D. Auroux – 26/8/09 – Tropical aspects of mirror symmetry I (1)

ADVERTISEMENT: graduate class on mirror symmetry TuTh 9:30-11 Evans Rm. 31
starts tomorrow but 1st lecture optional

Mirror symmetry = broad correspondence between symplectic geometry and complex algebraic geometry coming from duality b/w two versions of string theory.

In some sense, affine geometry / tropical geometry is what lies inbetween sympl. & complex geometries, at the heart of mirror symmetry.

Disclaimer: • I'm not a tropical geometer! • I will omit many attributions.

Today: • complex & symplectic mflds, some features of mirror symmetry
• mirror symmetry in tori

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§ Background on complex & symplectic geometry:

MS predicts existence of mirror pairs of CY manifolds

CY := kähler mfld with $\Omega^{n,0}_X \simeq \mathcal{O}_X$ $\rightsquigarrow$ (X, ω, J, Ω)
↳ sympl. + complex structures

ω: symplectic structure; J: complex structure; Ω: holom. vol. form

1) "B-model":

• complex manifold:

glue together open subsets of $\mathbb{C}^n$ by coord. changes = biholomorphisms
(differential is $\mathbb{C}$-linear) ie. have locally holom. coords $\{z_j = x_j + i y_j\}$

Important case (the only one relevant for us): alg. vars $/\mathbb{C}$ (polynomial expr's in z's)

• Calabi-Yau condition: existence of a global holom. volume form

locally $\Omega = f(z_1, .., z_n)\, dz_1 \wedge \dots \wedge dz_n$ (evaluates: tgt n-planes $\mapsto \mathbb{C}$).

$(\Omega \wedge \bar{\Omega} = |f|^2 . \mathrm{vol})$. This imposes a topological condition on X.

• we'll care about:

– complex submanifolds (e.g. alg. subvarieties: def^d by polynomial eq^{n}s)

– also consider holomorphic vector bundles, or coherent sheaves
(if you don't know about these: won't use today)

– Intersection theory

• $V_1, V_2 \subset X$ of complementary dim., $V_1 \cap V_2$ = points: $\#(V_1 \cap V_2)$?
more generally, $H^*(V_1 \cap V_2)$? (actually, Dolbeault $H^{0,*}$)

② 2) "A-model":

- <u>symplectic manifold</u>: ω = closed nondegenerate 2-form (ie: oriented tgt 2-planes $\to \mathbb{R}$; for closed surfaces $\int_\Sigma \omega$ is deform^{n} invariant, only sees homology class $[\Sigma]$).

  Local model: $\mathbb{R}^{2n}$, $\omega_0 = \sum dx_i\, dy_i$ [Darboux's theorem]

  Alternatively: symplectic mfld = glue together open subsets of $\mathbb{R}^{2n}$ by coordinate changes which preserve ω_0.

- <u>Lagrangian submanifolds</u>: $L^n \subset X^{2n}$ s.t. $\omega_{|L} = 0$

  loc. model $\mathbb{R}^n = \mathrm{span}(\partial_{x_i}) \subset (\mathbb{R}^{2n}, \omega_0)$

- <u>Intersection theory</u>: $L, L' \rightsquigarrow \#(L \cap L')$. Actually,

  <u>Floer homology</u> = "corrected" intersection theory, invt under "Hamiltonian" isotopies of L, L' (discards some intersections)

  $CF(L,L') = \mathbb{C}^{|L\cap L'|} \rightsquigarrow HF(L,L')$ of rank $\leq \#(L\cap L')$.

★ <u>Mirror symmetry philosophy</u>: (X, J, Ω, ω) Calabi-Yau
$(X^\vee, J^\vee, \Omega^\vee, \omega^\vee)$ mirror CY

($J^\vee, \Omega^\vee$ <u>complex</u> str. on $X^\vee$ depends on ω <u>sympl.</u> str. on X and vice-versa)

Roughly, $\left.\begin{array}{l}\text{complex subvars.}\\ \left(\begin{array}{l}\text{holom. bundles}\\ \text{coherent sheaves}\end{array}\right)\end{array}\right\}$ on X $\longleftrightarrow$ Lagrangian submflds in $X^\vee$ (+ local systems)

intersection theory (sections, sheaf cohomology) $\longleftrightarrow$ (corrected) intersection theory ie. (twisted) Floer homology.

and vice-versa.

