L_0, L_1, L_2, with corners at prescribed intersection points.