Mirror symmetry for blow-ups

(Abouzaid-Auroux-Katzarkov, in progress)

Goal: construct mirror of $\hat{X}_Y$ = blow-up of X along a codimension 2 subvariety $Y \subset X$ (need $Y \subset D \in |-K_X|$)

Motivation: a mirror of $\hat{X}_Y$ is almost as good as a mirror of Y.

- $D^b Coh(\hat{X}_Y) \simeq \langle D^b Coh(Y), D^b Coh(X) \rangle$ (semiorthogonal decomp.) (Bondal-Orlov)
- also expect $\mathcal{F}(\hat{X}_Y)$ related to $\mathcal{F}(Y)$ (esp. if $X = D \times \mathbb{C}$ and Y fiber of a pencil in D)

Simplification: assume (X, D) **toric** (but not Y).

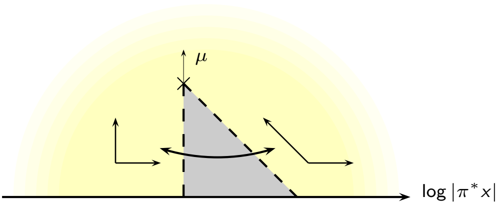
Motivating example: what's the mirror of a genus 2 curve Σ?
Answer: blow up $(\mathbb{CP}^1)^3$ along $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{0\}$, take mirror, restrict.

Blowing up a point

Local model in dim. 2:

$X = \mathbb{C}^* \times \mathbb{C}$, $D = \mathbb{C}^* \times \{0\}$, $\Omega = d\log x \wedge d\log y$, $\omega = \omega_0$
$\hat{X}$ = blowup at $(1,0)$, $\hat{D}$ = proper transform, $\hat{\Omega} = \pi^*\Omega$, $\hat{\omega} = \hat{\omega}_\epsilon$ ($\int_E \hat{\omega} = \epsilon$)

S^1 action ($y \mapsto e^{i\theta} y$) lifts, fixed point set $\hat{D} \cup \{pt\}$. $\mu :=$ moment map.
S^1-invariant S.Lag. fibration on $\hat{X} \setminus \hat{D}$: $L_{t_1,t_2} = \{\log|\pi^* x| = t_1,\ \mu = t_2\}$.



Blowing up a point (continued)

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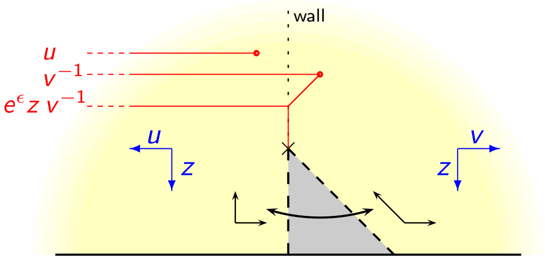


Classical: $uv = 1$ for $|z| < e^{-\epsilon}$ (above $\times$); $uv = e^\epsilon z$ for $|z| > e^{-\epsilon}$ (below $\times$).

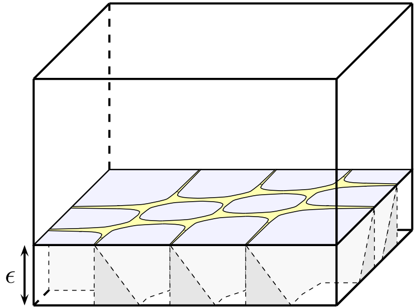
Corrected: $\{(u,v,z) \in \mathbb{C}^2 \times \mathbb{C}^*,\ uv = 1 + e^\epsilon z\}$ Superpotential: $W = z$

$(\Rightarrow$ blowup of $\mathbb{P}^1 \times \mathbb{P}^1$: $W = z + e^{-A} z^{-1} + u + e^{-B} v)$