★ <u>More precisely</u>, MS is a perturbative phenomenon near

"large cx. structure limit" for (J, Ω) $\longleftrightarrow$ "large volume limit" for ω

ie. degenerating family of cx. varieties $(X_t, t \in \Delta)$ $\qquad$ (symplectic areas $\to \infty$: $t = \exp(-\int_C \omega)$ parameter $\to 0$).

③ LCSL degeneration is where tropical geometry arises.

In general, might only make sense as formal families: scheme / $\mathbb{C}[[t]]$
t formal param.

HMS Conj (Kontsevich '94):

derived category of coherent sheaves on X $\quad D^bCoh(X)$
& derived Fukaya category of $X^\vee$ $\quad DF(X^\vee)$ $\quad$ are equivalent. & vice-versa

(derived cat = look at complexes of objects up to homotopy & quasi isom.
- very roughly, "formal combination" of objects).

[There are other predictions, e.g. Hodge theory/periods on $X \leftrightarrow$ GW invts of $X^\vee$]

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§ <u>Example: $(\mathbb{C}^*)^n$</u> $\leftarrow$ simplest instance where tropical geometry appears.

$(\mathbb{C}^*)^n$, std. complex structure, $\Omega = \prod d\log z_i$ $\longleftrightarrow$ $(\mathbb{C}^*)^n$, $\omega^\vee = \sum d\log z_i \wedge d\log \bar{z}_i$
$\left(= \sum d\log r_i \wedge d\theta_i\right)$

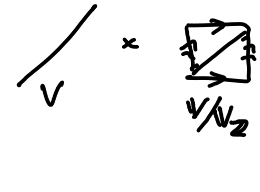
[$(\mathbb{C}^*)^n$ is its own mirror]

Things are best viewed in terms of the Log map $\quad (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$
$(z_i) \longmapsto (\log|z_i|)$

- LCSL corresponds to rescaling log map by some factor t ($\leftrightarrow$ tropical limit)
We'll start with fixed J, ω for simplicity
- geometric interpretation of MS := dualization in T^n coordinates $\theta_i \leftrightarrow p_i$
Complex side: $(\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n = T(\mathbb{R}^n)/\mathbb{Z}^n$ tangent bundle
Symplectic side: $(\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^{*n} = T^*\mathbb{R}^n/(\mathbb{Z}^n)^*$ cotangent bundle (IE: POINT + LIN. FORM ON TGT SPACE)

☆ $V \subset \mathbb{R}^n$ affine subspace of dim. k, s.t. underlying vector space $\mathbb{V}$
admits a basis of integer vectors: $\mathbb{Z}\text{-span}(v_1, \dots, v_k) = \mathbb{V} \cap \mathbb{Z}^n =: \mathbb{V}_\mathbb{Z}$

$\to$ <u>complexification</u>: $V^\mathbb{C} = TV/\mathbb{V}_\mathbb{Z} = \{(x, \theta) \,/\, x \in V, \theta \in \mathbb{V}/\mathbb{V}_\mathbb{Z}\} \subset T\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{C}^*)^n$
is always a complex submfld of $(\mathbb{C}^*)^n$
(tangent space = complexification of $\mathbb{V}$)



④ <u>ex:</u> $\{x_1 + 2x_2 = 1\} \subset \mathbb{R}^2$ $\longrightarrow$ $\{z_1 z_2^2 = e^{2\pi 1}\} \subset (\mathbb{C}^*)^2$

complexify: $\{x_1+2x_2=1,\ \theta_1+2\theta_2=0\}$, $x_j \in \mathbb{R}$, $\theta_j \in \mathbb{R}/\mathbb{Z}$; $z_j = \exp 2\pi(x_j + i\theta_j)$ [or rather family $z_1 z_2^2 = t^{-1}$]