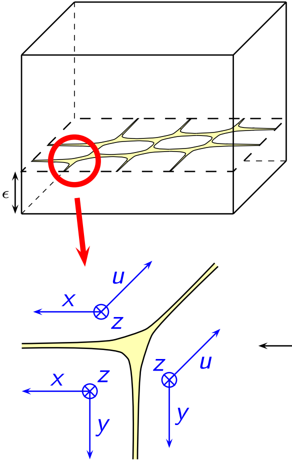
Blowing up a curve (A.-A.-K., in progress)

$X = (\mathbb{CP}^1)^3$, $D = \bigcup$ toric strata, $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{0\} \subset D \subset X$.
$\hat{X} =$ blowup along Σ, $\hat{D} =$ proper transform, $\hat{\Omega} = \pi^*\Omega$, $\int_E \hat{\omega} = \epsilon$.

- S^1-action (3^{rd} factor) lifts; new fixed point stratum $\simeq \Sigma$ at $\mu = \epsilon$. All reduced spaces $\simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, carry (S.??) Lag. torus fibrations.
- This gives a Lagr. T^3 fibration on $\hat{X} \setminus \hat{D}$, with discriminant locus $\simeq$ amoeba of Σ.



Blowing up a curve (continued)



Walls propagate "vertically" from the amoeba.
Chambers $\leftrightarrow$ components in its complement.

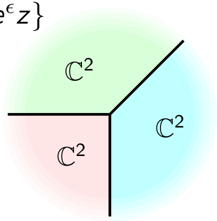
Need: instanton-corrected gluing across walls.

Work out local models (and glue them together)

Mirror: $\{xyu = 1 + e^{\epsilon} z\}$

$\{z = -e^{-\epsilon}\}$
$= \{xyu = 0\}$

three $\mathbb{C}^2$'s glued
along coord. axes



Local mirror: $\{(x, y, u, z) \in \mathbb{C}^3 \times \mathbb{C}^*,\ xyu = 1 + e^{\epsilon} z\}$
Superpotential: $W = z +$ other terms

Gluing pieces...

For open curve in $(\mathbb{C}^*)^2 \times \mathbb{C}$, superpotential $W = z$

$\Rightarrow$ singular fiber = union of toric surfaces glued along $\mathbb{P}^1$'s and $\mathbb{C}$'s

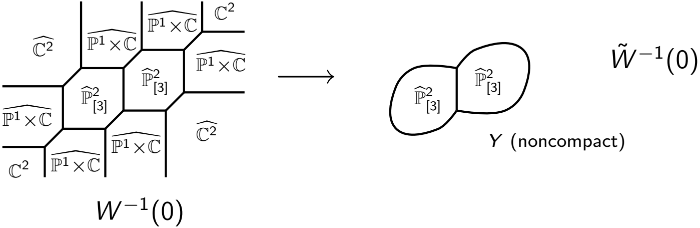
(combinatorics governed by tropicalization of Σ) crit. locus of W

Jeff Koons, *Balloon Dog* (photo Librado Romero - The New York Times)

From open curves to closed curves

Mirror to blowup of $(\mathbb{C}^*)^2 \times \mathbb{C}$ along $\Sigma \cap (\mathbb{C}^*)^2$: singular fiber = union of toric surfaces glued along $\mathbb{P}^1$'s and $\mathbb{C}$'s. For blowup of $(\mathbb{P}^1)^3$ along Σ:

- $\tilde{W} = W +$ 5 extra terms (from compactification divisors)
  $\Rightarrow$ noncompact components of sing. fiber merge, and critical points of $\tilde{W}$ become genus g trivalent configuration of $\mathbb{P}^1$'s
- $\hat{D}$ contains $c_1 < 0$ spheres, which contribute extra terms to $\tilde{W}$ (and further instanton corrections???). Expect: $\text{crit}\,\tilde{W}$ is still a trivalent configuration of $\mathbb{P}^1$'s.



(cf. Seidel: HMS for genus 2 case)