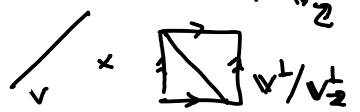
$\rightsquigarrow$ <u>conormal construction</u>: $N^*V = \{(x,p) \,/\, x \in V,\ p \in \underbrace{\mathbb{V}^\perp}_{\text{annihilator}}\} \subset T^*(\mathbb{R}^n)$

Lagrangian wrt std. sympl. form $\Sigma\, dx_j \wedge dp_j$

(tangent space $= (\mathbb{V}\times 0) \oplus (0 \times \mathbb{V}^\perp) \subset \underset{x}{\mathbb{R}^n} \times \underset{p}{\mathbb{R}^n}$).

$\mathbb{V}^\perp$ also has a basis of integer vectors; $\mathbb{V}^\perp_{\mathbb{Z}} := \mathbb{V}^\perp \cap (\mathbb{Z}^n)^*$ dual lattice

$V^L = N^*V/\mathbb{V}^\perp_{\mathbb{Z}} = \{(x,p) \,/\, x \in V,\ p \in \mathbb{V}^\perp/\mathbb{V}^\perp_{\mathbb{Z}}\} \subset T^*\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{C}^*)^n$



is always Lagrangian.

Mirror correspondence: $V^{\mathbb{C}} \longleftrightarrow V^L$.

★ <u>Ex.</u> case $V = pt \rightsquigarrow V^{\mathbb{C}} = pt \in (\mathbb{C}^*)^n$

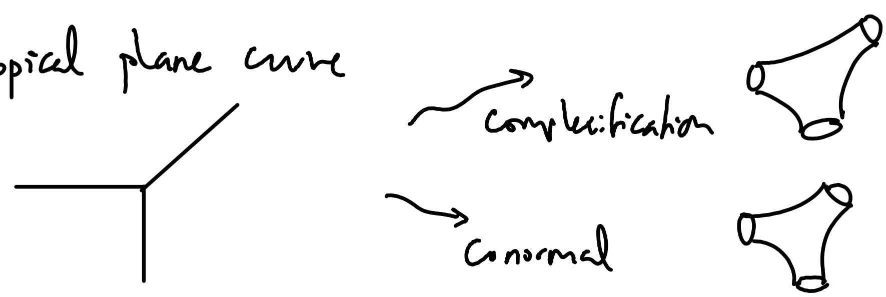
its mirror $V^L =$ torus $S^1(r_1) \times \dots \times S^1(r_n) \subset (\mathbb{C}^*)^n$ (fix x, but $p =$ anything).

★ Patch affine pieces together (when possible! let's say: $\underline{\underline{n=2}}$)

only well-understood case – in general $\exists$ obstructions to realization

+ not so easy to build a smooth Lagrangian

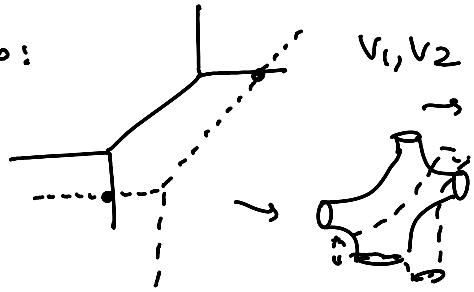
tropical plane curve



complexification: (family of) complex curves $\subset (\mathbb{C}^*)^2$

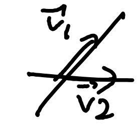
conormal: Lagrangian submanifold $\subset (\mathbb{C}^*)^2$

Intersections:



V_1, V_2 two tropical plane curves, in g$^{\text{al}}$ position

$\rightarrow V_1 \cap V_2 =$ finite set of pts with multiplicities $(= |\det(\vec{v}_1, \vec{v}_2)|$



integer vectors along edges

Then $\#(V_1 \cap V_2)_{trop} = \#(V_1^{\mathbb{C}} \cap V_2^{\mathbb{C}}) \underset{\text{ms}}{=} \#(V_1^L \cap V_2^L)$ (in θ or p direction, $\#(\mathbb{R}/\mathbb{Z}\,\vec{v}_1) \cap (\mathbb{R}/\mathbb{Z}\,\vec{v}_2) = |\det|$).